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Distortion Theorems for Bounded p-valent Functions

Twierdzenia o zniekształceniu dla funkcji p-listnych ograniczonych

Теорема о деформации для p-листной ограниченной функции

Introduction

The classical distortion theorems for univalent functions such as the one-quarter theorem and the theorems concerning the bounds for moduli of function and its derivate still hold, as pointed out by W. K. Hayman [2], [3], for more general classes of functions, e.g. for circumferentially mean univalent functions introduced by M. Biernacki [1] and even for a less restrictive class of weakly univalent functions [3].

That generalisation is inasmuch of importance, as it provides immediately distortion theorems for p-valent functions. The main advantage of Biernacki and Hayman classes is their behaviour under taking the p-th root: if $F(z)=z^p+A_{p+1}z^{p+1}+\dots$ is weakly (resp. circumferentially mean) p-valent in $|z|<1$, then

$$[F(z)]^{1/p}=z+\frac{1}{p}A_{p+1}z^2+\dots$$

is weakly (resp. circumferentially mean) univalent in $|z|<1$. The p-valent functions do not generally posses an analogous property. However, if the extremal functions obtained for the more general class of weakly p-valent functions are p-valent in the ordinary sense, the problem for p-valent functions can be settled in this way.

On the other hand, G. Pick [6] obtained some distortion theorems for bounded, univalent functions. In these theorems the extremal Pick function $w=f(M,z)=f_M(z)=z+2(1-1/M)z^2+\dots$ mapping $|z|<1$ onto

$|w| < M$ slit along the negative real axis from $-M$ to $-r(M) = [2M - 1 - 2\sqrt{M(M-1)}]$ plays a similar role as the Koebe function $w = z/(1-z)^2$ for unbounded functions.

In this paper we shall obtain distortion theorems for the class $S_M^{(1)}$ of functions regular, weakly univalent and bounded in the unit circle and such that $f(z) = z + a_2 z^2 + \ldots$, $|f(z)| < M$ for $|z| < 1$. This is an obvious generalisation of G. Pick results concerning the univalent functions. By taking p-th root as above, distortion theorems for the class $S_M^{(p)}$ ($F(z) = z^p + A_{p+1} z^{p+1} + \ldots$, $|F(z)| < M$) of regular bounded, weakly p-valent functions can be obtained. Making $M \to \infty$, we obtain some Hayman's results for unbounded weakly univalent and weakly p-valent functions.

1. A generalisation of the one-quarter theorem

A starting point in our considerations will be an analogue of the "one-quarter theorem" for bounded, weakly univalent functions. An extremely simple and elegant method of proof valid for this and similar problems has been introduced by Hayman [2]. For the sake of completeness and in regard of the fact that [2] is scarcely accessible ,we shall here sketch the proof.

The Hayman's method is based on the notion of inner conformal radius of a Riemann surface and on the determination of its maximal value by using the properties of circular symmetrization, a geometrical operation introduced by G. Pólya. If $g(W, P, P_0)$ is the Green function of the Riemann surface W with the pole at P_0 and w_0 (resp. w) is the value of local uniformizing parameter for P_0 (resp. for P), then $g(W, P, P_0) = \log |w - w_0|^{-1} + \log r(W, P_0) + o(1)$ for $P \to P_0$ and this equality defines the inner conformal radius $r(W, P_0)$ of W with respect to P_0. If W is the biunivoque conformal map of $|z| < 1$ by a regular function $f(z)$, then

$$g(W, P, P_0) = -\log \left| \frac{z(w) - z(w_0)}{1 - z(w)\overline{z(w_0)}} \right|, \qquad f[z(w)] = w,$$

and this implies $r(W, P_0) = (1 - |z_0|^2)|f'(z_0)|$ for P_0 being the map of z_0.

