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Holomorphic Non–Equivalence of Balls in Banach Spaces l_p and L_2 from the Geometrical Point of View

ABSTRACT. Let B_p $(p > 1)$ denotes the open unit ball in Banach space of all sequences of complex numbers with the usual l_p-norm. We prove that B_2 and B_p $(p \neq 2)$ are not holomorphically equivalent, and the same for B_2 and $B_2 \times B_2$.

1. Introduction. The Riemann Mapping Theorem states that two open, simply connected and bounded subsets of the complex plane $\mathbb{C}$ are holomorphically equivalent. In other complex Banach spaces the situation is more complicated. Methods applied in the study of holomorphic equivalence, or non-equivalence of domains usually depend on rather sophisticated tools (see e.g. [6]).

K. Goebel and S. Reich proved in [2] that the unit ball in complex Hilbert space can be seen as a "nice" metric space (so called ρ-uniformly convex space) and holomorphic feature and metric properties are strictly connected.

Using the idea of Goebel and Reich together with the classical theory of Schwarz-Pick systems of pseudometrics ([1], [5]) one can present an elementary proof of holomorphic non-equivalence of unit balls in even spaces. "Elementary" means here: based on concept of isometry between two metric spaces.

Key words and phrases. Holomorphic mapping, hyperbolic metric, biholomorphic equivalence .

Let $D \subset X$, $E \subset Y$ be nonempty, open and bounded subsets of complex normed linear spaces X, Y and let c_D, c_E be Carathéodory metrics in D and E, respectively, cf. [1], [5]. It is known that every holomorphic mapping $f: D \to E$ is nonexpansive in the sense that

$$c_E(f(x), f(y)) \leq c_D(x, y) \tag{1}$$

for any $x \in D$, $y \in E$. This implies

$$c_E(f(x), f(y)) = c_D(x, y) \tag{2}$$

for every biholomorphic mapping $f: D \to E$ and any points $x \in D$, $y \in E$. Consequently, if D and E are holomorphically equivalent then the corresponding metric spaces (D, c_D) and (E, c_E) are isometric.

Now let l^p $(1 < p < \infty)$ be the space of all sequences $x = (x_n)$ of complex numbers with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

and let B_p be the open unit ball in this space (in case $p = 2$ we will simply write B instead of B_2). Take $p \neq 2$. We show that B and B_p are not holomorphically equivalent, and the same is true for B and $B \times B$.

2. Auxiliary lemmas. Let ρ be the Carathéodory metric in B, cf. [2], [3]. Recall that the metric space (B, ρ) is unbounded and complete, and $\rho(0, x) = \tanh^{-1} \|x\|$ for every $x \in B$. Any two points $x, y \in B$ may be joined by the unique geodesic segment (isometric to the interval $[0, \rho(x, y)]$). Consequently, for any points $x, y \in B$ there exists the unique metric midpoint $u \in B$ such that

$$\rho(x, u) = \rho(y, u) = \rho(x, y)/2.$$

We will denote this point by $u = \frac{1}{2}(x \oplus y)$.

We need the following result from [2] (also see [3]), which, after a little reformulation, can be stated in the following form.

Lemma 1. *Let $a \in B$, $r > 0$, $0 \leq \varepsilon \leq 2$. Let $x, y \in B$ be such that $\rho(a, x) \leq r$, $\rho(a, y) \leq r$ and $\rho(x, y) \geq \varepsilon r$. Then*

$$\rho\left(a, \frac{1}{2}(x \oplus y)\right) \leq \tanh^{-1} \sqrt{\frac{\tanh^2 r - \tanh^2(\varepsilon r/2)}{1 - \tanh^2(\varepsilon r/2)}}. \tag{3}$$

Let us denote by ω the Poincaré-Bergman metric in the unit disc $\Delta \subset \mathbb{C}$ (which is equal to the Carathéodory metric in Δ, see [1]). Now we can prove the following two lemmas.

Lemma 2. *Take $1<p<2$ and let d denote the Carathéodory metric in B_p. Let $t>0$ be such that*

$$x=(t,t,0,0,...)\in B_p \quad \text{and} \quad y=(t,-t,0,0,...)\in B_p.$$

Then

$$d(x,y)\leq 2\tanh^{-1}\left(\frac{t}{(1-t^p)^{1/p}}\right). \tag{4}$$

Proof. We define holomorphic function $\varphi:\Delta\to B_p$ by the formula

$$\varphi(z)=\left(t,z(1-t^p)^{1/p},0,0,...\right).$$

If $x'=t/(1-t^p)^{1/p}$, $y'=(-t)/(1-t^p)^{1/p}$ then we have $\varphi(x')=x$, $\varphi(y')=y$ and

$$\omega(x',y')=2\tanh^{-1}\left(\frac{t}{(1-t^p)^{1/p}}\right).$$

Because of (1) the lemma is proved.

