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An inequality concerning series with decreasing positive terms

O pewnej nierówności dotyczącej szeregów o wyrazach dodatnich malejących

Об одном неравенстве, относящемся к рядам с положительными убывающими членами

1. Introduction.

Let $\sum_n u_n$ be a series (convergent, or not) such that $u_0 > 0$ and $u_k \geq 0$ $(k=1,2,...)$. Put

$$U_n = \sum_{k=0}^{n} u_k, \quad U_0 = u_0, \tag{1.1}$$

$$\sigma_n = \sum_{k=0}^{n} \frac{u_k}{U_k}, \tag{1.2}$$

$$\tau_n = \sum_{k=0}^{n} \frac{u_{n-k}}{U_k}. \tag{1.3}$$

It is well known that the sequences $\{U_n\}$ and $\{\sigma_n\}$ both converge, or both diverge (see [3], p. 299). Besides, it is easy to prove that $\{\tau_n\}$ *also converges for convergent* $\{U_n\}$ *and then* $\lim_n \tau_n = 1$.

In order to prove it, consider the following sequence-to-sequence transformation:

$$\eta_n = \left(\frac{1}{U_n} - \frac{1}{U_{n-1}}\right)\xi_0 + \left(\frac{1}{U_{n-1}} - \frac{1}{U_{n-1}}\right)\xi_1 + \dots + \left(\frac{1}{U_1} - \frac{1}{U_0}\right)\xi_{n-1} + \frac{1}{U_0}\xi_n \tag{1.4}$$

where $U_n > 0$ and $U_n \uparrow U$ (i. e. U_n increases and tends to the limit U). Obviously (1.4) is a Kojima (or convergence-preserving) transformation (see e. g. [1], p. 385). For convergent $\{\xi_n\}$ η_n tends to the limit

$$\eta = \frac{\lim_n \xi_n}{\lim_n U_n}.$$

Therefore $\lim_n \tau_n = 1$, since putting $\xi_n = U_n$ in (1.4) we have $\eta_n = \tau_n$.

If $\{U_n\}$ diverges, $\{\tau_n\}$ may be convergent (e. g. for $U_n = \sqrt{n+1}$) or divergent. (e. g. for $U_n = n+1$). We shall prove in the sequel that *for divergent* $\sum_n u_n$ *with positive and decreasing terms we have always* $\tau_n = O(\log n)$ *and that this bound cannot be decreased*. This result is an answer to the question raised by L. Jeśmanowicz.

2. We first give an example of a divergent sequence $\{U_n\}$ with $u_n \downarrow 0$ for which $\{\tau_n\}$ diverges and $\tau_n > (1/2 - \varepsilon) \log n$ ($\varepsilon > 0$, arbitrary) if n is large enough.

It is well known that, if $\frac{\alpha_n}{\beta_n} \to +\infty$ and $\sum \beta_n$ diverges ($\beta_n > 0$ for $n = 0, 1, 2, \ldots$). then also $\frac{\alpha_0 + \alpha_1 + \ldots + \alpha_n}{\beta_0 + \beta_1 + \ldots + \beta_n} \to +\infty$ („de l'Hôpital's rule" for sequences, an immediate consequence of Theorem 9, p. 52, [2]).

Suppose the sequence $\{U_n\}$ fulfills the conditions:

(2.1) $\{U_n\}$ increases strictly: $U_{n+1} - U_n = u_n > 0$,

(2.2) $u_n \to 0$ strictly decreasing,

$$\frac{\dfrac{1}{U_n}}{\dfrac{1}{u_{n+1}} - \dfrac{1}{u_n}} \to +\infty. \tag{2.3}$$

Putting $\alpha_n = \dfrac{1}{U_n}$, $\beta_n = \dfrac{1}{u_{n+1}} - \dfrac{1}{u_n}$ we see by the above remark that

$$\frac{\dfrac{1}{U_0} + \dfrac{1}{U_1} + \ldots + \dfrac{1}{U_n}}{\dfrac{1}{u_{n+1}} - \dfrac{1}{u_0}} = u_{n+1}\left(\frac{1}{U_0} + \ldots + \frac{1}{U_n}\right)\frac{1}{1 - \dfrac{u_{n+1}}{u_0}}$$

