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Smooth Approximation of Solutions of Cauchy-Riemann Systems

Abstract. It is shown here that Cauchy-Riemann systems and their solutions can be approximated by sequences of smooth Cauchy-Riemann systems and corresponding solutions such that these sequences satisfy certain additional conditions.

1. Introduction. Solutions of (generalized Cauchy-Riemann systems in general possess only weak regularity properties. Thus the situation often requires to consider an appropriate sequence of such systems and a corresponding sequence of solutions tending in an appropriate way to the original system and a prescribed solution of it, respectively. Of course, the problem also embodies the question of what is meant at the time by appropriate.

Here, Cauchy-Riemann system denotes a linear uniformly elliptic system of the form

$$f_{\bar z} = \nu(z) f_z + \mu(z) \overline{f_z}, \tag{1}$$

where

$$\nu\,,\,\mu \in L_\infty\,, \quad \|\,|\nu| + |\mu|\,\|_{L_\infty} := k < 1\,. \tag{2}$$

($L_s, L_{s,loc}$ always means $L_s(\mathbb{C})\,, L_{s,loc}(\mathbb{C})$, respectivly). We consider here approximation in bounded subdomains of the complex plane for three kinds

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of solutions of (1), namely (global) fundamental solutions (i.e. presence of just two logarithmic singularities, the one of them at ∞), generalized powers (i.e. exactly one singularity, of an entire order, possibly at ∞), and (ordinary, of course weak) solutions of (1) in a disk. Additinal to (2) we assume that

$$\nu(z) = \mu(z) = 0 \quad \text{for} \quad |z| > R^* \tag{3}$$

with a fixed positive R^*. At least in the third case this is no loss of generality. In both the other cases, (3) can be weakened essentially which, however, requires a certain amount of additional notations. That is why we here dispend with it.

As is well-known, every schlicht solution of (1) is a K-quasiconformal mapping with

$$K = \frac{1+k}{1-k}. \tag{4}$$

For brevity we call a system (1) also a (ν, μ)–system and a solution of (1) a (ν, μ)-solution. Concerning the approximation of a (ν, μ)–system by a sequence of (ν_n, μ_n)–systems we use the following conditions

(a) $\nu_n(z) \to \nu(z)$, $\mu_n(z) \to \mu(z)$ a.e. in $\mathbb{C}$, as $n \to \infty$,
(b) $|\nu_n(z)| + |\mu_n(z)| \le k \ \forall z \in \mathbb{C}$,
(c) $\mathrm{supp}(|\nu_n| + |\mu_n|) \subset \mathrm{supp}(|\nu| + |\mu|) + \{|z| < \varepsilon\}$ with any fixed $\varepsilon > 0$,
(d) $\nu_n, \mu_n \in C^\infty(\mathbb{C})$ for every $n \in \mathrm{N} = \{1, 2, ...\}$.

(As usual for two sets A, B, $A + B = \{x + y | x \in A, y \in B\}$).

Such sequences can easily be generated by means of convolution with mollifiers.

Of course, in the following considerations the complex Hilberttransformation T, symbolically

$$Tf(z) = -\frac{1}{\pi} \int \frac{f(\zeta)}{(\zeta - z)^2} \, d\sigma_\zeta \tag{5}$$

and the Cauchy transformation

$$Pf(z) = -\frac{1}{\pi} \int \frac{f(\zeta)}{\zeta - z} \, d\sigma_\zeta \tag{6}$$

play a crucial role.

The conditions (a)–(d) have to be completed if, e. g., pointwise convergence of the derivatives for a sequence of (ν_n, μ_n)–solutions at prescribed points is required. For such purposes we shall use here the Bojarski condition, cf. [2],

$$\frac{g(z) - g(z_0)}{z - z_0} \in L_p \quad \text{with a} \quad p > 2, \tag{7}$$

which is of course not the weakest one possible (the weakest on possible might be the condition of the Teichmüller–Wittich–Belinski distortion theorem, but this is still to be proved).

