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On Some Class of Functions Generated by Complex Functions of Bounded Variation

ABSTRACT. Let $\mathfrak{P}'_\alpha$ denote the family of functions p given by the integral

$$p(z) = \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z} d\mu(t), \qquad z \in K = \{z : |z| < 1\}$$

where μ is a complex function of bounded variation satisfying the condition $\left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \le \alpha$ for $\alpha \ge 1$.

In this paper we examine the properties of the class $\mathfrak{P}'_\alpha$ and give estimates of coefficients and of the real part in the class $\mathfrak{P}'_\alpha$.

1. Introduction. Let $\mathfrak{P}$ denote the well-known family of all functions of the form

$$p(z) = 1 + a_1 z + a_2 z^2 + \dots, \tag{1.1}$$

holomorphic in the disc $K = \{z : |z| < 1\}$ and satisfying the condition $\operatorname{Re} p(z) > 0$ for $z \in K$. Let S^C be the class of functions f holomorphic and univalent in the disc K, with the normalization $f(0) = f'(0) - 1 = 0$ and such that the image of the disc K is a convex domain. As well known (e.g.[6; p. 4]), a function p belongs to $\mathfrak{P}$ if and only if

$$p(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu(t), \qquad z \in K, \tag{1.2}$$

where

$$P(\varepsilon, z) = \frac{1+\varepsilon z}{1-\varepsilon z}. \tag{1.3}$$

Let $\mu \in M$, where M denotes the set of all real functions μ nondecreasing on the interval $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 1$. A function $f \in S^C$ if and only if $f(0) = 0$ and

$$f'(z) = \exp\left[-2\int_0^{2\pi} \log(1 - e^{-it}z)d\mu(t)\right], \tag{1.4}$$

where $\mu \in M$ and $z \in K$ ([6; p. 8]).

Paatero [3] extended the class S^C to the classes V_k of functions f for which f' can be expressed in the form (1.4) with $\mu \in M_k$, $k \geq 2$, where M_k consists of all real functions μ of bounded variation on $[0, 2\pi]$ such that $\int_0^{2\pi} d\mu(t) = 1$ and $\int_0^{2\pi} |d\mu(t)| \leq \frac{k}{2}$. Also, the classes P_k of functions (1.2) with $\mu \in M_k$, $k \geq 2$, are well-known, see [4].

V. Starkov [8] introduced the classes U'_α, of holomorphic functions f for which $f(0) = 0$ and f' has the form (1.4), where $\mu \in I_\alpha$, $\alpha \geq 1$, and I_α denotes the family of complex functions μ of bounded variation on $[0, 2\pi]$ satisfying the condition

$$\left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \leq \alpha. \tag{1.5}$$

It is evident that the class I_α reduces itself to the empty set for $\alpha < 1$. Moreover, I_1 is the family of nondecreasing real functions such that $\int_0^{2\pi} d\mu(t) \leq 1$.

In order to explain the geometrical sense of inequality (1.5), let us recall the definition of the universal linearly invariant family (Pommerenke [5]).

Let $\mathfrak{M}$ be some class of functions of the form $f(z) = z + \ldots$, holomorphic and locally univalent in K. We call $\mathfrak{M}$ a linearly invariant family if, for any Moebius self-mapping Φ of the disc K and any $f \in \mathfrak{M}$ also $\Lambda_\Phi[f(\cdot)] \in \mathfrak{M}$, where

$$\Lambda_\Phi[f(z)] = \frac{f(\Phi(z)) - f(\Phi(0))}{f'(\Phi(0))\Phi'(0)} = z + \ldots, \quad z \in K.$$

The number

$$\operatorname{ord}\mathfrak{M} = \sup_{f\in\mathfrak{M}} \frac{|f''(0)|}{2}$$

is called the order of the linearly invariant family $\mathfrak{M}$. In [5] it was proved that

$$\operatorname{ord}\mathfrak{M} = \sup_{f\in\mathfrak{M}} \sup_{z\in K} \left| -\bar{z} + \frac{1}{2}(1-|z|^2)\frac{f''(z)}{f'(z)} \right|.$$

We denote by U_α, $1 \le \alpha < \infty$, the union of all linearly invariant families $\mathfrak{M}$ whose order is not greater than α. It is known [5] that the universal linearly invariant family U_α is composed of all holomorphic and locally univalent functions $f(z) = z + \dots$ for which

$$\sup_{z\in K} \left| -\bar{z} + \frac{1}{2}(1-|z|^2)\frac{f''(z)}{f'(z)} \right| \le \alpha.$$

It turns out [8] that the above-mentioned class U'_α is a linearly invariant family of order α, and also $U'_\alpha \subsetneq U_\alpha$.

