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On a Horizontal Lift of a Linear Connection to the Bundle of Linear Frames

O podniesieniu horyzontalnym koneksji liniowej do wiązki reperów liniowych

The purpose of this paper is to define a horizontal lift $\overset{H}{\nabla}$ of a linear connection ∇ on M into the total space FM of the bundle of linear frames $\pi : FM \to M$.

We define n-vertical lifts X^{v_α} of type v_α, $\alpha = 1, \dots n$, to FM of a vector field X on M and in the standard way the horizontal lift X^H to FM of a vector field X on M. Later, we define a horizontal lift $\overset{H}{\nabla}$ on FM of a linear connection ∇ on M similarly as K. Yano for the tangent bundle in [3]. Our definition of the horizontal lift $\overset{H}{\nabla}$ on FM is different than the one of L. A. Cordero and M. de Leon in [1].

We find a torsion tensor and a curvature tensor of the connection $\overset{H}{\nabla}$ on FM. Next, we describe a geodesic curve on FM and a parallel displacement of the horizontal lift A^H or the vertical lifts A^{v_α}, $\alpha = 1, \dots n$ of the vector $A \in T_xM$ with respect to the connection $\overset{H}{\nabla}$.

We state that if ∇ is a metric connection for the metric tensor g on M, then the horizontal lift $\overset{H}{\nabla}$ is a metric connection for the diagonal lift G of g to FM.

1. Let M be an n-dimensional Hausdorff manifold of the class C^∞. Let T_xM be the tangent space at a point x. The ordered basis $\{X_\alpha\}_{\alpha=1,\dots,n}$ of the tangent space T_xM is a linear frame at x. We treat the linear frame $\{X_\alpha\}_{\alpha=1,\dots,n}$ at x as the image of the map

$$(1.1) \qquad u : \mathbb{R}^n \longrightarrow T_xM \quad , \quad u(e_\alpha) = X_\alpha \quad , \quad \alpha = 1, \dots n$$

where $\{e_\alpha\}_{\alpha=1,\dots,n}$ is the natural basis of $\mathbb{R}^n$.

Let FM be a set of all linear frames over M and let $\pi : FM \to M$, $\pi(u) = x$, be the canonical projection. Let (U, x^i) be a local chart on M, where U is the coordinate neighborhood of a point x. Each vector X_α of the frame u can be expressed uniquely in local chart (U, x^i) in the form $X_\alpha = x^i_\alpha \frac{\partial}{\partial x^i}$. Then, the induced local chart on FM is of the form $(\pi^{-1}(U), x^i, x^i_\alpha)$.

If (U, x^i) and $(U', x^{i'})$ are the local charts on M and $x^{i'} = x^{i'}(x^i)$ is a change of local coordinates, then for the induced local coordinates $(\pi^{-1}(U), x^i, x^i_\alpha)$ and $(\pi^{-1}(U'), x^{i'}, x^{i'}_\alpha)$ on FM we get :

$$x^{i'} = x^{i'}(x^i) \quad , \quad x^{i'}_\alpha = A^{i'}_i x^i_\alpha \quad , \quad A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i} \quad . \tag{1.2}$$

Thus we obtain the change of the basis of the tangent space $T_u FM$:

$$\begin{aligned} \frac{\partial}{\partial x^i} &= A^{i'}_i \frac{\partial}{\partial x^{i'}} + A^{i'}_{ij} x^j_\alpha \frac{\partial}{\partial x^{i'}_\alpha} \\ \frac{\partial}{\partial x^i_\alpha} &= \qquad\qquad A^{i'}_i \delta^\beta_\alpha \frac{\partial}{\partial x^{i'}_\beta} \quad . \end{aligned} \tag{1.3}$$

2. Let Γ be a linear connection on M as a connection in the linear frame bundle $\pi : FM \to M$ and ∇ be its covariant derivative. The horizontal distribution H_Γ and the vertical distribution V on $F(U) = \pi^{-1}(U)$ are spanned by vectors D_i and D_{i_α}, $i = 1, \dots n$, $\alpha = 1, \dots n$ defined in local induced coordinates $(\pi^{-1}(U), x^i, x^i_\alpha)$ by formulas respectively :

$$D_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij} x^j_\alpha \frac{\partial}{\partial x^k_\alpha} \quad , \quad D_{i_\alpha} = \frac{\partial}{\partial x^i_\alpha} \quad . \tag{2.1}$$

