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Image Areas of Functions in the Dirichlet Type Spaces and their Möbius Invariant Subspaces

ABSTRACT. For $p \in (0, +\infty)$ let D_p be the Dirichlet type space of functions f analytic in the unit disk $U = \{z : |z| < 1\}$ for which

$$||f||^2_{D_p} := \iint_U |f'(z)|^2 (1-|z|^2)^p \, dx dy < \infty .$$

Furthermore let Q_p be the Möbius invariant subspace of D_p consisting of those $f \in D_p$ with $\sup_{w \in U} ||f \circ \varphi_w||_{D_p} < \infty$, where $\varphi_w(z) = (w-z)/(1-\overline{w}z)$. In particular, let $Q_{p,0} = \{f \in Q_p : \lim_{|w| \to 1} ||f \circ \varphi_w||_{D_p} = 0\}$. In this paper we investigate the image areas of functions in D_p, Q_p and $Q_{p,0}$.

1. Introduction. Let $U = \{z : |z| < 1\}$ and $\partial U = \{z : |z| = 1\}$ denote the unit disk and the unit circle, respectively, and $dm(z)$ the Lebesgue measure on U. For z, $w \in U$, let

$$g(z, w) = \log \left| \frac{1 - \overline{w}z}{w - z} \right|$$

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be the Green function of U with pole at w. Throughout this paper we shall use A as a symbol for the class of functions analytic on U. We are interested in the Dirichlet–type spaces D_p, $p \in (0,\infty)$ and their subspaces Q_p invariant under analytic automorphisms of U.

Definition. Let $p \in [0,\infty)$ and $\varphi_w(z) = (w-z)/(1-\overline{w}z)$.

a) For $f \in A$ we say that $f \in D_p$ if

$$||f||_{D_p}^2 := \iint_U |f'(z)|^2(1-|z|^2)^p\, dm(z) < \infty .$$

b) For $f \in A$ we say that $f \in Q_p$ if

$$||f||_{Q_p}^2 = \sup_{w\in U} ||f\circ\varphi_w||_{D_p}^2 < \infty .$$

c) We say that $f \in Q_{p,0}$ if $\lim_{|w|\to 1} ||f\circ\varphi_w||_{D_p} = 0$.

Obviously, the spaces D_p, Q_p and $Q_{p,0}$ increase with increasing p.

For special values of p these spaces may be identified as follows: D_0 is the Dirichlet space D, D_1 is the Hardy space H^2, D_2 is the Bergman space B^2, Q_0 is D, $Q_{0,0}$ is the set of constant functions, Q_1 is the space of analytic functions with bounded mean oscillation on ∂U, i. e. BMOA, $Q_{1,0}$ is the space of analytic functions of vanishing mean oscillation on ∂U, i. e. VMOA, Q_2 is the Bloch space B and $Q_{2,0}$ is the little Bloch space B_0.

Furthermore

$$Q_p = \left\{ f : f \in A \text{ and } \sup_{w\in U} \iint_U |f'(z)|^2 g^p(z,w)\, dm(z) < \infty \right\};$$

and

$$Q_{p,0} = \left\{ f : f \in A \text{ and } \lim_{|w|\to 1} \iint_U |f'(z)|^2 g^p(z,w)\, dm(z) = 0 \right\}.$$

As references concerning these identifications, cf. [1], [2], [3], [4], [13], [14] and [15].

In this paper, we mainly study the characterization of functions f belonging to D_p, Q_p and $Q_{p,0}$ resp. by the area of the image domains $f(U)$.