Lemma 3. *Take $p>2$ and let d denote the Carathéodory metric in B_p. Let $t>0$ be such a number that*

$$x=(t,0,0,0,...)\in B_p \quad \text{and} \quad y=(0,t,0,0,...)\in B_p.$$

Then we have

$$d(x,y)\leq 2\tanh^{-1}\left(\frac{t}{2r_0}\right), \tag{5}$$

where r_0 is the unique solution of the equation

$$\left|\frac{t}{2}+r\right|^p+\left|\frac{t}{2}-r\right|^p=1,\quad r\geq 0. \tag{6}$$

Proof. Let us define holomorphic function $\psi:\Delta\to B_p$ by the formula

$$\psi(z)=\left(\frac{t}{2}+r_0z,\frac{t}{2}-r_0z,0,0,...\right).$$

Let us put $x'=t/(2r_0)$, $y'=(-t)/(2r_0)$. We have now $\psi(x')=x$, $\psi(y')=y$ and

$$\omega(x',y')=2\tanh^{-1}\left(\frac{t}{2r_0}\right).$$

Inequality (5) follows from (1).

3. Main results. We now prove the following theorem.

Theorem 1. *If $p \neq 2$ then B and B_p are not holomorphically equivalent.*

Proof. Assume the contrary. Then the metric spaces (B,ρ) and (B_p,d) are isometric. We consider two cases.

Case 1, $1 < p < 2$. Choose $t > 0$ such that $a, x, y \in B_p$, where

$$a = (t,t,0,\ldots), \quad x = (-t,t,0,\ldots), \quad y = (t,-t,0,\ldots).$$

Obviously $d(a,x) = d(a,y)$ and $0 = (0,0,\ldots) = \frac{1}{2}(x \oplus y)$. By the definition of the metric d ([1], [5]) we obtain:

$$d(x,y) = 2d(0,x) = 2\tanh^{-1}\|x\| = 2\tanh^{-1}\left(2^{1/p}t\right).$$

Putting $r = d(a,x)$, $\varepsilon r = d(x,y)$ and applying (3) we obtain

$$d(0,a) \leq \tanh^{-1}\sqrt{\frac{\tanh^2 r - \tanh^2(\varepsilon r/2)}{1-\tanh^2(\varepsilon r/2)}}$$

and further, after simple calculations:

$$(\tanh d(0,a))\left(2-\tanh^2 d(0,a)\right)^{1/2} \leq \tanh r.$$

Lemma 2 gives

$$2^{1/p}t\left(2-2^{2/p}t^2\right)^{1/2} \leq \tanh\left[2\tanh^{-1}\left(\frac{t}{(1-t^p)^{1/p}}\right)\right]$$

and

$$2^{1/p}\left(2-2^{2/p}t^2\right)^{1/2} \leq \frac{2(1-t^p)^{1/p}}{(1-t^p)^{2/p}+t^2}.$$

If t tends to zero, we obtain $2^{2/p} \leq 2$ or $p \geq 2$. This contradicts our assumption $(1 < p < 2)$.

Case 2, $p > 2$. Choose $t > 0$ such that $a, x, y \in B_p$, where

$$a = (0,t,0,\ldots), \quad x = (t,0,0,\ldots), \quad y = (-t,0,0,\ldots).$$

It is easy to observe that $d(a,x) = d(a,y)$ and $0 = (0,0,...) = \frac{1}{2}(x \oplus y)$. Moreover,

$$d(x,y) = 2d(0,x) = 2\tanh^{-1} t .$$

Let us put $r = d(a,x)$, $\varepsilon r = d(x,y)$. By means of (3) we conclude that

$$d(0,a) \leq \tanh^{-1} \sqrt{\frac{\tanh^2 r - \tanh^2(\varepsilon r/2)}{1 - \tanh^2(\varepsilon r/2)}}$$

and equivalently

$$2\tanh^2 d(0,a) - \tanh^4 d(0,a) \leq \tanh^2 r .$$

Let us denote by r_0 the unique positive solution of (6). Lemma 3 together with the above inequality yields

$$2t^2 - t^4 \leq \left(\frac{4tr_0}{4r_0^2 + t^2}\right)^2$$

or equivalently

$$2 - t^2 \leq \frac{16r_0^2}{(4r_0 + t^2)^2} .$$

If $t \to 0$, then $r_0 \to 2^{(-1)/p}$, so we obtain $2 \leq 2^{2/p}$, or $p \leq 2$. This contradicts our assumption ($p > 2$).

Theorem 2. *B and $B \times B$ are not holomorphically equivalent.*

Proof. The Carathéodory metric d in $B \times B$ is defined by

$$d((x,y),(a,b)) = \max\{\rho(x,a), \rho(y,b)\} ,$$

where ρ is the Carathéodory metric in , B cf. [1], [5]. To complete the proof it is enough to notice that in $(B \times B, d)$ metric segments are not unique. Thus metric spaces $(B \times B, d)$ and (B, ρ) are not isometric.

Remark. Theorem 2 was proved by S. Greenfield and N. Wallach [4] by using standard methods of complex analysis.

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