If we change the local induced coordinates (1,2), then the vectors D_i, D_{i_α} are related in the following way :

$$D_i = A^{i'}_i D_{i'} \quad , \quad D_{i_\alpha} = A^{i'}_i \delta^\beta_\alpha D_{i'_\beta} \quad . \tag{2.2}$$

The dual coframe on $\pi^{-1}(U)$ with respect to the frame $\{D_i, D_{i_\alpha}\}_{i,\alpha=1,\dots n}$ consists of the 1– forms $\{\eta^j, \eta^j_\beta\}_{j,\beta=1,\dots n}$ defined in local induced coordinates by formulas :

$$\eta^j = dx^j \quad ; \quad \eta^j_\beta = dx^j_\beta + \Gamma^j_{ik} x^k_\beta dx^i \quad . \tag{2.3}$$

There are two sets : $\{\theta^\alpha\}_{\alpha=1,\dots n}$ and $\{\omega^\alpha_\beta\}_{\alpha,\beta=1,\dots n}$ of the 1–forms on FM with local expression :

$$\theta^\alpha = x^\alpha_j \eta^j \quad , \quad \omega^\alpha_\beta = x^\alpha_j \eta^j_\beta \quad . \tag{2.4}$$

The 1–form $\theta = \theta^\alpha \otimes e_\alpha$ is a canonical 1–form on FM and $\{e_\alpha\}_{\alpha=1,\dots n}$ is a canonical basis of $\mathbb{R}^n$.
The 1–form $\omega = \omega^\alpha_\beta \otimes e^\beta_\alpha$ is a connection form on FM and $\{e^\beta_\alpha\}_{\alpha,\beta=1,\dots n}$ is a canonical basis of the Lie algebra $gl(n, \mathbb{R})$.

Definition 1. A horizontal lift of the vector field X on M into the total space FM the bundle of linear frames is a vector field X^H on FM defined by :

$$\pi_* X^H = X \quad , \quad \omega(X^H) = 0 \quad . \tag{2.5}$$

Definition 2. A vertical lift of type v_α, $\alpha = 1, \dots n$ of a vector field X on M into the total space FM is a vector field X^{v_α} on FM defined by :

$$\pi_* X^{v_\alpha} = 0 \quad , \quad \omega(X^{v_\alpha}) = [u^{-1}(X)]^\beta e^\alpha_\beta \quad , \tag{2.6}$$

where $[u^{-1}(X)]^\beta$ denotes β–coordinate of vector $u^{-1}(X) \in \mathbb{R}^n$ with respect to the basis $\{e_\alpha\}_{\alpha=1,\dots,n}$.

If a vector field X on M is of the form $X = X^i \dfrac{\partial}{\partial x^i}$ in local chart (U, x^i), then the horizontal lift X^H and the vertical lifts X^{v_α} , $\alpha = 1, \dots n$ in the induced local chart $\left(\pi^{-1}(U), x^i, x^i_\alpha\right)$ are of the form :

$$\begin{aligned} X^H &= X^i D_i \\ X^{v_\alpha} &= X^i \delta^\alpha_\beta D_{i_\beta} \end{aligned} \tag{2.7}$$

The vertical lifts of type v_α, $\alpha = 1, \dots n$ for the vectors $\dfrac{\partial}{\partial x^i}$ are of the form : $\left(\dfrac{\partial}{\partial x^i}\right)^{v_\alpha} = \dfrac{\partial}{\partial x^i_\alpha}$.

The vertical distribution V on FM is spanned by n–vertical lifts of type v_α : $\left(\dfrac{\partial}{\partial x^i}\right)^{v_\alpha} = \dfrac{\partial}{\partial x^i_\alpha}$ of n vectors of the natural basis.

