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Estimates of Constants Connected with Linearly Invariant Families of Functions

Abstract. In this note we present estimates of some constants connected with linearly invariant families of functions. We estimate the constant $\sup\{c_f : f = \log h', \text{ ord } h = \alpha\}$, where c_f is defined by (1.1). Moreover, we extend a result of Pfaltzgraff on the univalence of a certain integral. We also establish some results concerning the coefficients of Bloch functions.

1. Estimates of constants

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For an analytic function f on $\mathbb{D}$ we set

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|.$$

The Bloch space $\mathcal{B}$ is the set of all analytic functions f on $\mathbb{D}$ for which $\|f\|_{\mathcal{B}} < \infty$. The quantity $|f(0)| + \|f\|_{\mathcal{B}}$ defines a norm of the linear space $\mathcal{B}$ which, equipped with this norm, is a Banach space (see, e.g. [1], [7]).

Let

$$\mathcal{B}(0) = \{f : f \in \mathcal{B},\ f(0) = 0\},$$

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and

$$\mathcal{B}_1 = \{f : f \in \mathcal{B}(0),\ \|f\|_{\mathcal{B}} \le 1\}.$$

As usual, let S denote the class of functions $g(z) = z + \cdots$ that are analytic and univalent in $\mathbb{D}$. J.M. Anderson, J. Clunie and Ch. Pommerenke ([1]) put the following problem:

Find the best universal constant c, $c > 0$ in the representation

$$f(z) - f(0) = c \log g'(z),$$

where $f \in \mathcal{B}$ and $g \in S$. Of course, it suffices to solve this problem for the class $\mathcal{B}(0)$. Thus, if $f \in \mathcal{B}(0)$ and

$$c_f = \inf\{c : f = c\log g',\ g \in S\}, \tag{1.1}$$

then we are interested in finding

$$C = \sup_{f \in \mathcal{B}(0)} c_f.$$

Let us note that $\sup\{c : f = c\log g',\ g \in S\} = \infty$.

For $a \in \mathbb{D}$ let the Möbius function $\Phi_a : \mathbb{D} \to \mathbb{D}$ be defined by the formula

$$\Phi_a(z) = \frac{a+z}{1+\bar{a}z} e^{i\theta}, \qquad \theta \in \mathbb{R}.$$

If f is a function locally univalent in $\mathbb{D}$ then the order of f is defined in the following way

$$\operatorname{ord} f = \sup_{a \in D} \left| \left\{ \frac{f(\Phi_a(z)) - f(a)}{f'(a)(1-|a|^2)} \right\}_2 \right|,$$

where $\{h(z)\}_2$ denotes the second Taylor coefficient of the function $h(z) = z + \cdots$.

The universal linearly invariant (or universal Möbius invariant) family $\mathcal{U}_\alpha$ (see [6]) is the class of all functions $f(z) = z + \cdots$ analytic in $\mathbb{D}$ such that

1° $f'(z) \ne 0$ in $\mathbb{D}$,

2° $\operatorname{ord} f \le \alpha$.

The authors proved the following result.

Lemma 1.1 [4]. *$f \in \mathcal{B}$ if and only if there exists a function $h \in \bigcup_{\alpha<\infty} \mathcal{U}_\alpha$ such that $f(z) - f(0) = \log h'(z)$.*

This Lemma allows us to obtain some properties of Bloch functions in terms of the order of a corresponding function from $\bigcup_{\alpha<\infty} \mathcal{U}_\alpha$.

Let us foliate the class $\mathcal{B}(0)$ with respect to parameters α and consider the problem of finding

$$C_\alpha = \sup\{c_f : f = \log h',\ \text{ord}\ h = \alpha\},$$

where c_f is defined by (1.1).