Let now W be a Riemann surface over the w plane being the map of $|z| < 1$ by a regular weakly univalent function $w = f(z)$. This means that every circle $|w| = r$ is either covered simply by the points of W, or it leaves at least one point over this circumference uncovered. This implies (in view of the fact that univalency on the boundary involves the univalency in the interior of boundary) that the Riemann surface of a regular, weakly univalent function contains a simply covered circular disc $|w|<\delta$ and at least one point over each circumference $|w|=r\geqslant\delta$ remains uncovered. Let now W be the Riemann surface being the map of $|z|<1$ under $f(z)$, $f\in S_M^{(1)}$. We shall prove that

$$\delta\geqslant r(M)=M\,[2\,M-1-2\sqrt{M(M-1)}].$$

Let W_0 be a simply covered (but generally multiply connected) domain obtained by identifying the multiply covered points of W. The boundary of W_0 is contained in the boundary of W and hence, by the Lindelöf's principle, $g(W_0, P, P_0)\geqslant g(W, P, P_0)$ and therefore $r(W_0, 0)\geqslant r(W, 0)=1$. The circular symmetrization of W_0 with respect to the positive real axis (for definition and properties see: [2], [4], [5]) increases the inner conformal radius with respect to points on the symmetrization axis, and so, for the obtained simply connected domain W^*, we have $r(W^*, 0)\geqslant$ $\geqslant r(W_0, 0)\geqslant 1$. Since the enlargement of domain increases the Green function and thus the inner conformal radius, too, we finally obtain $r(K_M, 0)\geqslant 1$, where K_M is the circular disc $|w|<M$ slit along the segment $[-M, -\delta]$, of the real axis. An easy calculation gives $r(K_M, 0)=$ $=4\,\delta M^2/(M+\delta)^2$. Since $r(K_M, 0)\geqslant 1$, we have $\delta\geqslant M\,[2M-1-2\sqrt{M(M-1)}]$ and this is the desired result.

Added in proof: the method of majorization of the inner conformal radius is presented in W. K. Hayman, *Multivalent Functions*, Cambridge 1958, and the above given value of $r(M)$ may be derived immediately from the theorem 4.14, p. 88.

2. The lower bound for $|f(z)|$

Theorem 2.1. *If $f\in(S_M^{(1)}$, then we have*

$$-f(M, -|z|)\leqslant|f(z)| \tag{2.1}$$

where $f(M, z)$ is the Pick function (defined in Introduction). The obtained lower bound is sharp and is attained for $f=f_M$ and real, negative z, $-1<z<0$.

Proof. Since for $-1<z<0$ we have $0<-f(M, z)<r(M)$ is suffices to prove the theorem for z such that $|f(z)|=|d|<r(M)$. We may suppose that $|f(z)|$ attains its minimum on $|z|=\varrho$ just for $r=-\varrho$. We have therefore $f(-\varrho)=d$, $|d|<r_M$, $0<\varrho<1$. The circle $|z|<1$, slit along the negative real axis from -1 to $-\varrho$, is mapped by $w=f(M, z)$ onto the circle $|w|<M$ slit along the negative real axis from $-M$ to $f(M, -\varrho)$. The transformation given implicitely as follows:

$$\frac{z}{(1-z)^2}=\frac{4\,\varrho}{(1+\varrho)^2}\,\frac{t}{(1-t)^2}=\omega,$$

represents conformally the circle $|z|<1$ slit along the negative real axis from -1 to $-\varrho$ onto the circular disc $|t|<1$ (since both domains are mapped conformally onto the ω plane slit along the negative real axis from $-\infty$ to $-\varrho/(1+\varrho)^2$. The above given transformation has the form: $z=z(t)=4\varrho(1+\varrho)^{-2}t+\ldots$ The resulting transformation $\varphi(t)=f[z(t)]$ takes the form:

$$\varphi(t)=\frac{4\varrho}{(1+\varrho)^2}t+\cdots=At+\cdots \quad \text{where} \quad 0<A=\frac{4\varrho}{(1+\varrho)^2}<1$$

and defines a regular, weakly univalent function of t, $|t|<1$. Its modulus is bounded there by M, besides, it does not take the value d in $|t|<1$. Since the function $A^{-1}\varphi(t)=t+\ldots$ belongs to $S^{(1)}_{M/A}$ and does not take the value $A^{-1}d$ in the circle $|t|<1$, we have

$$A^{-1}|d|\geqslant r(M/A)=MA^{-1}|2MA^{-1}-1-2\sqrt{MA^{-1}(MA^{-1}-1)}|$$

and this implies

$$|f(-\varrho)|=|d|\geqslant MA^{-1}|2M-A-2\sqrt{M(M-A)}|.$$

But

$$f(M,z)=M\frac{1/2+zM^{-1}(1-z)^{-2}-\sqrt{1/4+zM^{-1}(1-z)^{-2}}}{zM^{-1}(1-z)^{-2}} \tag{2.2}$$

(we take here this branch of radical that is equal to $1/2$ for $z=0$) and therefore

$$-f(M,-\varrho)=A^{-1}M[2M-A-2\sqrt{M(M-A)}|$$

with $A=4\varrho(1+\varrho)^{-2}$. This proves that $|f(-\varrho)|\geqslant -f(M,-\varrho)$. The inequality (2.1) is proved.

