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On the Derivative of Bounded p-valent Functions

O pochodnej funkcji p-listnych ograniczonych

О производной p-листных ограниченных функций

1. Let $S_M^{(p)}$ denote the class of bounded, weakly p-valent functions, regular in the unit circle $|z|<1$ and such that for $F \in S_M^{(p)}$ and $|z|<1$ we have: $F(z)=z^p+A_{p+1}z^{p+1}+\ldots$, $|F(z)|<M$ $(M>1)$ (for the definition of weak p-valency see [2] or [4]). In the paper [4] the following estimation of $|F'(z)|$ for $F \in S_M^{(p)}$ has been obtained:

$$(1.1) \qquad |F'(z)| \leqslant \frac{pM(1+|z|)}{|z|(1-|z|)} \frac{|F(z)|}{M} \frac{1-[M^{-1}|F(z)|]^{1/p}}{1+[M^{-1}|F(z)|]^{1/p}}.$$

This bound is valid and sharp for all $0<|z|<1$, the equality being attained for $F(z)=[f(M^{1/p},z)]^p$, where $w=f(K,z)$ denotes the Pick function which represents the unit circle $|z|<1$ on the circle $|w|<K$ $(K>1)$ slit along the negative real axis from $-K$ to

$$-r(K)=-K\,[2K-1-2\sqrt{K(K-1)}].$$

The bound (1.1) depends, however, on $|F(z)|$. It would be desirable to find a bound depending on $|z|$ only and this can be done, at least for small values of z, quite similarly as in [3].

We now prove the following

Theorem 1. *If $F(z)\in S_M^{(p)}$ and $0<|z|<r(M,p)$, where*

$$(1.2) \qquad r(M,p)=A^{-1}(A+1/2-\sqrt{A+1/4}),$$

$$(1.3) \qquad A=\frac{1}{2}M^{1/p}p(\sqrt{1+p^2}+p).$$

$$(1.4)\qquad |F'(z)| \leqslant \frac{p(1+|z|)}{|z|(1-|z|)}[f(M^{1/p},|z|)]^p \frac{M^{1/p}-f(M^{1/p},|z|)}{M^{1/p}+f(M^{1/p},|z|)},$$

$f(K,z)$ being the Pick function.

We have

$$(1.5)\qquad r(M,p) \geqslant \sqrt{2}-1 = 0{,}4142,...,$$

for any M, p ($M>1$, $p\geqslant 1$) and besides,

$$(1.6)\qquad \lim_{M\to+\infty} r(M,p) = \lim_{p\to+\infty} r(M,p) = 1.$$

Under the above given assumptions the inequality (1.4) *is sharp, the equality being attained for $F(z)=[f(M^{1/p},z)]^p$ and real, positive z, $0<|z|<r(M,p)$.*

Proof. Putting $u=[M^{-1}|F(z)|]^{1/p}$ we see that

$$(1.1a)\qquad |F'(z)| \leqslant \frac{pM(1+|z|)}{|z|(1-|z|)}\frac{u^p(1-u)}{1+u}.$$

Now, the function

$$(1.7)\qquad \varphi_p(u) = \frac{u^p(1-u)}{1+u}, \qquad 0<u<1,$$

is a strictly increasing function of the variable u for $u \in (0,u_p)$ where

$$(1.8)\qquad u_p = \frac{1}{p}(\sqrt{1+p^2}-1), \qquad 0<u_p<1.$$

Clearly u_p is an increasing function of p, too. Let us now suppose that, M and p being fixed, $r=r(M,p)$ is the solution of the equation $f(M^{1/p},r)=$ $=M^{1/p}u_p$ with respect to r. Since $0<u_p<1$, this equation has always a unique solution $r=r(M,p)$ such that $0<r(M,p)<1$. Since the Pick function is real and increasing for $r\in(0,1)$, the inequality $\varrho<r(M,p)$ implies

$$(1.9)\qquad f(M^{1/p},\varrho) < M^{1/p}u_p = f(M^{1/p},r).$$

We have proved in [4] that for $F\in S_M^{(p)}$, $|F(z)|^{1/p} \leqslant f(M^{1/p},|z|)$. Therefore, if $|z|<r(M,p)$, then $(|F(z)|/M)^{1/p} \leqslant M^{-1/p}f(M^{1/p},|z|)<u_p$ by (1.9). Now the function $\varphi_p(u)$ is increasing for $u\in(0,u_p)$ and hence for $|z|<r(M,p)$ and $u=[M^{-1}|F(z)|]^{1/p}$, $\varphi_p(u)$ takes, $|z|$ being fixed, the maximal value for $u=M^{-1/p}f(M^{1/p},|z|)$. Putting $u=M^{-1/p}f(M^{1/p},|z|)$ in (1.1a) we obtain (1.4). This bound holds clearly for $0<|z|<r(M,p)$. An immediate calculation shows that for $0<z<r(M,p)$ and $F(z)=[f(M^{1/p},z)]^p$ an equality in (1.4) is attained.