tends to infinity and this implies that also $\tau_n = \frac{u_n}{U_0} + \dots + \frac{u_0}{U_n}$ does so. In particular, $U_n = \frac{n+a}{\log(n+a)}$ $(a > e^2)$ fulfills (2.1) — (2.3). We have $\tau_n >$

$$> u_n\left(\frac{1}{U_0} + \dots + \frac{1}{U_n}\right) > \frac{\log(n+a)-1}{[\log(n+a)]^2} \int_a^{n+a} \frac{\log x}{x}\, dx = \frac{1}{2}\, \frac{\log(n+a)-1}{\log n} \cdot$$

$$\cdot \left[1 - \frac{\log^2 a}{\log^2(n+a)}\right] \cdot \log n > \left(\frac{1}{2} - \varepsilon\right) \log n \quad \text{for } n \text{ large enough.}$$

3. We shall now prove that for any increasing and concave sequence $\{U_n\}$ $(U_0 > 0)$ we have

$$(3.3) \qquad \tau_n \leqslant 1 + \frac{1}{2} + \dots + \frac{1}{n+1} = O(\log n).$$

In order to prove it, we fix n and vary the first $n+1$ terms $U_0, \dots, U_n$ so that τ_n should attain maximum. We remark also that

$$\tau_n = \left(\frac{U_0}{U_n} + \dots + \frac{U_n}{U_0}\right) - \left(\frac{U_0}{U_{n-1}} + \dots + \frac{U_{n-1}}{U_0}\right).$$

Then the inequality (3.3) is contained in the following

Lemma. *Let* $x_0 = \delta > 0$ *and*

$$(3.1) \qquad x_0 = \delta \geqslant x_1 - x_0 \geqslant x_2 - x_1 \geqslant \dots \geqslant x_n - x_{n-1} \geqslant 0$$

If

$$(3.2) \qquad \tau_n = \left(\frac{x_0}{x_n} + \frac{x_1}{x_{n-1}} + \dots + \frac{x_n}{x_0}\right) - \left(\frac{x_0}{x_{n-1}} + \frac{x_1}{x_{n-2}} + \dots + \frac{x_{n-1}}{x_0}\right)$$

then

$$(3.3) \qquad \tau_n \leqslant 1 + \frac{1}{2} + \dots + \frac{1}{n+1}$$

for all $x_1, \dots x_n$ *fulfilling* (3.1). *This maximum is attained for* $x_k = (k+1)\delta$, $(k = 1, 2, \dots, n)$.

Proof. The set D of points of the n-dimensional space whose coordinates $x_1, \dots, x_n$ fulfill (3.1) is compact and τ_n being continuous on this set attains its maximum at a point $P \in D$. (3.1) implies

$$(3.4) \qquad \delta \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_n \leqslant (n+1)\delta,$$

Since $\frac{x_0}{x_n}+\frac{x_n}{x_0}$ is for $x_n \geqslant x_0$ a strictly increasing function of x_n, so $x_1, \ldots, x_{n-1}$ being fixed, τ_n attains its maximal value for x_n of possibly greatest value, i. e. $x_n = 2x_{n-1} - x_{n-2}$, for $2x_{n-1} - x_{n-2} \geqslant x_n$ by (3.1). Therefore it suffices to find the maximum of

$$\tau^{(1)} = \left(\frac{x_0}{2x_{n-1}-x_{n-2}}+\frac{x_1}{x_{n-1}}+\ldots+\frac{x_{n-1}}{x_1}+\frac{2x_{n-1}-x_{n-2}}{x_0}\right) - \left(\frac{x_0}{x_{n-1}}+\ldots+\frac{x_{n-1}}{x_0}\right).$$

Since $\left(\varphi+\frac{1}{\varphi}\right)' = \varphi'\left(1-\frac{1}{\varphi^2}\right)$, we have

$$\frac{\partial \tau^{(1)}}{\partial x_{n-1}} = \left[1-\left(\frac{x_0}{2x_{n-1}-x_{n-2}}\right)^2\right]\frac{2}{x_0}+\left[1-\left(\frac{x_1}{x_{n-1}}\right)^2\right]\frac{1}{x_1}-\left[1-\left(\frac{x_0}{x_{n-1}}\right)^2\right]\frac{1}{x_0} \geqslant 0$$

because (3.1) implies $\frac{x_0}{2x_{n-1}-x_{n-2}} \leqslant \frac{x_0}{x_{n-1}} \leqslant 1$, or

$$1-\left(\frac{x_0}{2x_{n-1}-x_{n-2}}\right)^2 \geqslant 1-\left(\frac{x_0}{x_{n-1}}\right)^2 \geqslant 0$$