2. Smooth approximation of fundamental solutions. In [4] existence and uniqueness of a special global fundamental solution has been shown:

Proposition 1. *Let* ν, μ *satisfy* (2), (3). *For every fixed* $z_0 \in \mathbb{C}$ *there exists a solution* $H(z, z_0, \nu, \mu)$ *of* (1) *in* $\mathbb{C} \setminus \{z_0\}$, *unique up to the branch of the logarithm, which admits a representation*

(i) $H(z, z_0, \nu, \mu) = \log(z - z_0) + r_\infty(z, z_0)$, *where* $r_\infty(z, z_0)$ *is single-valued and continuous in* $\mathbb{C} \setminus \{z_0\}$,

(ii) $r_\infty(z, z_0) \in L_{s,loc}$ *for every* $s \in [1, \infty)$,

(iii) $\lim_{z\to\infty} r_\infty(z, z_0) = 0$.

We call this $H(z, z_0, \nu, \mu)$ the fundamental solution of (1). By means of the Bers–Nirenberg representation theorem we have almost obviously

Corollary 1. *There exists a unique* K*-quasiconformal mapping* $\chi(z) = \chi(z; z_0)$ *of* $\mathbb{C}$ *onto itself such that*

(i) $\chi(z)$ *is conformal for* $|z| > R^*$,

(ii) $\chi(z) = z + O(1)$ *as* $z \to \infty$, $\chi(z_0) = 0$,

(iii) $H(z, z_0, \nu, \mu) = \log \chi(z)$.

Proof. Let χ be a schlicht solution of

$$\chi_{\bar{z}} = \left(\nu + \mu \frac{\overline{H_z}}{H_z}\right) \chi_z \tag{8}$$

in $\mathbb{C}$. Because of (3) it satisfies (i) and can be normalized to satisfy also (ii). Then

$$H(z, z_0, \nu, \mu) - \log \chi(z) := g(z)$$

is a (single-valued) solution of (8) in $\mathbb{C} \setminus \{z_0\}$ which is bounded as $z \to \infty$. Moreover, z_0 must be a removable singularity, cf. the conclusion in [4, p. 86]. Thus, $g(z) \equiv \text{const}$ by Liouville's theorem for solutions of (1), which means (iii) with the branches suitably chosen.

Theorem 1. *Let* ν, μ *satisfy* (2) *and* (3) *and let* $z_0 \in \mathbb{C}$ *be fixed. There exists sequences of* ν_n, μ_n *satisfying* (a)–(d) *above such that for the corresponding fundamental solutions*

$$H_n(z) := H(z, z_0, \nu_n, \mu_n), \quad H(z) := H(z, z_0, \nu, \mu)$$

(under suitable choice of the branches of the logarithm) holds
(i) $H_n(z) \to H(z)$ *locally uniformly in* $\mathbb{C} \setminus \{z_0\}$,
(ii) $H_{nz}(z) \to H_z(z)$ *(strongly) in* $L_{s,loc}$ for each $s \in [1,2)$, as $n \to \infty$.
Moreover,
(iii) *each* $H_n \in C^\infty(D)$ *for each simply-connected domain* D *with* $D \subset \mathbb{C} \setminus \{z_0\}$.

Proof. We may put $z_0 = 0$. By [4, p. 86], H_n, H admit the representation

$$(9) \qquad H_n(z) = \log z - PF_n(\cdot)(z), \quad H(z) = \log z - PF(\cdot)(z),$$

where $F(\cdot)$ is the unique solution of

$$(10) \qquad F(t) = -\left(\frac{\nu(t)}{t} + \frac{\mu(t)}{\bar{t}}\right) + \nu(t)TF(\cdot)(t) + \mu(t)\overline{TF(\cdot)(t)}$$

in L_q for every $q \in (2-\varepsilon_0, 2)$ with a positive ε_0 depending only on k, and F_n is the solution of (10) with ν, μ replaced by ν_n, μ_n, respectively. Hence $F_n \to F$ in such an L_q, because of (a)–(c) and (3). This means that

$$(11) \qquad H_{nz}(z) = \frac{1}{z} - TF_n(\cdot)(z) \to H_z(z) = \frac{1}{z} - TF(\cdot)(z)$$

in $L_{q,loc}$ with any $q \in [1,2)$, which proves (ii). Assertion (i) is clear because of Corollary 1 and well-known compactness criteria for mappings χ_n. The remaining assertion (iii) holds because of the well-known hypoellipticity of (1) in case of $\nu, \mu \in C^\infty$. Note that (i), (ii) hold even without (d).