We shall now deduce our statements concerning $r(M,p)$, $r(M,p)$ being defined as the unique solution of the equation

(1.10) $$f(M^{1/p},r)=M^{1/p}u_p\equiv\frac{1}{p}\left[\sqrt{1+p^2}-1\right]M^{1/p}.$$

Since the Pick function fulfills the equation

(1.11) $$f(K,r)\left[1-K^{-1}f(K,r)\right]^{-2}=r(1-r)^{-2}$$

we can replace (1.10) by an equivalent equation:

(1.12) $$M^{1/p}u_p(1-u_p)^{-2}=r(1-r)^{-2}.$$

By (1.8) we have

$$M^{1/p}u_p(1-u_p)^{-2}=\frac{1}{2}M^{1/p}p(\sqrt{1+p^2}+p)$$

and (1.12) takes the form

(1.12a) $$r(1-r)^{-2}=\frac{1}{2}M^{1/p}p(\sqrt{1+p^2}+p)$$

and hence

(1.2) $$r=r(M,p)=A^{-1}(A+1/2-\sqrt{A+1/4}),$$

where A is defined as in (1.3). (1.2) implies that r is a strictly increasing function of A. In order to obtain the greatest lower bound of $r(M,p)$, we need a little more information about A. A increases strictly with M and therefore, p being fixed and $\geqslant 1$, we have

$$A>\frac{1}{2}p(\sqrt{1+p^2}+p)\geqslant\frac{1}{2}(\sqrt{2}+1)=A_0.$$

Hence

$$\inf_{p\geqslant 1,\,M>1} r(M,p)=(2A_0)^{-1}(2A_0+1-\sqrt{4A_0+1})=\sqrt{2}-1.$$

This greatest lower bound is evidently sharp and the circle where (1.4) is valid, has a respectively small radius for $p=1$ and M slightly greater than 1. If $A\to+\infty$, then $r(M,p)\to 1-$ and this gives (1.6) in virtue of (1.3). The theorem 1. is proved.

Since

$$\sup_{0<u<1}\varphi_p(u)=\varphi_p(u_p)=(\sqrt{1+p^2}-p)\left(\frac{\sqrt{1+p^2}-1}{p}\right)^p$$

we see that

(1.13) $$|F'(z)|\leqslant\frac{pM(1+|z|)}{|z|(1-|z|)}(\sqrt{1+p^2}-p)\left(\frac{\sqrt{1+p^2}-1}{p}\right)^p.$$

2. The inequality (1.13) implies that $F'(z) = O((1-|z|)^{-1})$ for $F \epsilon S_M^{(p)}$ and $|z| \to 1$. It is, however, well known that the derivative of an arbitrary regular function $F(z)$ bounded in the unit circle is $O((1-|z|)^{-1})$. This is an immediate consequence of G. Pick's inequality

$$|F'(z)| \leqslant M \frac{1-|M^{-1}|F(z)||^2}{1-|z|^2}$$

valid for $F(z)$ regular in $|z|<1$ and such that $|F(z)|<M$ for $|z|<1$. Let $M(r,f)$ denote $\sup\limits_{|z|<r} |f(z)|$. Pick's inequality implies: $\overline{\lim\limits_{r\to 1-}} (1-r) M(r, F'_r) \leqslant M(1, F)/2$. Besides, it is easy to see that for the function $w = e^{i \log \frac{1+z}{1-z}}$ (which is regular and bounded in the unit circle $|z|<1$ and represents it conformally on the universal covering surface of the annulus $e^{-\pi 2} < |w| < e^{\pi 2}$) we have

$$|f'(z)| = \frac{2|f(z)|}{|1-z^2|}$$

and for $0 < z = r < 1$ we have

$$(1-r) M(r, f') \geqslant \left| e^{i \log \frac{1+r}{1-r}} \right| \cdot \frac{2(1-r)}{1-r^2} = \frac{2}{1+r} > 1.$$

Therefore "O" cannot be replaced by "o" under the sole assumption fo boundedness. Let us now suppose that $F(z)$ being bounded, fulfils a supplementary condition, such as p-valency or more generally, areally mean p-valency in the sense of D. C. Spencer, [5]. Then we can replace "O" by "o" in the statement $F'(z) = O((1-|z|)^{-1})$. The bounded functions fulfilling such a condition represent clearly the unit circle on a Riemann surface of finite area. We next prove that for functions with this property we have $F'(z) = o((1-|z|)^{-1})$. This means that the boundedness of the area covered by $F(z)$ exercises a greater influence upon the growth of the derivative then the boundedness of the function itself.