2. Definition and basic properties of the class $\mathfrak{P}'_\alpha$.

Definition 2.1. Let $\mathfrak{P}'_\alpha$, $\alpha \ge 1$, denote the class of functions given by formula (1.2), where μ are elements of the class I_α defined earlier.

By definition, the following properties hold.

Property 2.1. *The inclusion $\mathfrak{P} \subset \mathfrak{P}'_1$ takes place.*

Property 2.2. *If $1 \le \alpha_1 < \alpha_2$, then $\mathfrak{P}'_{\alpha_1} \subset \mathfrak{P}'_{\alpha_2}$.*

We have also

Theorem 2.1. *The set of functions of the form (1.2), generated by piecewise constant functions $\mu \in I_\alpha$, is dense in $\mathfrak{P}'_\alpha$.*

Proof. Let Q denote the set of functions described in the above theorem, i.e. the set of functions p of the form $p = \sum_{k=1}^{n} P(e^{-it_k}, z)a_k$, $t_k \in [0, 2\pi]$, $a_k \in \mathbb{C}$,

$$\left|\sum_{k=1}^{n} a_k - 1\right| + \sum_{k=1}^{n} |a_k| \le \alpha, \qquad n = 1, 2, \dots . \tag{2.1}$$

Of course, $Q \subset \mathfrak{P}'_\alpha$.

Take an arbitrary fixed function $p \in \mathfrak{P}'_\alpha$. Then

$$p(z) = \int_0^{2\pi} P(e^{-it}, z) d\mu(t), \qquad z \in K,$$

where $\mu \in I_\alpha$. Let $\mu(t) = \mu_1(t) + i\mu_2(t)$ for $t \in [0, 2\pi]$. The functions $\mu_1(t)$, as well as $\mu_2(t)$ can be approximated by step functions on the interval $[0, 2\pi]$ ([7; p. 282]). So, there exists a sequence (μ_n) of complex piecewise constant functions, uniformly convergent to the function μ. Hence, for any $\varepsilon > 0$, there exists an N such that, for each $n > N$ and each $t \in [0, 2\pi]$, we have

$$|\mu_n(t) - \mu(t)| < \varepsilon.$$

Let $\Gamma = \{z : z = \mu(t), \ \ t \in [0, 2\pi]\}$. For any n, define a piecewise constant function ν_n which has the same points of discontinuity as μ_n and takes the values $z_1 = \mu(t_1), \dots, z_{m_n} = \mu(t_{m_n})$, $0 \le t_1 < t_2 < \dots < t_{m_n} \le 2\pi$, where t_i, $i = 1, \dots, m_n$, are points from different constancy intervals of μ. Put, moreover, $\nu_n(0) = \mu(0)$ and $\nu_n(2\pi) = \mu(2\pi)$, which causes no loss of generality. Of course, for $n > N$ and $t \in [0, 2\pi]$, we have

$$|\mu_n(t) - \nu_n(t)| < \varepsilon.$$

Thus, for $n > N$ and $t \in [0, 2\pi]$,

$$|\mu(t) - \nu_n(t)| < 2\varepsilon.$$

This means that the sequence (ν_n) is uniformly convergent to the function μ on the interval $[0, 2\pi]$. Besides, the construction of the sequence (ν_n) implies that

$$\int_0^{2\pi} d\nu_n(t) = \int_0^{2\pi} d\mu(t)$$

and

$$\int_0^{2\pi} |d\nu_n(t)| \le \int_0^{2\pi} |d\mu(t)|.$$

Thus $\nu_n \in I_\alpha$ because

$$\left|\int_0^{2\pi} d\nu_n(t) - 1\right| + \int_0^{2\pi} |d\nu_n(t)| \le \left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \le \alpha \qquad \text{for } n = 1, 2, \dots .$$

Put $p_n(z) = \int_0^{2\pi} P(e^{-it}, z) d\nu_n(t)$, $z \in K$, $n = 1, 2, \dots$. The uniform convergence of the sequence (ν_n) to the function μ implies the almost uniform convergence of the sequence (p_n) to the function p in the disc K.