Therefore $\tau^{(1)}$ attains its maximum for the possibly greatest value of x_{n-1}. Since by (3.1) $x_{n-2}-x_{n-3} \geqslant x_{n-1}-x_{n-2}$ or $2x_{n-2}-x_{n-3} \geqslant x_{n-1}$, therefore $x_{n-1} = 2x_{n-2}-x_{n-3}$, $x_n = 2x_{n-1}-x_{n-2} = 3x_{n-2}-2x_{n-3}$. Substituting these values into (3.2) we obtain

$$\tau^{(2)} = \left(\frac{x_0}{3x_{n-2}-2x_{n-3}}+\frac{x_1}{2x_{n-2}-x_{n-3}}+\frac{x_2}{x_{n-2}}+\ldots+\frac{x_{n-2}}{x_2}+\right.$$
$$\left.+\frac{2x_{n-2}-x_{n-3}}{x_1}+\frac{3x_{n-2}-2x_{n-3}}{x_0}\right)-\left(\frac{x_0}{2x_{n-2}-x_{n-3}}+\frac{x_1}{x_{n-2}}+\ldots+\right.$$
$$\left.+\frac{x_{n-2}}{x_1}+\frac{2x_{n-2}-x_{n-3}}{x_0}\right).$$ We have similarly

$$\frac{\partial \tau^{(2)}}{\partial x_{n-2}} = \left[1-\left(\frac{x_0}{3x_{n-2}-2x_{n-3}}\right)^2\right]\frac{3}{x_0}+\left[1-\left(\frac{x_1}{2x_{n-2}-x_{n-3}}\right)^2\right]\frac{2}{x_1}+$$
$$+\left[1-\left(\frac{x_2}{x_{n-2}}\right)^2\right]\frac{1}{x_2}-\left[1-\left(\frac{x_0}{2x_{n-2}-x_{n-3}}\right)^2\right]\frac{2}{x_0}-\left[1-\left(\frac{x_1}{x_{n-2}}\right)^2\right]\frac{1}{x_1} \geqslant 0$$

since (3.1) implies $3x_{n-2} - 2x_{n-3} \geqslant 2x_{n-2} - x_{n-3}$, or

$$1 - \left(\frac{x_0}{3x_{n-2} - 2x_{n-3}}\right)^2 \geqslant 1 - \left(\frac{x_0}{2x_{n-2} - x_{n-3}}\right)^2 \text{ and}$$

$$2x_{n-2} - x_{n-3} \geqslant x_{n-2} \quad \text{or} \quad 1 - \left(\frac{x_1}{2x_{n-2} - x_{n-3}}\right)^2 \geqslant 1 - \left(\frac{x_1}{x_{n-2}}\right)^2$$

$\tau^{(2)}$ attains its maximum for the greatest value of x_{n-2}, i. e. $x_{n-2} = 2x_{n-3} - x_{n-4}$. Therefore $x_{n-2} = 2x_{n-3} - x_{n-4}$, $x_{n-1} = 2x_{n-2} - x_{n-3} = 3x_{n-3} - 2x_{n-4}$, $x_n = 2x_{n-1} - x_{n-2} = 4x_{n-3} - 3x_{n-4}$. Substituting these values into (3.2) we obtain the function $\tau^{(3)}$ of the variables $x_1, \ldots, x_{n-3}$ and an analogous computation shows that it is also an increasing function of the variable x_{n-3} and so it assumes the maximum for the possibly greatest value of x_{n-3}. Then $x_n - x_{n-1} = x_{n-1} - x_{n-2} = x_{n-2} - x_{n-3} = x_{n-3} - x_{n-4}$. The continuation of this procedure gives us that all the differences $x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$ are equal and x_1 shall be of possibly greatest value, i. e. $x_0 = \delta$, $x_1 = 2\delta, \ldots x_n = (n+1)\delta$. Then

$$\max \tau_n = \frac{x_0}{x_n} + \frac{x_1 - x_0}{x_{n-1}} + \ldots + \frac{x_n - x_{n-1}}{x_0} = \frac{1}{n+1} + \frac{1}{n} + \ldots + 1$$

and this is the desired result.