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2. Results. For $f \in A$, $w \in U$ and $r \in (0,1]$ let $U_r(w) = \{z \in U : |\varphi_w(z)| < r\}$, in particular $U_r = U_r(0)$. If we denote by

$$A(f(U_r(w))) = \iint_{U_r(w)} |f'(z)|^2 \, dm(z)$$

the area of $f(U_r(w))$ on the Riemann surface $f(U)$, we get immediately that $f \in D = D_0$ if and only if $\sup_{0<r\leq 1} A(f(U_r)) < \infty$. A similar characterization of the functions in D_p is delivered by

Theorem 1. *Let $p \in (0,\infty)$ and $f \in A$. Then $f \in D_p$ if and only if*

$$\int_0^1 A(f(U_r))(1-r)^{p-1} dr < \infty . \tag{1}$$

Proof. Using the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we get

$$\int_0^1 A(f(U_r))(1-r)^{p-1} dr = 2\pi \sum_{n=1}^{\infty} \frac{n^2}{2n} |a_n|^2 \int_0^1 r^{2n}(1-r)^{p-1} dr$$

and

$$\iint_U |f'(z)|^2 (1-|z|^2)^p dm(z) = 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1}(1-r^2)^p dr ,$$

$$\int_0^1 \frac{r^{2n}}{2n}(1-r)^{p-1} dr = \frac{1}{p} \int_0^1 r^{2n-1}(1-r)^p dr .$$

The inequalities

$$\int_0^1 r^{2n-1}(1-r)^p dr \leq \int_0^1 r^{2n-1}(1-r^2)^p dr \leq 2^p \int_0^1 r^{2n-1}(1-r)^p dr$$

immediately show that the desired equivalence is valid.

Remarks.

1) The case $p = 1$ of Theorem 1 is the case $\lambda = 2$ of Theorem 1 in [10].
2) Furthermore, applying Corollary 1 in [11], which says that for

$$g(x) = \sum_{n=0}^{\infty} b_n x^n , \, b_n \geq 0 , \, x \in (0,1) ,$$

the inequalities

$$\int_0^1 (1-x)^{p-1} g(x)dx < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-(p+1)} \left(\sum_{k=1}^{n} b_k \right) < \infty$$

are equivalent, we just find that $f \in D_p$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if

$$\sum_{n=1}^{\infty} n^{-(p+1)} \left(\sum_{k=1}^{n} k|a_k|^2 \right) < \infty .$$

Note that the special cases $p = 1$ and $p > 1$ were given in [10, Corollary 2] and [6, Proposition 2.21], respectively.

Next, we denote by $a(f(U_r(w)))$ the area of the projection of $f(U_r(w))$ from the Riemann surface into the complex plane, i.e.

$$a(f(U_r(w))) = \iint_{f(U_r(w))} dm(z) .$$

It is trivial that $a(f(U_r(w))) \leq A(f(U_r(w)))$ and hence $f \in D_p$ implies

$$\int_0^1 a(f(U_r))(1-r)^{p-1} dr < \infty . \tag{2}$$

We will see below that the converse is not valid for any $p \in (0, \infty)$.

Example 2. For $p \in (0,1)$ this is a consequence of the fact that there exist functions $f \in A$, continuous on the closure of U such that

$$\iint_U |f'(z)| dm(z) = \infty , \tag{3}$$

as proved by Rudin in [12]. The Schwarz inequality and (3) imply

$$\begin{aligned} \infty &= \iint_U |f'(z)|(1-|z|^2)^{p/2}(1-|z|^2)^{-p/2} dm(z) \\ &\leq \iint_U |f'(z)|(1-|z|^2)^p dm(z) \cdot \iint_U (1-|z|^2)^{-p} dm(z) \end{aligned}$$

which proves that $f \notin D_p$. On the other hand, the continuity of f on the closure of U implies the boundedness of $a(f(U))$ and thus (2).

Example 3. Let $Z = \{m + in : (m,n) \in \mathbb{Z}^2\}$ and $f \in A$ be such that $f(U) = \mathbb{C} \setminus Z$. Since $f(U)$ is a Bloch domain i. e. $\mathbb{C} \setminus Z$ does not contain arbitrarily large euclidean disks, f is a Bloch function, that is (c. f. [8])

$$\sup_{z \in U} (1-|z|^2)|f'(z)| < \infty .$$

This implies that there exists a constant C such that

$$|f(z)| \leq C \log \frac{2}{1-|z|}, \quad z \in U.$$

From this we deduce

$$\int_0^1 a(f(U_r))dr \leq C^2\pi \int_0^1 \left(\log \frac{2}{1-r}\right)^2 dr < \infty.$$

On the other hand, we see that Z has zero capacity, so $f \notin H^2 = D_1$. So (2) in the case $p = 1$ is valid for this function f not in D_1.