3. The upper bound for $|f(z)|$

We now solve the following extremal problem. Given a Riemann surface W_φ being the map of $|t|<1$ under the transformation $z=\varphi(t)$, where $\varphi(t)$ is regular and weakly univalent in $|t|<1$ and does not attain the value $z=0$ there, and, besides, $|\varphi(t)|<1$. We shall determine the maximal value of the inner conformal radius of such a surface W_φ, (having at least one inner point over z_0, z_0 being fixed, $0<|z_0|<1$) with respect to a point lying over z_0. The succesive operations of identifying the different sheets of the Riemann surface, then the circular symmetrization with respect to the axis emanating from $z=0$ through z_0, applied quite similarly as in sect. 1, increase the inner conformal radius at z_0. Finally we obtain a simply connected domain which does not contain the radius of the unit circle $|z|<1$ opposite to z_0. The subsequent enlargement of domain increases the inner conformal radius, and therefore we see that it attains greatest possible value for W_φ being the circular disc slit along the radius being the prolongation of the radius containing z_0. We can take z_0 such that $0<z_0<1$ and then the extremal function is defined by the relation $z/(1-z)^2=(1+t)^2/(1-t)^2$ and represents the unit circle $|t|<1$ onto the unit circle $|z|<1$ slit along the negative real axis. We have $r(W_\varphi, z_0)=(1-|t|^2)|d\varphi/dt|$ and therefore $r(W_\varphi, z_0)=4z_0(1-z_0)/(1+z_0)$ is the desired maximal value.

If we consider an analogous problem for Riemann surfaces contained in the circle $|w|<M$ $(M\geqslant 1)$, it is easy to see that the maximal value of the inner conformal radius of such a surface W_f with respect to the point lying over w_0 is equal to $4|w_0|(1-M^{-1}|w_0|)(1+M^{-1}|w_0|)$. The obtained maximal value at once gives us an upper bound for the logarithmic derivative:

Theorem 3.1. *If $w=w(t)$ is a regular, weakly univalent and bounded function of t, where $|t|<1$, such that 0 $|w(t)|<M$ for $|t|<1$,*

$$\frac{1}{|w(t)|}\left|\frac{dw}{dt}\right|<\frac{4}{1-|t|^2}\,\frac{1-M^{-1}|w(t)|}{1+M^{-1}|w(t)|}. \tag{3.1}$$

The obtained upper bound is sharp, the equality being attained for $w(t)$ mapping the unit circle $|t|<1$ onto $|w|<M$ slit along the radius and for t such that the corresponding value of w is situated on the prolongation of the slit.

Let us now suppose that $f(z)$ is an arbitrary, bounded and weakly univalent function, such that $|f(z)|<M$ for any z in the unit circle and $f(0)=0$. Obviously the compounded transformation $\varphi(t)=f[z(t)]$, where $z=z(t)$ maps $|t|<1$ onto $|z|<1$ slit along the radius $-1\leqslant z\leqslant 0$, defines a bounded, non-vanishing, weakly univalent function. If the points $t=0\longleftrightarrow z=\varrho$ and $t=\tau\longleftrightarrow z=\varrho_1$ (where $\tau>0$) correspond to each other, then it is easy to verify that $0<\varrho<\varrho_1<1$. The inequality (3.1) gives

$$\int_0^\tau \frac{(1+M^{-1}|w(t)|)\,d|w(t)|}{|w(t)|(1-M^{-1}|w(t)|)}\leqslant 4\int_0^\tau\frac{dt}{1-t^2}.$$