We have :

Proposition 1. *For the horizontal lift X^H and the vertical lifts X^{v_α}, $\alpha = 1, \dots n$ into FM of vector fields X, Y on M, we have the relations:*

$$\begin{aligned} [X^H, Y^H] &= [X, Y]^H - \gamma R(X, Y) \quad , \quad & [X^H, Y^{v_\beta}] &= (\nabla_X Y)^{v_\beta} \\ [X^{v_\beta}, Y^H] &= [X, Y]^{v_\beta} - (\nabla_X Y)^{v_\beta} \quad , \quad & [X^{v_\alpha}, Y^{v_\beta}] &= 0 \end{aligned} \tag{2.8}$$

where $\gamma R(X, Y) = R_{ijk}{}^l x^k_\alpha X^i Y^j D_{l_\alpha}$ and $R_{ijk}{}^l$ are the components of the curvature tensor of the linear connection Γ with the covariant derivative ∇.

3. Let ∇ be a covariant derivative on M of the linear connection Γ.

Definition 3. A horizontal lift of a linear connection ∇ on M into the total space FM the bundle of linear frames is a linear connection $\overset{H}{\nabla}$ defined by :

$$\begin{aligned} \overset{H}{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H \quad , \quad & \overset{H}{\nabla}_{X^H} Y^{v_\alpha} &= (\nabla_X Y)^{v_\alpha} \\ \overset{H}{\nabla}_{X^{v_\alpha}} Y^H &= 0 \quad , \quad & \overset{H}{\nabla}_{X^{v_\alpha}} Y^{v_\beta} &= 0 \quad , \end{aligned} \tag{3.1}$$

for all vector fields X and Y on M.

The components of the horizontal lift $\overset{H}{\nabla}$ of a connection ∇ on M with components $\Gamma_{ij}^{\ \ k}$ in the natural frame $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are of the form in the adapted frame $\{D_i, D_{i_\alpha}\} = D_J$,

$$\overset{H}{\nabla}_{D_J} D_K = \overset{H}{\Gamma}{}^{L}_{JK} D_L$$

$$\begin{array}{llll} \overset{H}{\Gamma}{}^{h}_{ij} = \Gamma^{h}_{ij} \quad , & \overset{H}{\Gamma}{}^{h_\beta}_{ij} = \overset{H}{\Gamma}{}^{h}_{ij_\alpha} = 0 \quad , & \overset{H}{\Gamma}{}^{h_\beta}_{ij_\alpha} = \delta^{\beta}_{\alpha} \Gamma^{h}_{ij} \quad , & \\ \overset{H}{\Gamma}{}^{h}_{i_\alpha j} = 0 \quad , & \overset{H}{\Gamma}{}^{h_\beta}_{i_\alpha j} = 0 \quad , & \overset{H}{\Gamma}{}^{h}_{i_\alpha j_\beta} = 0 \quad , & \overset{H}{\Gamma}{}^{h_\gamma}_{i_\alpha j_\beta} = 0 \quad . \end{array} \tag{3.2}$$

We have the relations :

$$(D_i, D_{i_\alpha}) = \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j_\beta}\right) \cdot \begin{bmatrix} \delta^j_i & 0 \\ -\Gamma^{j}_{ik} x^k_\beta & \delta^j_i \delta^\alpha_\beta \end{bmatrix} \tag{3.3}$$

$$D_J = \partial_\mu \cdot B^\mu_J$$

Thus, the components of the linear connection $\overset{H}{\nabla}$ in the natural frame $\partial_\mu = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i_\alpha}\right)$ are of the form : $\overset{H}{\nabla}_{\partial_\mu} \partial_\nu = \overset{H}{F}{}^{r}_{\mu\nu} \partial_r$,