3. Smooth approximation of generalized powers. Generalized powers which are to be normalized by asymptotic expansions, require certain additional conditions on ν, μ. We restrict ourselves here to the Bojarski condition (7). Without loss of generality we now put $z_0 = 0$ until further notice.

We shall say that a sequence of functions g_n satisfies a uniform Bojarski condition at 0, if there exist constants C, p where C is positive and $p > 2$ such that

$$(12) \qquad \left\| \frac{g(z) - g(0)}{z} \right\|_{L_p} < C \quad \forall n.$$

Lemma 1. *Let* ν, μ *satisfy* (2) *and* (7) *(with* $z_0 = 0$ *). There exist sequences* ν_n, μ_n *satisfying the conditions* (a)–(d) *as well as uniform Bojarski conditions at* 0.

Proof. Choose a monotone null sequence of positive numbers r_n and put

$$\kappa_n(z) = \begin{cases} \nu(0) & \text{for} \quad |z| \leq r_n \\ \nu(z) & \text{for} \quad |z| > r_n \end{cases}$$

(note that $\nu(0)$ is well-defined by (7)). Next choose a mollifier $m(z)$ and put, e. g.,

$$m_{n,l}(z) = \frac{4l^2}{r_n^2}\, m\left(\frac{2lz}{r_n}\right), \quad n, l \in \mathrm{N}.$$

Then let

$$\nu_{n,l} = \kappa_n * m_{n,l}(z),$$

where $*$ means convolution. Then

$$\nu_{n,l}(0) = \nu(0) \quad \forall\, n, l \in \mathrm{N}.$$

Now let first $l_n \in \mathrm{N}$ satisfy

$$\frac{r_n}{2l_n} < r_n - r_{n+1}.$$

Then we have

$$\left\| \frac{\nu_{n,l_n}(z) - \nu(0)}{z} \right\|_{L_p} = \left\| \frac{\nu_{n,l_n}(z) - \nu(0)}{z} \right\|_{L_p(\{|z| > r_{n+1}\})}$$

$$\leq \left\| \frac{\nu_{n,l_n}(z) - \kappa_n(z)}{z} \right\|_{L_p(\{r_{n+1} < |z| < r_n\})} + \left\| \frac{\nu_{n,l_n}(z) - \kappa_n(z)}{z} \right\|_{L_p(\{|z| . r_n\})}$$

$$+ \left\| \frac{\nu_{n,l_n}(z) - \nu(0)}{z} \right\|_{L_p}.$$

For each fixed n, l_n can be chosen in such a way that, additionally, each of the first two terms on the right–hand side of the last inequality is less than, e.g., 1. Then

$$\nu_n(z) := \nu_{n,l_n}(z)$$

satisfies

$$\left\| \frac{\nu_n(z) - \nu_n(0)}{z} \right\|_{L_p} \leq \left\| \frac{\nu(z) - \nu(0)}{z} \right\|_{L_p} + 2 \quad \forall\, n.$$

The same procedure can be applied to $\mu(z)$. The remaining assertions of the lemma are obvious.

Remark. The same procedure can be applied also in case of the condition of the Teichmüller–Wittich–Belinski distortion theorem.

Theorem 2. *Let ν, μ satisfy* (2), (3) *and* (7), *let j be a nonzero integer and c be any nonzero constant. Then there exist sequences ν_n, μ_n such that for the generalized powers*

$$F_n(z) := [cz^j]_{(\nu_n,\mu_n)}, \quad F(z) := [cz^j]_{(\nu,\mu)}$$

holds

(i) $F_n(z) \to F(z)$ *uniformly in compact subsets of* $\mathbb{C}\setminus\{0\}$ *(of* $\mathbb{C}$ *if* $j \geq 1$*)*,

(ii) $F_{nz}(z)z^{-j+1} \to F_z(z)z^{-j}$ *weakly in* $L_{p,loc}$ *for each* $p \in \left[1, \frac{2K}{K-1}\right)$,

and

(iii) *the functions* $F_n(z)z^{-j}$ *are uniformly bounded in* $\mathbb{C}$ *as* $n \to \infty$.