An integration gives

$$\log\frac{M|\varphi(\tau)|}{[M-|\varphi(\tau)|]^2}-\log\frac{M|\varphi(0)|}{[M-|\varphi(0)|]^2}\leqslant\log\left(\frac{1+\tau}{1-\tau}\right)= \tag{3.2}$$

$$=\log\frac{\varrho_1}{(1-\varrho_1)^2}-\log\frac{\varrho}{(1-\varrho)^2}.$$

But $\varphi(0)=f(\varrho)$, $\varphi(\tau)=f(\varrho_1)$ and therefore (3.2) gives

$$\frac{M|f(\varrho_1)|(1-\varrho_1)^2}{\varrho_1[M-|f(\varrho_1)|]^2}\leqslant\frac{M|f(\varrho)|(1-\varrho)^2}{\varrho[M-|f(\varrho)|]^2}. \tag{3.3}$$

This means that on the positive real axis

$$\frac{|f(r)|(1-r)^2}{r[1-M^{-1}|f(r)|]^2}$$

is a decreasing function of r. The same is clearly true for any other ray and we obtain

Theorem 3.2. *The formula*

$$\frac{|f(re^{i\theta})|(1-r)^2}{r[1-M^{-1}|f(re^{i\theta})|]^2}$$

represents a decreasing function of r $(0<r<1)$, θ being real and fixed, for any bounded $(|f(z)|<M)$, weakly univalent function $f(z)$, such that $f(0)=0$.

If $M\to\infty$, we obtain the well known result to H a y m a n [3]. Besides, under the above assumptions, the *finite limit*

$$\lim_{r\to1-}\frac{|f(re^{i\theta})|(1-r)^2}{r[1-M^{-1}|f(re^{i\theta})|]^2}$$

exists for any real and fixed θ. If $M(r,f)=\sup\limits_{0\leqslant\theta\leqslant2\pi}|f(re^{i\theta})|$, then we can choose θ_1 such that $|f(\varrho_1e^{i\theta_1})|=M(\varrho_1,f)$ and therefore, if $0<\varrho<\varrho_1<1$, then we have

$$\frac{M(\varrho,f)(1-\varrho)^2}{\varrho[1-M^{-1}M(\varrho,f)]^2}\geqslant\frac{|f(\varrho e^{i\theta_1})|(1-\varrho)^2}{\varrho[1-M^{-1}|f(\varrho e^{i\theta_1})|]^2}\geqslant\frac{|f(\varrho_1e^{i\theta_1})|(1-\varrho_1)^2}{\varrho_1[1-M^{-1}|f(\varrho_1e^{i\theta_1})|]^2}=$$

$$=\frac{M(\varrho_1,f)(1-\varrho_1)^2}{\varrho_1[1-M^{-1}M(\varrho_1,f)]^2}$$

since $x(1-M^{-1}x)^{-2}$ increases for $0<x<M$.

Hence

$$\frac{M(r,f)(1-r)^2}{r[1-M^{-1}M(r,f)]^2}$$

is a decreasing function of r, $0<r<1$, for any bounded, weakly univalent function $f(z)$, such that $|f(z)|<M$, $f(0)=0$.

Supposing that $f'(0)=1$, we see that the above considered decreasing function does not exceed its right-hand side limit at $r=0$. This implies

$$\frac{M(r,f)}{[1-M^{-1}M(r,f)]^2}\leqslant\frac{r}{(1-r)^2}.$$

The Pick function (2.2) can be also defined by the equation

$$\frac{f(M,z)}{|1-M^{-1}f(M,z)|^2}=\frac{z}{(1-z)^2} \tag{3.4}$$

and we see that for $0<z=r<1$

$$\frac{M(r,f)}{|1-M^{-1}M(r,f)|^2}\leqslant\frac{f(M,r)}{|1-M^{-1}f(M,r)|^2}$$

and in view of monotonity of $x(1-M^{-1}x)^{-2}$ we have $M(r,f)\leqslant f(M,r)$.

We have proved the

Theorem 3.3. *If $f\in S_M^{(1)}$, then $|f(z)|\leqslant f(M,|z|)$, the obtained upper bound is sharp, the equality being attained by the Pick function $f(M,z)$ for positive real z.*

The inequality (3.1) enables us to obtain a sharp upper bound of the logarithmic derivative for bounded, weakly univalent functions without the restriction that $f(z)$ does not vanish. Suppose that $f(0=0$. The weak univalence implies that $f(z)\neq 0$ for $z\neq 0$ and $f'(0)\neq 0$. Suppose that $z=z(t)$ maps the unit circle $|t|<1$ onto the unit circle $|z|<1$ slit along the radius, so, that $\varrho=z(0)>0$, $\varrho_1=z(\tau)>\varrho,(0<\tau<1)$.