Let us observe note that the functions p_n constructed above are elements of the set Q. Indeed, if we denote by $a_1, \dots, a_m$ the jumps of ν_n at the points of discontinuity $\tau_1, \dots, \tau_m$, then we obtain

$$p_n(z) = \sum_{k=1}^{m} P(e^{-i\tau_k}, z) a_k,$$

with a_k satisfying (2.1). This ends the proof. □

Corollary 2.1. *If $p(z) \in \mathfrak{P}'_\alpha$, then $p(e^{i\theta} z) \in \mathfrak{P}'_\alpha$ for $\theta \in \mathbb{R}$.*

Corollary 2.2. *If $p(z) \in \mathfrak{P}'_\alpha$, then $p(rz) \in \mathfrak{P}'_\alpha$ for $r \in [-1, 1]$.*

Corollary 2.3. *If $p \in \mathfrak{P}'_\alpha$, then $p \circ \omega \in \mathfrak{P}'_\alpha$, where ω is a Schwarz function (i.e. ω is holomorphic in K, $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in K$).*

Simple examples of functions of the class $\mathfrak{P}'_\alpha$ are:

1) $p_1(z) = \frac{(\alpha-1)i}{2} \frac{1+z}{1-z}$, $z \in K$, $\alpha > 1$, maps conformally the disc K onto $\{w : \operatorname{Im} w > 0\}$;
2) $p_2(z) = \frac{1-\alpha}{2} \frac{1+z}{1-z}$, $z \in K$, $\alpha > 1$, maps conformally the disc K onto $\{w : \operatorname{Re} w < 0\}$;
3) $p_3(z) = \frac{(\alpha-1)iz}{1-z^2} = \frac{(\alpha-1)i}{4} \frac{1+z}{1-z} + \frac{(1-\alpha)i}{4} \frac{1-z}{1+z}$, $z \in K$, $\alpha > 1$, maps the disc K onto $\mathbb{C} \setminus \{(-\infty, -\frac{1}{2}(\alpha-1)] \cup [\frac{1}{2}(\alpha-1), +\infty)\}$.

Theorem 2.2. *The class $\mathfrak{P}'_\alpha$ is compact in the topology of almost uniform convergence in K.*

Proof. Let $p_n(z) = \int_0^{2\pi} P(e^{-it}, z)d\mu_n(t)$, $z \in K$, $\mu_n \in I_\alpha$ for $n = 1, 2, \dots$, whereas P is defined by formula (1.3). Using the method applied in [8], we shall prove that from the sequence (p_n) one can choose a subsequence almost uniformly convergent in K to a function of the class $\mathfrak{P}'_\alpha$.

Take into consideration the sequence(μ_n) and denote $|\mu_n(t)| = \alpha_n(t)$, $\arg \mu_n(t) = \varphi_n(t)$, $t \in [0, 2\pi]$. Without loss of generality let us assume $\mu_n(0) = 2\alpha$. Hence by (1.5) it follows that $\alpha_n(t)$ and $\varphi_n(t)$ are functions of bounded variation on the interval $[0, 2\pi]$. By of Helly's selection principle ([1; p. 196]), from the sequences (α_n) and (φ_n) one can choose subsequences (α_{n_k}) and (φ_{n_k}) such that $\alpha_{n_k}(t) \to \alpha_0(t)$ and $\varphi_{n_k}(t) \to \varphi_0(t)$ for $t \in [0, 2\pi]$, where α_0 and φ_0 are functions of bounded variation. Therefore the sequence (μ_{n_k}) is convergent to the function $\mu_0(t) = \alpha_0(t)e^{i\varphi_0(t)}$ for $t \in [0, 2\pi]$.