Theorem 2. *If the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is regular in the unit circle $|z|<1$ and if the Riemann surface being the map of $|z|<1$ by $f(z)$ is of finite area, then $M(r, f') = o((1-r)^{-1})$ for $r \to 1-$.*

Proof. Let $A(r, f)$ denote the area of the Riemann surface being the map of $|z|<r$ by $f(z)$. We have

$$A(r, f) = \pi \sum_{n=1}^{\infty} n |a_n| r^{2n}.$$

Since $A(r, f)$ is bounded as a function of r, $r \in (0, 1)$, the series $\sum_n n|a_n|^2$ converges. Given arbitrary $\varepsilon > 0$, we can choose $N \geqslant 2$ such that

$$\sum_{n=N}^{\infty} n|a_n|^2 < \frac{\varepsilon^2}{4}.$$

Then

$$\sum_{n=N}^{\infty} n|a_n| r^{n-1} = \sum_{n=N}^{\infty} \sqrt{n}|a_n| \sqrt{n} r^{n-1} \leqslant \sqrt{\sum_{n=N}^{\infty} n|a_n|^2} \cdot \sqrt{\sum_{n=N}^{\infty} nr^{2n-2}} <$$

$$< \frac{\varepsilon}{2} \sqrt{\sum_{n=1}^{\infty} nr^{2n-2}} = \frac{\varepsilon}{2} \frac{1}{1-r^2}.$$

Thus we have

$$(1-r) \sum_{n=N}^{\infty} n|a_n| r^{n-1} < \frac{\varepsilon}{2} \frac{1}{1+r} < \frac{\varepsilon}{2}$$

for every $r \in (0, 1)$ if $N = N(\varepsilon)$ is large enough. Having fixed N as above, we choose r_0 so that

$$(1-r) \sum_{n=1}^{N-1} n|a_n| < \frac{\varepsilon}{2} \quad \text{for any} \quad r \in (r_0, 1).$$

Then a fortiori

$$(1-r) \sum_{n=1}^{N-1} n|a_n| r^{n-1} < \frac{\varepsilon}{2} \quad \text{for} \quad r \in (r_0, 1)$$

and

$$(1-r) M(r, f') \leqslant (1-r) \sum_{n=1}^{\infty} n|a_n| r^{n-1} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon \quad \text{for any} \quad r \in (r_0, 1).$$

This proves our statement.

We have even proved somewhat more: under the above stated assumption the majorizing function for the derivative

$$\mathfrak{M}(r, f') = \sum_{n=1}^{+\infty} n|a_n| r^{n-1} \quad \text{is} \quad o\left(\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1.$$

Corollary 1. *If $F(z)$ is regular, bounded and p-valent, or, more generally areally mean p-valent, then $A(r, F) \leqslant \pi p M^2(1, F)$ and therefore $M(r, F') = o((1-r)^{-1})$. The same is true for the class of circumferentially mean p-valent functions [1], but not for the class of weakly p-valent functions.*

[*Counter-example:* for $f(z) = 2e^{\pi^2} + e^{i \log \frac{1+z}{1-z}}$ being weakly univalent we have $(1-r) M(r, f') \geqslant 1$].

If W is a simply connected Riemann surface being the 1—1 map of $|z| < 1$ under $f(z)$, $f(z)$ being regular in the unit circle, then the inner conformal radius $r(W, P_0)$ of W with respect to P_0 is equal to

$(1-|z_0|^2)|f'(z_0)|$. We suppose here that the point P_0 of W corresponds to the point z_0 of the unit circle $|z|<1$. If the point P_0 tends to the ideal boundary $Fr(W)$ of W, then the corresponding point z_0 tends to the boundary of the unit circle by the topological invariance of boundary.

Corollary 2. *Since for functions whose values cover Riemann domains of finite area* $r(W,P_0)/(1+|z_0|) = (1-|z_0|)\,|f'(z_0)|$ *tends to zero in virtue of Theorem* 2, *we see that*

$$\lim r(W,P_0)=0, \qquad P_0 \to Fr(W).$$

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Streszczenie

Wychodząc z uzyskanego poprzednio oszacowania na pochodną logarytmiczną, otrzymuję dokładne oszacowanie od góry pochodnej funkcji $f(z)=z^p+\ldots$ p-listnej, ograniczonej ($|f|<M$) i regularnej w kole $|z|<1$, ważne dla $0\leqslant|z|\leqslant r(M,p)$ gdzie $r(M,p)\geqslant 0{,}4142\ldots$ oraz $r(M,p)\to 1$ zarówno przy $p\to+\infty$, jak i przy $M\to\infty$.

Ponadto wykazuję, że dla funkcji, których powierzchnia Riemanna posiada skończone pole, a więc w szczególnym przypadku dla funkcji polowo p-listnych i ograniczonych, jest $f'(z)=o(1/(1-|z|))$.

Резюме

Исходя из ранее полученной оценки логарифмической производной, я получаю точную верхнюю оценку производной р-листной функций $f(z)=z^p+\cdots$ ограниченной и регулярной в круге $|z|<1$, пригодную для $0\leqslant|z|\leqslant r(M,p)$ где $r(M,p)\geqslant 0{,}4142\ldots$, и $(M,p)\to 1$ так при $p\to+\infty$, как и при $M\to+\infty$.

Сверх того я показываю, что для функций, которых римановы поверхности имеют конечные площади, — следовательно, в частности для функций площадно p-листных и ограниченных — имеет место $f'(z)=o(1/(1-|z|))$.