Proof. We may assume that $\nu(0) = \mu(0) = 0$, which can be achived by two affine mappings (cf. e.g. [4, p. 51]) not affecting the assertions of the theorem and preserving condition (7) (however the new ν, μ, which are in any case constant in a neighborhood of ∞, do not necessarily possess compact supports). Let ν_n, μ_n be corresponding sequences according to Lemma 1. For each corresponding $F_n(z)$ we may assume a representation

$$F_n(z) = c(\chi_n(z))^j$$

where χ_n is a schlicht solution of

$$\chi_{n\bar{z}} = \left(\nu_n(z) + \mu_n(z)\frac{\overline{F_{nz}}}{F_{nz}}\right)\chi_{nz}$$

in $\mathbb{C}$, normalized by

$$\chi_n(z) = z + O(|z|^{1+\alpha}) \tag{13}$$

where, by [4, Theorem II.5.2] and [3, p. 231],

$$|O(|z|^{1+\alpha})| \leq M|z|^{1+\alpha}, \; |z| < R' \tag{14}$$

for any fixed positive R' with the same positive constants M, α for every n (a more detailed consideration shows that this holds here even with $R' = \infty$). In particular, the χ_n must be locally uniformly bounded. Hence, there is a subsequence of χ_n (which we then take as the whole sequence) converging locally uniformly to a quasiconformal mapping $\chi(z)$ of $\mathbb{C}$ with an asymptotic expansion (13) at $z = 0$. Then, of course, the corresponding $F_n(z)$ tend to an $F^*(z) := c(\chi(z))^j$. Since F^* is the locally uniform limit of (ν_n, μ_n)–solutions F_n in $\mathbb{C}\setminus\{0\}$ (of course, even in $\mathbb{C}$ if $j \geq 1$), then

F^* is a (ν,μ)–solution there, hence $F^*(z) = [cz^j]_{(\nu,\mu)}$, which proves (i) of Theorem 2.

Further,

$$F_{nz}(z) = cj\chi_{nz}\,\frac{F_n(z)}{\chi_n(z)}.$$

By (13), (14),

$$\text{(15)}\qquad \frac{F_n(z)}{\chi_n(z)}\,z^{-j+1} \to \frac{F(z)}{\chi(z)}\,z^{-j+1}$$

locally uniformly in $\mathbb{C}$. By [1], χ_{nz} belongs to $L_{p,loc}$ for every $p \in \left[1, \frac{2K}{K-1}\right)$, and $\|\chi_{nz}\|_{L_p(\{|z|<R\})}$ is uniformly bounded by a constant depending only on R, p, K (because of the local uniform boundedness of the χ_n). Since the χ_n converge locally uniformly to χ, the χ_{nz} then tend weakly to χ_z in $L_p(\{|z|<R\})$ for each such p and each finite R. This, together with (15), proves (ii). Concerning the remaining part (iii) of Theorem 2 we only have to remove "locally" in the statement with (15). This can be done simply by returning to the original ν,μ and corresponding $\nu_n\mu_n$ and observing the analyticity of the expressions in (iii) at ∞.

4. Smooth approximation of (ν,μ)–solutions in a disk. We want to prove

Theorem 3. *Let ν,μ satisfy (2), (3), and let f be a (ν,μ)–solution in $\{|z|<R_0\}$. Then, for any fixed $R \in (0,R_0)$ there exist sequences ν_n,μ_n satisfying the conditions (a)–(d) above and a sequence of (ν_n,μ_n)–solutions f_n in $\{|z|<R\}$ such that*

$$f_n(z) \to f(z) \quad \text{uniformly in} \quad \{|z|<R\}.$$

If, additionally, (7) is satisfied at $z_0 = 0$ then, additionally,

$$f_{nz}(0) \to f_z(0).$$

Proof. We fix an $R' \in (R,R_0)$, put

$$\nu_0(z) = \begin{cases} \nu(z) & \text{for } |z|<R' \\ 0 & \text{for } |z|\ge R', \end{cases}$$

and define $\mu_0(z)$ in the same way. Obviously there are two sequences ν_n,μ_n satisfying (a)–(d) with respect to ν,μ and two sequences ν_{0n},μ_{0n} satisfying (a)–(d) with respect to ν_0,μ_0 such that, moreover,

$$\nu_{0n}(z) = \nu_n(z), \quad \mu_{0n}(z) = \mu_n(z) \quad \forall n \in \mathbb{N}$$