We show that $\mu_0 \in I_\alpha$. Since μ_{n_k} are functions of the class I_α,

$$\int_0^{2\pi} |d\mu_{n_k}(t)| \le \alpha - \Big| \int_0^{2\pi} d\mu_{n_k}(t) - 1 \Big| \stackrel{def}{=} L_{n_k}.$$

By the above (compare the proof of Helly's theorem in [1])

$$\int_0^{2\pi} |d\mu_0(t)| \le \liminf_{k\to\infty} \int_0^{2\pi} |d\mu_{n_k}(t)| \le \lim_{k\to\infty} L_{n_k},$$

and, consequently,

$$\int_0^{2\pi} |d\mu_0(t)| \le \alpha - \Big| \int_0^{2\pi} d\mu_0(t) - 1 \Big|.$$

Let us consider a subsequence (p_{n_k}) of the sequence (p_n) such that $p_{n_k}(z) = \int_0^{2\pi} P(e^{-it}, z)d\mu_{n_k}(t)$, $z \in K$, and denote

$$p_0(z) = \int_0^{2\pi} P(e^{-it}, z)d\mu_0(t), \qquad z \in K.$$

Ewidently $p_0 \in \mathfrak{P}'_\alpha$. By Helly's theorem,

$$\lim_{k\to\infty} p_{n_k}(z) = p_0(z), \qquad z \in K.$$

Since the subsequence (p_{n_k}) is sequence of locally bounded functions in K, we obtain by Vitali's theorem that the sequence (p_{n_k}) is almost uniformly convergent to the function p_0 in the disc K. This ends the proof. □

Theorem 2.3. *The class $\mathfrak{P}'_\alpha$ is convex.*

Proof. Take arbitrary fixed functions of the class $\mathfrak{P}'_\alpha$.

$$p_j(z) = \int_0^{2\pi} P(e^{-it}, z)d\mu_j(t), \qquad \mu_j \in I_\alpha, \quad z \in K, \quad j = 1, 2,$$

with P being defined by formula (1.3). Let

$$p_\Theta(z) = \Theta p_1(z) + (1-\Theta)p_2(z), \qquad 0 \le \Theta \le 1, \quad z \in K.$$

Obviously

$$p_\Theta(z) = \int_0^{2\pi} P(e^{-it}, z)d(\Theta\mu_1(t) + (1-\Theta)\mu_2(t)), \qquad z \in K,$$

and

$$\Big|\int_0^{2\pi} d(\Theta\mu_1(t) + (1-\Theta)\mu_2(t)) - 1\Big| + \int_0^{2\pi} |d(\Theta\mu_1(t) + (1-\Theta)\mu_2(t))|$$
$$\le \Theta\Big|\int_0^{2\pi} d\mu_1(t) - 1\Big| + (1-\Theta)\Big|\int_0^{2\pi} d\mu_2(t) - 1\Big|$$
$$+ \Theta\int_0^{2\pi} |d\mu_1(t)| + (1-\Theta)\int_0^{2\pi} |d\mu_2(t)| \le \Theta\alpha + (1-\Theta)\alpha = \alpha.$$

Thus $p_\Theta \in \mathfrak{P}'_\alpha$ for $0 \le \Theta \le 1$ and this ends the proof. □

Corollary 2.4. *The class $\mathfrak{P}'_\alpha$ is arcwise connected and, in consequence, connected.*

3. Estimates of certain functionals in the class $\mathfrak{P}'_\alpha$. Using elementary methods, we can obtain estimates of some functionals in the class $\mathfrak{P}'_\alpha$.

Let $p \in \mathfrak{P}'_\alpha$, $\alpha \ge 1$, and let $\{p\}_k$, $k = 0, 1, \ldots$, denote the k-th Taylor coefficient of p at zero.