The obtained result implies an inequality for power series. Put

$$\sum_{n=0}^{\infty} u_n z^n = u(z), \qquad \sum_{n=0}^{\infty} U_n z^n = U(z), \qquad \sum_{n=0}^{\infty} \frac{1}{U_n} z^n = U^*(z)$$

where $\{u_n\}$ is a decreasing sequence of positive numbers and $U_n = u_0 + u_1 + \ldots + u_n$, the series being evidently convergent for $|z| < 1$. Since

$$\frac{u_0}{U_n} + \frac{u_1}{U_{n-1}} + \ldots + \frac{u_n}{U_0} = \tau_n \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{n+1}, \text{ therefore}$$

$$\sum_{n=0}^{\infty} \tau_n r^n \leqslant \frac{1}{r}(1 + r + r^2 + \ldots)\left(\frac{r}{1} + \frac{r^2}{2} + \ldots\right) = \frac{1}{r}\,\frac{1}{1-r}\log\frac{1}{1-r}$$

$$\text{or} \quad u(r)\,U^*(r) \leqslant \frac{1}{r(1-r)}\log\frac{1}{1-r},$$

for real and positive $r < 1$. In other words

$$U(r)U^*(r) \leqslant \frac{1}{r(1-r)^2} \cdot \log \frac{1}{1-r}, \quad \text{or}$$

$$(U_0 + U_1 r + U_2 r^2 + \ldots)\left(\frac{1}{U_0} + \frac{1}{U_1} r + \frac{1}{U_2} r^2 + \ldots\right) \leqslant \frac{1}{r(1-r)^2} \log \frac{1}{1-r}$$

for increasing, concave $\{U_n\}$ with positive terms and with the equality for $U_n = n + 1$ only.

REFERENCES

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[3] Knopp K., *Theorie und Anwendung der unendlichen Reihen*. Springer. Berlin u. Heidelberg 1947.

Streszczenie

Połóżmy $U_n = u_0 + u_1 + \ldots + u_n$ ($u_0 > 0$, $u_k \geqslant 0$ dla $k = 1, 2, \ldots$). Wykazuję, że gdy $\{U_n\}$ jest zbieżny, to $\tau_n = \frac{u_0}{U_n} + \frac{u_1}{U_{n-1}} + \ldots + \frac{u_n}{U_0} \to 1$, oraz podaję przykład ciągu U_n takiego, że $u_n \downarrow 0$, zaś $\tau_n > \left(\frac{1}{2} - \varepsilon\right) \log n$. Mamy jednak zawsze dla ciągu rosnącego i wklęsłego o wyrazach dodatnich $\{U_n\}$:

$$\tau_n \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{n+1}$$

i jest to oszacowanie możliwie najlepsze. Wynika stąd nierówność

$$(U_0 + U_1 r + \ldots + U_n r^n + \ldots)\left(\frac{1}{U_0} + \frac{1}{U_1} r + \ldots + \frac{1}{U_n} r^n + \ldots\right) \leqslant$$

$$\leqslant \frac{1}{r(1-r)^2} \log \frac{1}{1-r}$$

dla $0 < r < 1$, „=“ jedynie dla $U_n = n + 1$.

Резюме

Положим $U_n = u_0 + u_1 + \ldots + u_n$ ($u_0 > 0$, $u_k \geqslant 0$ для $k = 1, 2, \ldots$) Я доказываю, что если $\{U_n\}$ сходится, то $\tau_n = \frac{u_0}{U_n} + \frac{u_1}{U_{n-1}} + \ldots + \frac{u_n}{U_0} \to 1$, и даю пример такой последовательности $\{U_n\}$, что $u_n \downarrow 0$, а $\tau_n > \left(\frac{1}{2} - \varepsilon\right) \log n$. Однако всегда для последовательности $\{U_n\}$ растущей и вогнутой с положительными членами имеем

$$\tau_n \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{n+1},$$

и эта оценка возможно лучшая. Отсюда вытекает неравенство

$$(U_0 + U_1 r + \ldots + U_n r^n + \ldots)\left(\frac{1}{U_0} + \frac{1}{U_1} r + \ldots + \frac{1}{U_n} r^n + \ldots\right) \leqslant \frac{1}{r(1-r)^2} \log \frac{1}{1-r}$$

для $0 < r < 1$. Знак $=$ имеет место только при $U_n = n + 1$.