in $B := \{|z| < R\}$. If (7) holds for ν, μ, then we may additionally suppose that these sequences each satisfy a uniform Bojarski condition at $z_0 = 0$. Then

$$g(z) := f(z) + P(\nu_0 f_z + \mu_0 \overline{f_z})(z) =: Vf(z)$$

is analytic in $B' := \{|z| < R'\}$ and continuously in $B_0 = \{|z| < R_0\}$ and $\in W^1_{p,loc}(B_0)$ for certain $p > 2$. Futher let

$$V_n f(z) := f(z) + P(\nu_{0n} f_z + \mu_{0n} \overline{f_z})(z).$$

Each transformation V_n has an inverse U_n in spaces which, in any case, contain the images of f under V_n, V, in particular the above mentioned g. Moreover, cf. [4, chap. IV.1]

$$f_n(z) := U_n g(z) \in C(B_0) \cap W^1_{p,loc}(B_0),\ p \in [2, 2 + \varepsilon_1),$$

whith a positive ε_1 depending only on k, and f_n is a $(\nu_n, \mu_n)(= (\nu_{0n}, \mu_{0n}))$-solution in B. For such $U_n g$ we have

$$U_n g(z) = g(z) + \frac{1}{\pi} \int_{B_0} \left[\Phi_{1n}(t,z) g_t(t) + \Phi_{2n}(t,z) \overline{g_t(t)} \right] d\sigma_t$$

where

$$2\Phi_{1n}(t,z) = F_n(t,z) + G_n(t,z),\ 2\Phi_{2n}(t,z) = \overline{F_n(t,z) - G_n(t,z)},$$

$$F_n(t,z) = -\left(\frac{\nu_{0n}(t)}{t-z} + \overline{\frac{\mu_{0n}(t)}{t-z}} \right) + \nu_{0n}(t) T F_n(\cdot, z)(t)$$
$$+ \overline{\mu_{0n}(t) T F_n(\cdot, z)(t)},$$

and where $G_n(t,z)$ is defined by an equation of the same shape, cf. [4, p.83]. Since

$$\frac{\nu_{0n}(t)}{t-z} \to \frac{\nu_0(t)}{t-z}, \quad \frac{\mu_{0n}(t)}{t-z} \to \frac{\mu_0(t)}{t-z}$$

as $n \to \infty$ in L_q for the same q as with (10) above, and that uniformly for all z from any fixed bounded subset of $\mathbb{C}$, we obtain convergence of $\Phi_{ln}(\cdot, z) \to \Phi_l(\cdot, z)$, $l = 1, 2$, in each such L_q, uniformly for all z from an arbitrarily fixed bounded subset of $\mathbb{C}$. Here Φ_l corresponds to ν_0, μ_0 in the same way as Φ_{ln} to ν_{0n}, μ_{0n}, and

$$Ug(z) := g(z) + \frac{1}{\pi} \int_{B_0} \left[\Phi_1(t,z) g_t(t) + \Phi_2(t,z) \overline{g_t(t)} \right] d\sigma_t = V^{-1} g(z) = f(z).$$

Because of (c) above with ε chosen less than $(R_0 - R')/2$ we have

$$\operatorname{supp} \Phi_{l_n}(\cdot, z) \subset \{|z| < (R_0 + R')/2\}$$

Hence,

$$U_n g(z) - U g(z) \to 0$$

uniformly even in B_0 (even in every compact subset of $\mathbb{C}$ if we consider the difference to be defined everywhere in $\mathbb{C}$). This proves the first part of Theorem 3.

Let now, additionally, (7) be satisfied for ν, μ at $z_0 = 0$. Then each $f_n(z)$ admits an expansion

$$f_n(z) = f_n(0) + f_{nz}(0) \cdot z + f_{n\overline{z}}(0) \cdot \overline{z} + O(|z|^{1+\alpha}) \tag{16}$$

at 0. Because of the uniform Bojarski conditions at 0 for ν_n, μ_n and the uniform boundedness of the f_n in B we have

$$\left|O(|z|^{1+\alpha})\right| \leq M|z|^{1+\alpha} \tag{17}$$

with the same positive constants M, α for each f_n , cf. [4, Theorem II.5.2] (with $D = \{0\}$ there). (16), (17) imply, of course, the remaining assertion of Theorem 3.

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