ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA

VOL. XLVIII, 6 SECTIO A 1994

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Universal Teichmüller Space and Fourier Series

Abstract. This is an abridged version of a paper by the author [7]. It was presented as a talk with the same title at the Workshop on Complex Analysis (Lublin, June 1994).

1. Ahlfors – Bers equivalence relation

The notion of the universal Teichmüller space $T(1)$ has its origin in the fundamental papers by Teichmüller [11] and Ahlfors [1]. Teichmüller was the first to deal with quasiconformal (abbr.: qc) mappings between Riemann surfaces. Let W be a fixed compact Riemann surface of genus g. The number g may be evaluated from the equality $s_0 - s_1 + s_2 = 2 - 2g$, where s_k denotes the number of k-dimensional simplexes in an arbitrary simplicial decomposition of W. Suppose $\widetilde{W} = f(W)$ is a homeomorphism and $z = h(p)$, $\widetilde{z} = \widetilde{h}(q)$ are local parameters for $p \in W$, $q \in \widetilde{W}$. The mapping f is said to be qc iff $\widetilde{z} = \widetilde{h} \circ f \circ h^{-1}(z)$ is qc, whenever the mapping $h \circ f \circ h$ is defined. Teichmüller realized the necessity of distinguishing between different homotopy classes of such mappings. If W is a fixed compact Riemann surface and $W_k = f_k(W)$ are qc images of W then W_1, W_2 are said to be in the same homotopy class if and only if $f_2 \circ f_1^{-1}$ is homotopic to identity. By identifying conformally equivalent Riemann surfaces in each homotopy class we obtain what is now called *Teichmüller space* $T(W)$ *generated by* W.

While trying to put in [1] some Teichmüller statements on a firm basis Ahlfors used the representation of W of a finite genus $g > 1$ as a quotient surface $\mathbb{D}/G$. Here the unit disk $\mathbb{D}$ is the universal covering surface of W and G is the Fuchsian group of covering Möbius automorphisms of $\mathbb{D}$. We may associate with G its fundamental region whose interior points and suitably identified boundary points represent in a one–to–one manner the Riemann surface W. The qc mapping $f : W \to \widetilde{W}$ can be lifted to $\mathbb{D}$ as a qc automorphism $\varphi = h \circ f \circ h^{-1}$ of $\mathbb{D}$. The complex dilatation μ of φ remains unchanged after $z = h(p), p \in W$, has been replaced by $S(z) = z^*$, $S \in G$, because z^* may be also considered as a local parameter for the same point $p \in W$. This means that μ is actually a Beltrami differential of type $(-1, 1)$, i.e.

$$\mu(z)\,\overline{dz}/dz = \mu^*(z^*)\,\overline{dz^*}/dz^* \; . \tag{1.1}$$

Since f is qc, we have

$$\|\mu\|_\infty = \|\mu^*\|_\infty < 1 \; . \tag{1.2}$$

Ahlfors proved that any qc automorphism of $\mathbb{D}$ has a homeomorphic extension on $\overline{\mathbb{D}}$ and so does φ. We can normalize φ so that

$$t_j = \exp(2\pi i j/3), \quad j = 0, 1, 2 \; , \tag{1.3}$$

are fixed points of φ.

Let us now replace $\widetilde{W}$ and φ by W_k and φ_k resp., where $k = 1, 2$. As shown by Ahlfors [1], Riemann surfaces W_1, W_2 of finite genus $g > 1$ belong to the same homotopy class if and only if the lifted mappings φ_k satisfy $\varphi_1(t) \equiv \varphi_2(t)$ on $\mathbb{T} = \partial\mathbb{D}$. If μ_k stands for the complex dilatation of φ_k then the last identity may be considered as an equivalence relation between μ_1 and μ_2.

Write $B = \{(\mu : \mathbb{D} \to \mathbb{C}) : \|\mu\|_\infty < 1\}$. Given $\mu \in B$, let us denote by f^μ the unique qc automorphism of $\mathbb{D}$ with complex dilatation μ and fixed points t_j given by (1.3). We say that *the Ahlfors – Bers equivalence relation $\mu \sim \nu$ holds for $\mu, \nu \in B$ if $f^\mu(t) \equiv f^\nu(t)$ on* $\mathbb{T}$.

Consequently, W_1 and W_2 are in the same homotopy class in $T(W)$ if and only if complex dilatations μ_k of lifted mappings φ_k satisfy $\mu_1 \sim \mu_2$. This justifies the following

Definition . If $[\mu] := \{\nu \in B : \nu \sim \mu\}$ then the universal Teichmüller space $T(1)$ is defined as the set $\{[\mu] : \mu \in B\}$.

In fact, any homotopy class in $T(W)$ is associated with some $\mu \in B$ satisfying (1.1) and hence we may write $T(W) \subset T(1)$. Given an arbitrary $\mu \in B$, we may consider the mapping $w = f^{\mu}(z)$ as a qc mapping of a Riemann surface $(\mathbb{D}, z)$ onto $(\mathbb{D}, w)$ and the homotopy relation can be defined formally in the same way.