$$\begin{array}{l} \overset{H}{F}{}^{r}_{\mu\nu} = B^r_K \overset{H}{\Gamma}{}^{K}_{JL} \tilde{B}^J_\mu \tilde{B}^L_\nu - (D_J B^r_L) \tilde{B}^J_\mu \tilde{B}^L_\nu \quad , \quad B^\mu_J \tilde{B}^J_\nu = \delta^\mu_\nu \, , \\ \overset{H}{F}{}^{k}_{ij} = \Gamma^{k}_{ij} \quad , \quad \overset{H}{F}{}^{k_\alpha}_{ij} = \partial_i \Gamma^{k}_{jl} x^l_\alpha + \Gamma^{k}_{li} \Gamma^{l}_{jm} x^m_\alpha - \Gamma^{k}_{lm} \Gamma^{l}_{ij} x^m_\alpha \, , \\ \overset{H}{F}{}^{k_\alpha}_{ij_\beta} = \Gamma^{k}_{ij} \delta^\beta_\alpha \quad , \quad \overset{H}{F}{}^{k_\alpha}_{i_\beta j} = \Gamma^{k}_{ij} \delta^\beta_\alpha \, , \\ \overset{H}{F}{}^{k}_{ij_\alpha} = 0 \quad , \quad \overset{H}{F}{}^{k}_{i_\alpha j_\beta} = 0 \quad , \quad \overset{H}{F}{}^{k_\gamma}_{i_\alpha j_\beta} = 0 \quad . \end{array} \tag{3.4}$$

A torsion tensor $\overset{H}{T}$ of the linear connection $\overset{H}{\nabla}$ is of the form :

$$\begin{array}{lll} \overset{H}{T}(X^H, Y^H) = \left(T(X,Y)\right)^H + \gamma R(X,Y) & , & \overset{H}{T}(X^H, Y^{v_\beta}) = 0 \\ \overset{H}{T}(X^{v_\beta}, Y^H) = \left(T(X,Y)\right)^{v_\beta} & , & \overset{H}{T}(X^{v_\alpha}, Y^{v_\beta}) = 0 \quad . \end{array} \tag{3.5}$$

A curvature tensor $\overset{H}{R}$ of the linear connection $\overset{H}{\nabla}$ is of the form :

$$\begin{array}{lll} \overset{H}{R}(X^H, Y^H) Z^H = \left(R(X,Y)Z\right)^H & , & \overset{H}{R}(X^H, Y^H) Z^{v_\beta} = \left(R(X,Y)Z\right)^{v_\beta} \\ \overset{H}{R}(X^{v_\alpha}, Y^H) Z^H = 0 & , & \overset{H}{R}(X^H, Y^{v_\alpha}) Z^H = 0 \\ \overset{H}{R}(X^{v_\alpha}, Y^{v_\beta}) Z^H = 0 & , & \overset{H}{R}(X^{v_\alpha}, Y^{v_\beta}) Z^{v_\gamma} = 0 \end{array} \tag{3.6}$$

Thus, we have :

Proposition 2. *The horizontal lift* $\overset{H}{\nabla}$ *is torsionless connection on* FM *iff a linear connection* ∇ *on* M *satisfies :* $T = 0$ *and* $R = 0$ *. A curvature tensor* $\overset{H}{R}$ *of the* *horizontal lift $\overset{H}{\nabla}$ on FM vanishes iff a curvature tensor R of a linear connection ∇ on M vanishes.*

4. Let $\tilde{C}$ be a geodesic curve on the total space FM with respect to the horizontal lift $\overset{H}{\nabla}$. In induced coordinates the equations of a geodesic curve $\tilde{C} : I \longrightarrow FM$ $\tilde{C} : t \longrightarrow \tilde{C}(t) = \left(x^i(t), x^i_\alpha(t)\right) = \left(x^\lambda(t)\right)$ are of the form :

$$\frac{d^2x^\lambda}{dt^2} + \overset{H}{F}{}_{\mu\nu}^{\lambda}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 . \tag{4.1}$$

Thus, using the formulas (3,4) for $\overset{H}{F}{}_{\mu\nu}^{\lambda}$ we get :

$$\begin{aligned} &\frac{d^2x^i}{dt^2} + \Gamma_{jk}^{\ i}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \\ &\frac{d^2x^i_\alpha}{dt^2} + (\partial_l\Gamma_{mk}^{\ i}x^k_\alpha + \Gamma_{jl}^{\ i}\Gamma_{mk}^{\ j}x^k_\alpha - \Gamma_{jk}^{\ i}\Gamma_{lm}^{\ j}x^k_\alpha)\frac{dx^k}{dt}\frac{dx^m}{dt} + \\ &\quad + \Gamma_{lm}^{\ i}\frac{dx^l}{dt}\frac{dx^m_\alpha}{dt} + \Gamma_{ml}^{\ i}\frac{dx^m_\alpha}{dt}\frac{dx^l}{dt} = 0 . \end{aligned} \tag{4.2}$$