Theorem 3.1. *If $p \in \mathfrak{P}'_\alpha$, $\alpha \geq 1$, then*

$$|\{p\}_k| \leq 2\alpha \qquad \text{for } k = 1, 2, \ldots . \tag{3.1}$$

Equality takes place for the function (1.2) *generated by*

$$\mu = \frac{1+\alpha}{2}\delta_0 + \frac{1-\alpha}{2}\delta_{\pi/k},$$

where

$$\delta_s(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq s, \\ 1 & \text{for } s < t < 2\pi, \quad 0 \leq s < 2\pi. \end{cases} \tag{3.2}$$

Proof. Let $p(z) = \int_0^{2\pi} P(e^{-it}, z)d\mu(t)$, $z \in K$, $\mu \in I_\alpha$, whereas P is defined by formula (1.3). It is evident that

$$\{p\}_k = 2\int_0^{2\pi} e^{-ikt}d\mu(t), \qquad k = 1, 2, \ldots .$$

We have

$$|\{p\}_k| = \left|2\int_0^{2\pi} e^{-ikt}d\mu(t)\right| \leq 2\int_0^{2\pi} |d\mu(t)| \leq 2\alpha, \quad k = 1, 2, \ldots,$$

and thus, the estimates (3.1). □

From Theorem 3.1 and Corollaries 2.1 and 2.2 we get

Corollary 3.1. *For $\alpha \geq 1$ and $k = 1, 2, \ldots$, the set V_k of values of the functional $p \to \{p\}_k$ on the class $\mathfrak{P}'_a$ is the disc of radius 2α and centre 0.*

Theorem 3.2. *For $\alpha > 1$, the set V_0 of all coefficient $\{p\}_0$, $p \in \mathfrak{P}'_a$, is the ellipse*

$$\frac{(\operatorname{Re} A - \frac{1}{2})^2}{\frac{\alpha^2}{4}} + \frac{(\operatorname{Im} A)^2}{\frac{\alpha^2-1}{4}} \leq 1. \tag{3.3}$$

In the case $\alpha = 1$, the set V_0 reduces to the interval $[0, 1]$.

Proof. It follows from (1.2) that for each function $p \in \mathfrak{P}'_\alpha$, $\alpha \geq 1$, there is a $\mu \in I_\alpha$ such that

$$\{p\}_0 = \int_0^{2\pi} d\mu(t).$$

For $\alpha > 1$ we have

$$|\{p\}_0 - 1| + |\{p\}_0| = \left|\int_0^{2\pi} d\mu(t) - 1\right| + \left|\int_0^{2\pi} d\mu(t)\right|$$
$$\leq \left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \leq \alpha.$$

Thus $\{p\}_0$ lies inside or on the boundary of the ellipse with foci at the points 0 and 1 and major axis of length α. This means that $\{p\}_0$ belongs to the set described by inequality (3.3).

Let A_0 be an arbitrary fixed complex number satisfying (3.3). Note that

$$A_0 = \int_0^{2\pi} d(A_0\delta_s)(t)$$

where δ_s is given by formula (3.2). Moreover, $A_0\delta_s \in I_\alpha$ for each $s \in [0, 2\pi]$. So, A_0 is the constant term of the Taylor expansion of a function from the class $\mathfrak{P}'_\alpha$ generated by $\mu = A_0\delta_s$. Thus $A_0 \in V_0$.

Let $\alpha = 1$. The second part of Theorem 3.2 results directly from the properties of the class I_1. □

Theorem 3.3. *If $p \in \mathfrak{P}'_\alpha$, $\alpha \geq 1$, then*

$$\text{(3.4)} \quad \frac{1+r}{1-r}\,\frac{1-\alpha}{2} \leq \operatorname{Re}[p(z)] \leq \frac{1+r}{1-r}\,\frac{\alpha+1}{2}, \qquad |z| = r, \quad z \in K.$$

Equality in (3.4) occurs for the functions (1.2) generated by

$$\mu = \frac{1 \pm \alpha}{2}\,\delta_{\arg z},$$

where $0 \leq \arg z < 2\pi$, δ_s is defined by (3.2) and + (resp. −) is taken for the upper (resp. lower) bound.

Proof. With respect to Theorem 2.1, we can confine our considerations to the dense subclass Q, i.e. to the class of functions of the form (1.2) generated by piecewise constant functions from the set I_α.