This mapping has the following implicite form:

$$\frac{z}{(1-z)^2}=\frac{\varrho}{(1-\varrho)^2}\left(\frac{1+t}{1-t}\right)^2$$

Therefore

$$(1-\tau^2)\left(\frac{dz}{dt}\right)_{t=\tau}=\frac{4\varrho_1(1-\varrho_1)}{1+\varrho_1}$$

and because the compounded function $w=f[z(t)]$ is clearly weakly univalent and does not vanish in $|t|<1$, we have in view of (3.1):

$$(1-\tau^2)[f(\varrho_1)]^{-1}\left|\frac{df}{dz}\right|_{z=\varrho_1}\left(\frac{dz}{dt}\right)_{t=\tau}=$$

$$=\frac{4\varrho_1(1-\varrho_1)}{|f(\varrho_1)|(1+\varrho_1)}\left|\frac{df}{dz}\right|_{z=\varrho_1}\leqslant\frac{4(1-M^{-1}|f(\varrho_1)|)}{1+M^{-1}|f(\varrho_1)|}.$$

The estimation on the real axis, however, implies an analogous estimation on the circle $|z|=$ const. and putting $\varrho_1=|z|$ we obtain

$$\frac{1}{|f(z)|}\left|\frac{df(z)}{dz}\right|\leqslant\frac{1+|z|}{|z|(1-|z|)}\,\frac{1-M^{-1}|f(z)|}{1+M^{-1}|f(z)|}. \tag{3.5}$$

We have therefore

Theorem 3.4. *If $w=f(z)$ is regular, bounded ($|f(z)|<M$) and weakly univalent in the unit circle $|z|<1$, and if $f(0)=0$, then the inequality* (3.5) *holds.*

This inequality is sharp, the equality being attained for the Pick function $f(M,z)$ and real, positive z.

The inequality (3.5) gives at the same time the solution of an extremal problem for bounded, weakly univalent Riemann surfaces. Suppose that W_f is a Riemann surface being the map of $|z|<1$ under $w=z+a_2z^2+\ldots=f(z)$, $f(z)$ being bounded ($|f(z)|<M$) weakly univalent and regular in the unit circle. We shall determine

$$\sup_{f\in S_M^{(1)}} r(W_f, P),$$

P being a point of W_f lying over $w=w_0$, $0<|w_0|<M$. We have by (3.5)

$$r(W_f, P)=(1-|z_0|^2)|f'(z_0)|\leqslant\frac{(1+|z_0|)^2}{|z_0|}\frac{|w_0|(1-M^{-1}|w_0|)}{1+M^{-1}|w_0|}$$

where $w_0=f(z_0)$, w_0 being the value of local uniformizing parameter for P. The expression $(1+|z_0|)^2/|z_0|$ is a strictly decreasing function of $|z_0|$ and it attains, w_0 being given, the greatest possible value for this function which increases most rapidly, i.e. for the Pick function (resp. for a function obtained from it by trivial modifications). Then

$$\frac{|z_0|}{(1-|z_0|)^2}=\frac{|w_0|}{(1-M^{-1}|w_0|)^2}$$

and hence

$$\frac{(1+|z_0|)^2}{|z_0|}=\frac{(1-|z_0|)^2}{|z_0|}+4=\frac{(1-M^{-1}|w_0|)^2}{|w_0|}+4\,.$$

This means that

(3.6) $$r(W_f, P)=r(W_f, w_0)\leqslant\frac{1-M^{-1}|w_0|}{1+M^{-1}|w_0|}\left[(1-M^{-1}|w_0|)^2+4|w_0|\right]$$

or

(3.7) $$(1-|z_0|^2)|f'(z_0)|\leqslant\frac{1-M^{-1}|w_0|}{1+M^{-1}|w_0|}\left[(1-M^{-1}|w_0|)^2+4|w_0|\right].$$

The obtained upper bound is sharp, the equality being attained for $f=f_M$ and $0<w_0<M$. In particular, (3.7) gives for $M\to+\infty$:

(3.8) $$(1-|z|^2)|f'(z)|\leqslant 1+4|f(z)|$$

for any regular and weakly univalent function $f(z)=z+a_2z^2+\ldots$ This inequality contains the precise upper bounds for the moduli of function $f\in S_\infty^{(1)}$ and its derivative. Similarly, an integration of (3.7) leads to the theorem 3.3.