We denote :

$$\frac{Dx^i_\alpha}{dt} = \frac{dx^i_\alpha}{dt} + \Gamma_{lm}^{\ i}x^m_\alpha\frac{dx^l}{dt} . \tag{4.3}$$

Then if we assume that :

$$\frac{d^2x^i}{dt^2} + \Gamma_{jk}^{\ i}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \quad , \quad \Gamma_{jk}^{\ i} = \Gamma_{kj}^{\ i} \tag{4.4}$$

we obtain :

$$\begin{aligned} \frac{D^2x^i_\alpha}{dt^2} &= \frac{D}{dt}\left(\frac{Dx^i_\alpha}{dt}\right) = \\ &= \frac{d^2x^i_\alpha}{dt^2} + (\partial_l\Gamma_{mk}^{\ i} + \Gamma_{mj}^{\ i}\Gamma_{lk}^{\ j} - \Gamma_{jk}^{\ i}\Gamma_{lm}^{\ j})x^k_\alpha\frac{dx^l}{dt}\frac{dx^m}{dt} + \\ &\quad + \Gamma_{lm}^{\ i}\frac{dx^l}{dt}\frac{dx^m_\alpha}{dt} + \Gamma_{ml}^{\ i}\frac{dx^m_\alpha}{dt}\frac{dx^l}{dt} . \end{aligned} \tag{4.5}$$

Thus, we have :

Proposition 3. *A geodesic curve on the total space FM with respect to the horizontal lift $\overset{H}{\nabla}$ of a linear connection ∇ on M with tensor torsion $T = 0$, has in induced coordinates on FM the equations of the form :*

$$\frac{d^2x^i}{dt^2} + \Gamma_{lm}^{\ i}\frac{dx^l}{dt}\frac{dx^m}{dt} = 0 \quad , \quad \frac{D^2x^i_\alpha}{dt^2} = 0 . \tag{4.6}$$

Proposition 4. *A curve $\tilde{C}$ on the total space FM of the bundle of linear frames is a geodesic curve with respect to the horizontal lift $\overset{H}{\nabla}$, if projection $C = \pi(\tilde{C})$ on M is a geodesic curve on M with respect to ∇ and the second covariant derivative of each vector $X_\alpha(t) = x^k_\alpha(t) \left.\dfrac{\partial}{\partial x^k}\right|_{C(t)}$ of the frame u along C defined by $\tilde{C}$ vanishes.*

Let $\tilde{X}$ be a vector field defined along the curve $\tilde{C}$ on FM. The vector field $\tilde{X} = \tilde{X}^\lambda \dfrac{\partial}{\partial x^\lambda}$ along the curve $\tilde{C}(t) = \left(x^\lambda(t)\right)$ is parallel with respect to the horizontal lift $\overset{H}{\nabla}$ if the equations are satisfied :

$$\frac{d\tilde{X}^\lambda}{dt} + \overset{H}{F}{}^{\lambda}_{\mu\nu} \frac{dx^\mu}{dt} \tilde{X}^\nu = 0 \quad . \tag{4.7}$$

If a vector field $\tilde{X} = X^i \dfrac{\partial}{\partial x^i_\alpha}$ on FM is the vertical lift of type v_α of a vector field $X = X^i \dfrac{\partial}{\partial x^i}$ on M, then the equations (4,7) reduce to :

$$\frac{dX^i}{dt} + \Gamma^{\ i}_{jk} X^k \frac{dx^j}{dt} = 0 \quad . \tag{4.8}$$

If a vector field $\tilde{X} = X^i D_i$ on FM is the horizontal lift of a vector field $X = X^i \dfrac{\partial}{\partial x^i}$ on M, then the equations (4,7) reduce to :

$$\begin{aligned} &\frac{dX^i}{dt} + \Gamma^{\ i}_{jk} X^k \frac{dx^j}{dt} = 0 \quad , \\ -&\Gamma^{\ i}_{jk} x^k_\alpha \left(\frac{dX^j}{dt} + \Gamma^{\ j}_{lm} X^m \frac{dx^l}{dt} \right) = 0 \, . \end{aligned} \tag{4.9}$$