Example 4. By modifying a bit the proof of Lemma 2 in [11] one may show that for any $\gamma \in (0, \infty)$ there exists a constant K_1 such that

$$\sum_{n=0}^{\infty} 2^{n\gamma} r^{2^n} \leq K_1 |\log r|^{-\gamma}, \quad r \in (0,1).$$

Hence for the functions $f(z) = \sum_{n=0}^{\infty} 2^{n\gamma} z^{2^n}$ we get

$$\sup_{|z|\leq r} |f(z)| \leq \sum_{n=0}^{\infty} 2^{n\gamma} r^{2^n} \leq K_1 |\log r|^{-\gamma}.$$

If we choose $\gamma < p/2$ we derive

$$\int_0^1 (1-r)^{p-1} a(f(U_r))dr \leq K_1^2 \pi \int_0^1 (1-r)^{p-1} |\log r|^{-2\gamma} dr < \infty.$$

Considering the criterion for f to be a member of D_p given in Remark 2 above, we see that there is a constant K_2 such that in our case

$$\sum_{n=1}^{\infty} n^{-(p+1)} \left(\sum_{k=1}^{n} k|a_k|^2\right) \geq K_2 \sum_{n=0}^{\infty} 2^{n(2\gamma+1-p)}.$$

This sum is divergent for $\gamma > (p-1)/2$. So choosing $\gamma \in ((p-1)/2, p/2)$, $f \notin D_p$, but (2) is valid.

Since we have seen in the proof of Theorem 1 that

$$p2^p \int_0^1 A(f(U_r))(1-r)^{p-1} dr \geq ||f||_{D_p}^2 \geq p \int_0^1 A(f(U_r))(1-r)^{p-1} dr,$$

using the identities $U_r(w) = \varphi_w^{-1}(U_R) = \varphi_w(U_r)$, we may formulate the following corollary to Theorem 1.

Corollary 5. *Let* $p \in (0,\infty)$ *and* $f \in A$.

a) $f \in Q_p$ *if and only if*

$$\sup_{w\in U} \int_0^1 A(f(U_r(w)))(1-r)^{p-1}dr < \infty ,$$

b) $f \in Q_{p,0}$ *if and only if*

$$\lim_{|w|\to 1} \int_0^1 A(f(U_r(w)))(1-r)^{p-1}dr = 0 .$$

As in the discussion after Theorem 1 we see that for $f \in Q_p$

$$\sup_{w\in U} \int_0^1 a(f(U_r(w)))(1-r)^{p-1}dr < \infty \tag{4}$$

and for $f \in Q_{p,0}$

$$\lim_{|w|\to 1} \int_0^1 a(f(U_r(w)))(1-r)^{p-1}dr = 0 . \tag{5}$$

So far as the converse in the case $p \in (0,1)$ is concerned, the function $f(z) = \exp\left(\frac{z+1}{z-1}\right) \in H^\infty \setminus Q_p$ cf. [7, Corollary 4.2] delivers a counterexample.

For the case $p = 1$ the universal covering map (see [9]) f from U onto the universal covering surface of $\mathbb{C} \setminus Z$ (see Example 2) belongs to $B \setminus BMOA = B \setminus Q_1$, and as in Example 2 we see that (4) holds.

For $p \in (0,1]$ we don't know whether (5) implies $f \in Q_{p,0}$.

For $p \in (1,\infty)$ (4) implies $f \in Q_p = B$ and (5) implies $f \in Q_{p,0} = B_0$.

This is easily seen remarking that for fixed $r \in (0,1)$ (4) implies

$$\sup_{w\in U} a(f(U_r(w))) < \infty$$

and (5) implies

$$\lim_{|w|\to 1} a(f(U_r(w))) = 0 .$$

This according to Theorem 1 and Theorem 2 in [5] implies $f \in Q_p$ resp. $f \in Q_{p,0}$ (compare [16], too).

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