There are many other ways of expressing the Ahlfors – Bers equivalence relation. Given $\mu \in B$, let us denote by f_μ the unique qc automorphism of the extended plane $\widehat{\mathbb{C}}$ whose complex dilatation is equal to μ at $z \in \mathbb{D}$ and vanishes on $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, whereas $f_\mu(t_j) = t_j$ for $j = 0, 1, 2$. Then

$$\mu \sim \nu \Longleftrightarrow f_\mu|\mathbb{D}^* = f_\nu|\mathbb{D}^* \text{ , cf. [8; p.99] .} \tag{1.4}$$

It is also possible to define the Ahlfors – Bers equivalence relation without referring to the boundary correspondence. Suppose $\mu \in B$ and put

$$\widetilde{\mu}(z) = \begin{cases} \mu(z) , & z \in \mathbb{D} \\ 0 , & z \in \mathbb{D}^* . \end{cases}$$

The singular integral equation

$$\varphi = \widetilde{\mu} + \widetilde{\mu} S\varphi , \tag{1.5}$$

where $S\varphi$ stands for the Hilbert–Beurling transform of φ, has a unique $L^2(\mathbb{C})$–solution φ_μ whose support is contained in $\overline{\mathbb{D}}$. Moreover,

$$\widetilde{f}_\mu(z) := z - \frac{1}{\pi} P.V. \iint_{\mathbb{D}} \frac{\varphi_\mu(\zeta) d\xi d\eta}{\zeta - z} \text{ , } \zeta = \xi + i\eta \text{ ,} \tag{1.6}$$

is the unique quasiconformal self–mapping of $\widehat{\mathbb{C}}$, of the form $z + o(1)$ as $z \to \infty$, whose complex dilatation is equal to $\widetilde{\mu}$ a.e., see [4], [9]. One can easily derive from (1.4)

Theorem 1 [7]. *If μ, $\nu \in B$ then $\mu \sim \nu$ holds if and only if*

$$\widetilde{f}_\mu|\mathbb{D}^* = \widetilde{f}_\nu|\mathbb{D}^* . \tag{1.7}$$

As a corollary of (1.6) and (1.7) we obtain the following analytic criterion of the Ahlfors–Bers equivalence [7].

If μ, $\nu \in B$ then $\mu \sim \nu$ if and only if

$$\iint_{\mathbb{D}} \varphi_\mu(z) z^n \, dx dy = \iint_{\mathbb{D}} \varphi_\nu(z) z^n \, dx dy \ , \ z = x + iy \ , \tag{1.8}$$

holds for $n = 0, 1, 2, \ldots$.

It follows from (1.6) that $\Gamma = \widetilde{f}_\mu(\mathbb{T})$ is a quasicircle whose transfinite diameter $d(\Gamma) = 1$, whereas the condition $\lim_{z\to\infty}[\widetilde{f}_\mu(z) - z] = 0$ means that the conformal centre of gravity of Γ coincides with the origin. Such a quasicircle is said to be normalized. Consequently, there is a one–to–one correspondence between equivalence classes $[\mu]$ and normalized quasicircles Γ.

2. T(1) and Fourier series

According to the definition of the universal Teichmüller space there exists a one–to–one correspondence between the class $[\mu]$, $\mu \in B$, and automorphisms (= sense preserving homeomorphic self–mappings) h of $\mathbb{T}$ that admit a qc extension on $\mathbb{D}$:

$$f^\mu(t) \equiv h(t) \ , \quad t \in \mathbb{T} \ . \tag{2.1}$$

In order to characterize h we need a counterpart of the classical Beurling– Ahlfors theorem [3] for the unit disk which is quoted here as

Lemma A [5]. *An automorphism h of the unit circle $\mathbb{T}$ admits a qc extension on the unit disk if and only if there exists M such that the inequality*

$$|h(\alpha_1)|/|h(\alpha_2)| \le M \tag{2.2}$$

holds for all pairs α_1, α_2 of disjoint adjacent open subarcs α_1, α_2 of $\mathbb{T}$ such that $|\alpha_1| = |\alpha_2|$. Here $|\alpha|$ stands for the length of an arc $\alpha \subset \mathbb{T}$.