Thus, we have :

Proposition 5. *The parallel displacement of the vertical lift A^{v_α} of type v_α or the horizontal lift A^H of a vector $A \in T_{C(0)}M$ along the curve $\tilde{C}$ on FM with respect to the horizontal lift of a linear connection ∇ coincides with the vertical lift of type v_α or the horizontal lift of the parallel displacement of a vector A along the curve $C = \pi(\tilde{C})$ with respect to a linear connection ∇ on M :*

$$\tilde{C}(A^{v_\alpha}) = \left(C(A)\right)^{v_\alpha} \quad , \quad \tilde{C}(A^H) = \left(C(A)\right)^H \quad . \tag{4.10}$$

5. Let g be a metric tensor on M. We consider a diagonal lift G of g to FM with respect to a linear connection ∇. If in local chart (U, x^i) the metric tensor g is of the form $g = g_{ij}\, dx^i \otimes dx^j$, then the diagonal lift G of g with respect to adapted coframe (2,3) : $\{\eta^i, \eta^i_\alpha\}$ on FM is of the form :

$$G = g_{ij}\ \eta^i \otimes \eta^j + \delta^{\alpha\beta} g_{ij}\ \eta^i_\alpha \otimes \eta^j_\beta \quad . \tag{5.1}$$

We have the following formulas for the covariant derivative of the diagonal lift G of g with respect to the horizontal lift $\overset{H}{\nabla}$ of ∇ :

$$
(5.2)\qquad
\begin{array}{ll}
(\overset{H}{\nabla}_{X^H} G)(Y^H, Z^H) = (\nabla_X g)(Y, Z) & , \quad (\overset{H}{\nabla}_{X^H} G)(Y^{v_\alpha}, Z^H) = 0 \\
(\overset{H}{\nabla}_{X^H} G)(Y^{v_\alpha}, Z^{v_\beta}) = \delta^{\alpha\beta} (\nabla_X g)(Y, Z) & , \quad (\overset{H}{\nabla}_{X^{v_\alpha}} G)(Y^H, Z^H) = 0 \\
(\nabla_{X^{v_\alpha}} G)(Y^{v_\beta}, Z^H) = 0 & , \quad (\overset{H}{\nabla}_{X^{v_\alpha}} G)(Y^{v_\beta}, Z^{v_\gamma}) = 0 \quad .
\end{array}
$$

Thus, we have :

Proposition 6. *If g is a metric tensor on a manifold M and ∇ is its metric connection, then the horizontal lift $\overset{H}{\nabla}$ is a metric connection on FM for the diagonal lift G of g.*

REFERENCES

[1] Cordero , L. A. , de Leon , M. , *Horizontal lift of connection to the frame bundle* , Boll. Un. Mat. Ital., 6 (1984), 223–240

[2] Kurek , J. , *On a horizontal lift of a linear connection into linearized tangent bundle of second order* , An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. (N.S.), to appear.

[3] Yano , K. , Isihara , S. , *Tangent and Cotangent Bundles* , Marcel Dekker Inc., New York 1973

STRESZCZENIE

W pracy tej definiuje się podniesienie horyzontalne $\overset{H}{\nabla}$ koneksji liniowej ∇ na rozmaitości M do wiązki reperów liniowych FM analogicznie jak K.Yano [3] do wiązki stycznej.

W tym celu określa się w nowy sposób dla pola wektorowego X na M n podniesień wertykalnych X^{v_α}, $\alpha = 1, \ldots n$ oraz w standardowy sposób podniesienie horyzontalne X^H.

Wyznacza się tensor skręcenia, tensor krzywizny, geodezyjne, przeniesienie równoległe dla podniesienia horyzontalnego $\overset{H}{\nabla}$ na FM.

SUMMARY

In this paper a horizontal lift $\overset{H}{\nabla}$ of a linear connection ∇ on a manifold M into the total space FM of the bundle of linear frames $\pi : FM \to M$, in a way similar to that of K.Yano, is defined.

The torsion tensor and the curvature tensor of the connection $\overset{H}{\nabla}$ has been determined, as well as geodesics and parallel displacement of the horizontal lift with respect to $\overset{H}{\nabla}$ are determined.