Theorem 1.1. *For each $\alpha \in [1, \infty)$*

$$2\alpha \geq C_\alpha \geq \begin{cases} \alpha - 1, & \text{if} \quad \alpha \in [1, 2] \\ (\alpha+1)/3, & \text{if} \quad \alpha \in [2, \infty). \end{cases}$$

Proof. Let $\alpha \in [1, \infty)$ be fixed and let h be an arbitrary, locally univalent function of order α. Then there exist a constant $c > 0$ and a function $g \in S$ such that

$$\log h' = c \log g' = f \in \mathcal{B}. \tag{1.2}$$

If for a fixed h we put in (1.2) $c = c_f - \varepsilon$, $\varepsilon > 0$, then the function g cannot be univalent. Therefore

$$g(z) = \int_0^z (h'(\zeta))^{1/(c_f-\varepsilon)}\, d\zeta \notin S.$$

Thus, by a result of Pfaltzgraff ([5]) we get

$$\frac{1}{c_f - \varepsilon} > \frac{1}{2\alpha}.$$

Since ε arbitrary, we get

$$c_\alpha \leq 2\alpha.$$

Moreover, it follows from (1.2), cf. [6], that

$$\alpha = \operatorname{ord} h = \sup_{z\in\mathbb{D}} \left| \frac{g''(z)}{g'(z)} \frac{1-|z|^2}{2} c - \bar{z} \right| =$$

$$\sup_{z\in\mathbb{D}} \left| \left(\frac{g''(z)}{g'(z)} \frac{1-|z|^2}{2} - \bar{z} \right) c + (c-1)\bar{z} \right| \leq c \operatorname{ord} g + |c-1| \leq 2c + |c-1|.$$

Let us consider two cases.

(i) If $c \geq 1$, then $c \geq (\alpha+1)/3$. Thus

$$c \geq \max\{1, (\alpha+1)/3\}.$$

(ii) If $0 < c < 1$, then $c \geq \alpha - 1$. It follows from (i) and (ii) that

$$c \geq \begin{cases} \alpha - 1, & \text{if} \quad \alpha \in [1,2] \\ (\alpha+1)/3, & \text{if} \quad \alpha \in [2,\infty). \end{cases}$$

The above inequality is true for c_f (thus for C_f, too), because in (1.2) we can choose c sufficiently close to c_f. □

Now, observe that it follows from the definition of the constant C that $C \geq C_\alpha$ for all $\alpha \in [1,\infty)$. So, using Theorem 1.1 we get $C = \infty$.

Let us remark that for $\alpha = 1$ Theorem 1.1 gives a lower estimate of C_α. There exists a function $f_0 \in \mathcal{B}(0)$ such that $c_{f_0} = 0$. Indeed, if $h_0(z) = g_0(z) = z$, then $f_0(z) = c \log g_0'(z)$ for all $c > 0$. Thus $c_{f_0} = 0$.

For each complex number λ the non-linear operator $\mathbf{T}_\lambda$,

$$\mathbf{T}_\lambda(f)(z) = \int_0^z (f'(\zeta))^\lambda \, d\zeta$$

maps the class of functions $f(z) = z + \cdots$ analytic and locally univalent in $\mathbb{D}$ into itself. J.A. Pfaltzgraff ([5], also see [3]) showed that $\mathbf{T}_\lambda(f)$ is univalent, if $f \in \mathcal{U}_\alpha$ and $|\lambda| \leq 1/(2\alpha)$. W.C. Royster ([8], also see [3]) has shown that $\mathbf{T}_\lambda(S) \not\subset S$ for each complex $\lambda \neq 1$ in the range $|\lambda| > 1/3$. The next theorem extends the above results result.