The inequality $|f(z)|/|z|\leqslant f_M(|z|)/|z|$ gives immediately the estimation of the second coefficient. We can suppose that $a_2\geqslant 0$. Then we have

$$1+a_2\varrho+O(\varrho^2)\leqslant 1+2\left(1-\frac{1}{M}\right)\varrho+O(\varrho^2)$$

and for $\varrho\to 0$ we obtain

$$|a_2|\leqslant 2(1-M^{-1}) \tag{3.9}$$

the equality being attained for the Pick function.

4. Distortion theorems for bounded, p-valent functions

The above obtained results may be used to obtain analogous results for bounded, weakly p-valent functions. We say that a function $F(z)$ is weakly p-valent, if either a circumference $|w|=r$ is covered exactly p-times, or it contains at least one point which is covered at most $p-1$ times. It is easy to verify that, if $F(z)=z^p+A_{p+1}z^{p+1}+\ldots$ is regular and weakly p-valent in $|z|<1$, then $f(z)=[F(z)]^{1/p}=z+1/p\,A_{p+1}z^2+\ldots$ is also regular and weakly univalent in $|z|<1$, [3]. Besides, if $|F(z)|<M$, then $|f(z)|<M^{1/p}$. After these preliminaries the analogous results for bounded, weakly p-valent functions can be stated without proof, being simple corollaries of previous theorems. We shall throughout suppose that $F(z)=z^p+A_{p+1}z^{p+1}+\ldots$ is a regular, bounded ($|F(z)|<M$, $M>1$) and weakly p-valent function defined in the unit circle. Then we have.

Corollary 1. *The values of $F(z)$ cover exactly p-times the open circle $|w|<[r(M^{1/p})]^p$, $r(M)$ being defined in sect.* 1.

Corollary 2. *For z lying on the circumference* $|z|=$ const *we have*

$$|-f(M^{1/p},-|z|)|^p\leqslant|F(z)|\leqslant|f(M^{1/p},|z|)|^p.$$

Corollary 3.

$$\left|\frac{F'(z)}{F(z)}\right|\leqslant\frac{p(1+|z|)}{|z|(1-|z|)}\,\frac{1-|M^{-1}|F(z)||^{1/p}}{1+|M^{-1}|F(z)||^{1/p}}.$$

Corollary 4.

$$|A_{p+1}|\leqslant 2p(1-M^{-1/p}).$$

The above bounds are sharp, the equality being attained for $F(z)=[f(M^{1/p},z)]^p$. Since the extremal function is p-valent, the bounds are valid and sharp for bounded p-valent functions, too.

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Streszczenie

W pracy tej posługując się metodą Haymana-Pólyi majoryzacji konforemnego promienia wewnętrznego powierzchni Riemanna otrzymuję dokładne oszacowania od dołu i od góry na moduł funkcji słabo p-listnych ograniczonych, a także dokładne oszacowanie od góry na pochodną logarytmiczną funkcji p-listnych ograniczonych postaci

$$f(z)=z^p+A_{p+1}z^{p+1}+\dots$$

Otrzymane wyniki stanowią z jednej strony uogólnienie analogicznych wyników Haymana otrzymanych dla funkcji słabo p-listnych nieograniczonych, z drugiej zaś strony uogólnienie klasycznych rezultatów Picka odnoszących się do funkcji jednolistnych ograniczonych.

Резюме

В этой работе, пользуясь методом Хеймэна-Полиа майоризации конформного внутреннего радиуса поверхности Риманна, я получаю точную оценку снизу и сверху для модуля функций слабо p-листных, ограниченных, а также точную оценку сверху для логарифмической производной ограниченных p-листных функций вида

$$f(z)=z^p+A_{p+1}z^{p+1}+\dots$$

Полученные результаты представляют, с одной стороны, обощение аналогичных результатов Хеймэна, полученных для слабо p-листных, неограниченных функций, а с другой стороны обощение классических результатов Пика, относящихся к функциям однолистным органиченным.