Let $p \in Q$. Then $p(z) = \sum_{k=1}^{n} P(e^{-it_k}, z)a_k$, $n = 1, 2, \ldots$. Of course,

$$\left|\sum_{k=1}^{n} a_k - 1\right| + \sum_{k=1}^{n} |a_k| \leq \alpha.$$

Let us observe that $P(e^{-it_k},z) = \frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2}\eta_k$, $|z| = r$, $|\eta_k| = 1$ for $k = 1, \dots, n$.

Hence we get

$$\begin{aligned}
\operatorname{Re}[p(z)] &= \operatorname{Re}\Big[\sum_{k=1}^{n} P(e^{-it_k},z)a_k\Big] \\
&= \frac{1+r^2}{1-r^2}\operatorname{Re}\Big[\sum_{k=1}^{n} a_k\Big] + \frac{2r}{1-r^2}\operatorname{Re}\Big[\sum_{k=1}^{n} a_k\eta_k\Big] \\
&\le \frac{1+r^2}{1-r^2}\Big|\sum_{k=1}^{n} a_k\Big| + \frac{2r}{1-r^2}\sum_{k=1}^{n} |a_k| \\
&\le \frac{1+r^2}{1-r^2}\Big|\sum_{k=1}^{n} a_k\Big| + \frac{2r}{1-r^2}\Big(\alpha - \Big|\sum_{k=1}^{n} a_k - 1\Big|\Big).
\end{aligned}$$

Denote

$$\beta = \Big|\sum_{k=1}^{n} a_k - 1\Big| + \Big|\sum_{k=1}^{n} a_k\Big|. \tag{3.5}$$

Then $1 \le \beta \le \alpha$, $2\big|\sum_{k=1}^{n} a_k\big| - 1 \le \beta$ and

$$\begin{aligned}
\operatorname{Re}[p(z)] &\le \frac{1+r^2}{1-r^2}\Big|\sum_{k=1}^{n} a_k\Big| + \frac{2r}{1-r^2}\Big(\alpha - \beta + \Big|\sum_{k=1}^{n} a_k\Big|\Big) \\
&= \frac{1+r}{1-r}\Big|\sum_{k=1}^{n} a_k\Big| + \frac{2r}{1-r^2}(\alpha-\beta) \\
&\le \frac{1+r}{1-r}\,\frac{1+\beta}{2} + \frac{2r}{1-r^2}(\alpha-\beta) \\
&= \frac{1-r}{1+r}\,\frac{\beta-\alpha}{2} + \frac{1+r}{1-r}\,\frac{1+\alpha}{2} \le \frac{1+r}{1-r}\,\frac{1+\alpha}{2}.
\end{aligned}$$

Analogously, we can obtain an estimate 'from below'

$$\operatorname{Re}[p(z)] = \frac{1+r^2}{1-r^2}\operatorname{Re}\Big[\sum_{k=1}^{n} a_k\Big] + \frac{2r}{1-r^2}\operatorname{Re}\Big[\sum_{k=1}^{n} a_k\eta_k\Big]$$

$$\geq \frac{1+r^2}{1-r^2}\,\mathrm{Re}\Big[\sum_{k=1}^{n} a_k\Big] - \frac{2r}{1-r^2}\sum_{k=1}^{n}|a_k|$$

$$\geq \frac{1+r^2}{1-r^2}\,\mathrm{Re}\Big[\sum_{k=1}^{n} a_k\Big] - \frac{2r}{1-r^2}\Big(\alpha - \Big|\sum_{k=1}^{n} a_k - 1\Big|\Big)$$

$$= \frac{1+r^2}{1-r^2}\,\mathrm{Re}\Big[\sum_{k=1}^{n} a_k\Big] - \frac{2r}{1-r^2}\Big(\alpha - \beta + \Big|\sum_{k=1}^{n} a_k\Big|\Big).$$

From (3.5) we conclude that $\sum_{k=1}^{n} a_k$ is a point of the ellipse with foci at 0 and 1 and with major axis of length β.