Theorem 1.2. *For all* $\alpha > 1$ *and each complex* $\mu \neq 1$ *and* $|\mu| > 2[3(\alpha-1)]^{-1}$*:*

$$\mathbf{T}_{\mu}(\mathcal{U}_{\alpha}) \not\subset S.$$

Proof. W.C. Royster ([8], also see [3]) considered the function

$$F(z) = \exp[\nu \log(1-z)], \quad \nu \neq 0,$$

which is univalent in $\mathbb{D}$ for $|\nu + 1| \leq 1$ or $|\nu - 1| \leq 1$. Thus the function

$$g(z) = \frac{F(z) - F(0)}{F'(0)} \in S. \tag{1.3}$$

He proved that for these functions the integral $\int_0^z (g'(\zeta))^{\lambda}\, d\zeta$ does not belong to S, if $|\lambda| > 1/3$, $\lambda \neq 1$.

Let us denote by g_- the function of the form (1.3) with the parameter $\nu = -1$ and let

$$h_{\gamma}(z) = \int_0^z (g_-'(\zeta))^{\gamma}\, d\zeta, \qquad \gamma < 0.$$

For the function h_{γ} we have ([6])

$$\text{ord } h_{\gamma} = \sup_{z\in\mathbb{D}} \left|\frac{1-|z|^2}{2}\gamma\frac{g_-''(z)}{g_-'(z)} - \bar{z}\right| = \sup_{z\in\mathbb{D}} \left|\frac{1-|z|^2}{1-z}\gamma - \bar{z}\right| =$$

$$2|\gamma| + 1 = 1 - 2\gamma.$$

Let $\alpha = 1 - 2\gamma$. Then

$$h_{(1-\alpha)/2} \in \mathcal{U}_{\alpha},$$

for all $\alpha > 1$. Now, let us consider the function

$$h_{\mu(1-\alpha)/2}(z) = \int_0^z (g_-'(\zeta))^{\mu(1-\alpha)/2}\, d\zeta = \int_0^z (h_{(1-\alpha)/2}'(\zeta))^{\mu}\, d\zeta,$$

where $\mu \in \mathbb{C}$. By the result of W.C. Royster ([8]) we get $h_{\mu(1-\alpha)/2} \notin S$, if $\mu(1-\alpha)/2 \neq 1$ and $|\mu(1-\alpha)|/2 > 1/3$. □

2. Coefficients of Bloch functions

In this section we deal with coefficients of functions from the class $\mathcal{B}_1$. F.G. Avhadiev and I. Kayumov ([2]) gave the following result

Theorem AK. *If $f \in \mathcal{B}_1$ then for every non-increasing sequence $\delta_n \geq 0$ we have*

$$\sum_{k=1}^{\infty} k|a_k|^2 \delta_k \leq e \sum_{k=1}^{\infty} \delta_k .$$

The following theorem improves the above result.

Theorem 2.1. *If $f \in \mathcal{B}_1$ then*

(i) $$\sum_{k=1}^{\infty} k^2 |a_k|^2 \left(\sum_{l=2}^{\infty} \frac{\delta_l}{l^2 + l} \left(\frac{l-1}{l+1} \right)^{(k-1)/2} \right) \leq \sum_{k=2}^{\infty} \delta_k,$$

for $\delta_k \geq 0$,

(ii) $$\sum_{k=1}^{\infty} k|a_k|^2 \sum_{l=1}^{\infty} \left(\frac{l}{l+1} \right)^k (\delta_l - \delta_{l+1}) \leq \sum_{k=1}^{\infty} \delta_k,$$

for $\delta_k \geq 0$ such that $\sum_{k=1}^{\infty} \delta_k < \infty$,

(iii) $$\sum_{k=1}^{\infty} k|a_k|^2 \left(\sum_{l=1}^{n} \left(\frac{l}{l+1} \right)^n (\delta_l - \delta_{l+1}) + \delta_{n+1} \left(\frac{n}{n+1} \right)^k \right) \leq \sum_{k=1}^{n} \delta_k,$$

for all positive n and real δ_k such that $\sum_{k=1}^{n} \delta_k \geq 0$.