An automorphism h of $\mathbb{T}$ satisfying (2.2) is said to be *an M-quasisymmetric function on* $\mathbb{T}$ and then we write $h \in Q(M)$. If $h(e^{i\theta}) = \exp(i\varphi(\theta))$ then $\varphi(\theta) = \theta + \sigma(\theta)$ is an M-quasisymmetric function on $\mathbb{R}$ with the same M as in (2.2), cf. [5]. Thus φ satisfies the familiar M-condition of Beurling–Ahlfors [3], [2] :

$$(2.9) \qquad M^{-1} \leq \frac{\varphi(\theta+d)-\varphi(\theta)}{\varphi(\theta)-\varphi(\theta-d)} \leq M\,, \quad 0 \neq d\,, \quad \theta \in \mathbb{R}\,.$$

The difference $\varphi(\theta) - \theta =: \sigma(\theta)$ is a continuous, 2π-periodic function of bounded variation and, consequently, it may be represented by its Fourier series and considered as a deviation of $\varphi(\theta)$ from the identity. The class of all 2π-periodic function σ such that $\varphi(\theta) = \theta + \sigma(\theta)$ satisfies (2.3) is denoted by $E(M)$, whereas $E_0(M)$ stands for its subclass consisting of functions σ vanishing at $2\pi k/3$, $k \in \mathbb{Z}$. Evidently, there is a one-to-one correspondence between $\sigma \in E_0(M)$ and $[\mu] \in T(1)$.

Any $\sigma \in E(M)$ has a Fourier series representation

$$(2.4) \qquad \sigma(x) = c_0 + (2i)^{-1} \sum_{n=1}^{\infty} (c_n e^{inx} - \bar{c}_n e^{-inx})\,.$$

In order to obtain estimates of σ and $|c_n|$ we use the following

Lemma B [6; (2.7), (2.13)]. *If h is M-quasisymmetric on $\mathbb{R}$ and $h(x) - x$ vanishes at the end-points of an interval I then*

$$(2.5) \qquad |h(x) - x| \leq |I|(M-1)/(M+1) \quad \textit{for any } x \in I$$

and

$$(2.6) \qquad \int_I |h(x) - x|\, dx \leq \frac{1}{2}|I|^2 (M-1)/(M+1)\,.$$

The inequality (2.6) implies at once that for $\sigma \in E_0(M)$

$$(2.7) \qquad |c_0| \leq \frac{\pi}{3}(M-1)/(M+1)\,.$$

We have established in [7]

Theorem 2. *If* $x + \sigma(x)$ *is* M*-quasisymmetric on* $\mathbb{R}$ *and* σ *has the expansion* (2.4) *then*

$$n|c_n| \leq 2(M-1)/(M+1)\,. \tag{2.8}$$

The inequality (2.8) enables us to improve slightly an estimate of the sum $\sum_{n=1}^{\infty} |c_n|$ as obtained by M. Nowak, cf. [10, (3.2)]. We have

Theorem 3. *If* $x + \sigma(x)$ *is* M*-quasisymmetric on* $\mathbb{R}$ *and* σ *has the expansion* (2.4) *then*

$$\sum_{n=1}^{\infty} |c_n| < \pi\sqrt{2} \sum_{n=1}^{\infty} \Big[\Big(\frac{M}{M+1} \Big)^n - 2^{-n} \Big]^{1/2} . \tag{2.9}$$

Mrs. Nowak succeeded to find the estimate

$$\sum_{n=2}^{\infty} |c_n| \leq \pi\sqrt{2} \sum_{n=2}^{\infty} \Big[\Big(\frac{M}{M+1} \Big)^n - 2^{-n} \Big]^{1/2} ,$$

cf. [10, p.98]. Now, due to (2.8), we have

$$|c_1| \leq 2\frac{M-1}{M+1} < \pi \Big(\frac{M-1}{M+1} \Big)^{1/2} = \pi\sqrt{2} \Big(\frac{M}{M+1} - \frac{1}{2} \Big)^{1/2}$$

and (2.9) readily follows.

The estimates (2.8), (2.9) hold for $\sigma \in E(M)$. It is plausible that they could be improved for $\sigma \in E_0(M)$.

REFERENCES

[1] Ahlfors, L.V., *On quasiconformal mappings*, J. Analyse Math. 3 (1954), 1–58.

[2] Ahlfors, L.V., *Lectures on quasiconformal mappinqs*, Van Nostrand, Princeton - Toronto - New York - London 1966.

[3] Beurling, A., and L.V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125–142.

[4] Bojarski, B., *Generalized solutions of a system of first order differential equations of elliptic type with discontinuous coefficients*, [Russian], Mat. Sb. 43 (85) (1957), 451-503.

[5] Krzyż, J.G., *Quasicircles and harmonic measure*, Ann. Acad. Sci. Fenn. Ser. A. I Math. 12 (1987), 19–24.

[6] Krzyż, J.G., *Harmonic analysis and boundary correspondence under quasiconformal mappings*, ibid. 14 (1989), 225–242.

[7] Krzyż, J.G., *Universal Teichmüller space and Fourier series*, to appear ibid.

[8] Lehto, O., *Univalent functions and Teichmüller spaces*, Springer - Verlag, New York Inc. 1987.

[9] Lehto, O., and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer - Verlag, Berlin - Heidelberg - New York 1973.

[10] Nowak, M., *Some new inequalities for periodic quasisymmetric functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 43 (1989), 93–100.

[11] Teichmüller, O., *Extremale quasiconforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss., math.- naturw. Kl. 22 (1939), 1–197.

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