Then

$$\sum_{k=1}^{n} a_k = \frac{1+\beta\cos t}{2} + i\frac{\sqrt{\beta^2-1}\,\sin t}{2}, \qquad 0 \leq t \leq 2\pi,$$

and hence

$$\mathrm{Re}\Big[\sum_{k=1}^{n} a_k\Big] = \frac{1+\beta\cos t}{2}, \qquad \Big|\sum_{k=1}^{n} a_k\Big| = \frac{\beta+\cos t}{2}$$

and

$$\mathrm{Re}[p(z)] \geq \frac{1+r^2}{1-r^2}\,\frac{1+\beta\cos t}{2} - \frac{2r}{1-r^2}\Big(\alpha - \beta + \frac{\beta+\cos t}{2}\Big)$$

$$= \frac{1}{2(1-r^2)}\{1+r^2-4\alpha r+2\beta r+[\beta(1+r^2)-2r]\cos t\}$$

$$\geq \frac{1-r}{1+r}\,\frac{\alpha-\beta}{2} + \frac{1+r}{1-r}\,\frac{1-\alpha}{2} \geq \frac{1+r}{1-r}\,\frac{1-\alpha}{2}. \quad \square$$

4. Concluding remarks. It follows from the considerations carried out that there are substantial differences between the well-known family $\mathfrak{P}$ of Carathéodory functions with positive real part and the classes $\mathfrak{P}'_\alpha$. Of course, from Property 2.1, Theorem 3.1 and Theorem 3.3 we get classical estimates: $|\{p\}_k| \leq 2$, $\mathrm{Re}\,p(z) \leq \dfrac{1+|z|}{1-|z|}$.

However, on account of substantial differences between the sets M and I_α, the families $\mathfrak{P}$ and $\mathfrak{P}'_\alpha$ are essentially distinct.

This also implies suitable conclusions concerning applications of the classes $\mathfrak{P}$ and $\mathfrak{P}'_\alpha$. It is known, for instance, that if the function $f(z) = z + a_2z^2 + \dots$ holomorphic in K satisfies the condition $f' \in \mathfrak{P}$, then f is univalent in K. So, there arises a natural problem of investigating the properties of such functions.

This problem, in our case, can be formulated as follows: examine the class of functions $f(z) = a_1z + a_2z^2 + \dots$ holomorphic in K and such that $f' \in \mathfrak{P}'_\alpha$, $\alpha > 1$. Note that the functions f_k satisfying the conditions (see examples 1–3 from Section 2 of the paper)

$$f_1'(z) = \frac{(\alpha-1)i}{2}\frac{1+z}{1-z}, \qquad f_2'(z) = \frac{1-\alpha}{2}\frac{1+z}{1-z},$$
$$f_3'(z) = \frac{(\alpha-1)iz}{1-z^2}, \qquad z \in K,$$

are univalent in K in the case $k = 1, 2$, whereas f_3 is not univalent since, $f_3'(0) = 0$. Hence, among other things, the further investigations concerning the properties of the class P'_α considered in the paper and its applications seem interesting. This, however, was not the aim of the present studies.

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References

[1] Natanson, I. P., *Theorie der Funktionen einer reelen Veränderlichen*, Akademic Verlag, Berlin, 1954.

[2] Noshiro, K., *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ., Ser. I **2** (1934/5), 129–155.

[3] Paatero, V., *Über die konforme Abbildungen von Gebieten, deren Ränder von Beschränkter Drehung sind*, Ann. Acad. Sci. Fenn., Ser. A **33** (1933), no. 9, 1–79.

[4] Pinchuk, B., *Functions of bounded boundary rotation*, Israel J. Math. **10** (1971), 6–16.

[5] Pommerenke, Ch., *Linear-invariante Familien analytischer Funktionen I*, Math. Ann. **155** (1964), 108–154.

[6] Schober, G., *Univalent Functions - Selected Topics*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[7] Sikorski, R., *Differential and integral calculus*, PWN, Warszawa, 1977 (in Polish).
[8] Starkov, V. V., *On some sub-classes of linearly invariant families*, Dep. VINITI **3341** (1981), 1–50. (in Russian)

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