Proof. Let $f \in \mathcal{B}_1$. By the Parseval formula we obtain

(2.1) $$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta = \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2(k-1)} \leq \frac{1}{(1-r^2)^2},$$

for $r \in [0,1)$. Now, put $t = 1/(1-r^2) \in [1,\infty)$. Let us consider a function g, $g(t) = \delta_n$ for $t \in [n, n+1)$. The series

$$\sum_{k=1}^{\infty} k^2|a_k|^2 \left(\frac{t-1}{t}\right)^{k-1} \frac{g(t)}{t^2}$$

is uniformly convergence on all compact subsets of $[1,\infty)$. By (2.1), we get

$$\sum_{n=1}^{\infty} k^2|a_k|^2 \int_1^{n+1} \left(\frac{t-1}{t}\right)^{k-1} \frac{g(t)}{t^2}\,dt \le \int_0^{n+1} g(t)\,dt = \sum_{l=1}^{n} \delta_l. \tag{2.2}$$

Moreover we have

$$\int_1^{n+1} g(t)\left(1-\frac{1}{t}\right)^{k-1}\frac{dt}{t^2} = \sum_{l=1}^{n} \delta_l \int_l^{l+1} \left(1-\frac{1}{t}\right)^{k-1}\frac{dt}{t^2} =$$

$$= \frac{1}{k}\sum_{l=1}^{n} \delta_l \left[\left(\frac{l}{l+1}\right)^k - \left(\frac{l-1}{l}\right)^k\right]. \tag{2.3}$$

The arithmetic - geometric means inequality for $0 < a < b$ gives

$$b^k - a^k \ge k(b-a)a^{(k-1)/2}b^{(k-1)/2}.$$

Therefore

$$\left(\frac{l}{l+1}\right)^k - \left(\frac{l-1}{l}\right)^k \ge k\frac{1}{l(l+1)}\left(\frac{l-1}{l+1}\right)^{(k-1)/2}.$$

Thus it follows from (2.2) that

$$\sum_{k=1}^{\infty} k^2|a_k|^2 \left(\sum_{l=2}^{n} \frac{\delta_l}{l(l+1)}\left(\frac{l-1}{l+1}\right)^{(k-1)/2}\right) \le \sum_{l=1}^{n} \delta_l.$$

For $n \to \infty$ we get the inequality (i).

Now, using (2.3) we obtain

$$\sum_{k=1}^{\infty} k|a_k|^2 \left(\sum_{l=1}^{n} \left(\frac{l}{l+1}\right)^k (\delta_l - \delta_{l+1}) + \delta_{n+1} \left(\frac{n}{n+1}\right)^k \right) \le \sum_{k=1}^{n} \delta_k.$$

If $n \to \infty$ then

$$\sum_{k=1}^{\infty} k|a_k|^2 \sum_{l=1}^{\infty} \left(\frac{l}{l+1}\right)^k (\delta_l - \delta_{l+1}) + \lim_{n\to\infty} \delta_{n+1} \sum_{k=1}^{\infty} k|a_k|^2 \left(\frac{n}{n+1}\right)^k \le$$

$$\le \sum_{k=1}^{\infty} \delta_k.$$

We will show that the above limit is equal 0.

Let us observe that

$$A_f(r) = \frac{1}{\pi} \int_{|z|<r} |f'(xe^{i\theta})|^2 |dz| = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k} \le \frac{r^2}{1-r^2}.$$

Then

$$\delta_n \cdot A_f(\sqrt{1-1/n}) \le \delta_n \cdot (n-1).$$

Because the series $\sum_{k=1}^{\infty} \delta_n$ is convergent we have $\delta_n \cdot (n-1) \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \delta_{n+1} \sum_{k=1}^{\infty} k|a_k|^2 \left(\frac{n}{n+1}\right)^k = \lim_{n\to\infty} \delta_n A_f(\sqrt{1-1/n} = 0$$

and we get the inequality (ii). The proof of the inequality (iii) is analogous. □

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