ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA

VOL. L, 8 SECTIO A 1996

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An Example of a Non–Sasakian Five-Dimensional Contact Manifold

ABSTRACT. This paper presents an example of a five-dimensional contact manifold which is not Sasakian.

1. Introduction. Let M be a $(2n+1)$-dimensional differentiable manifold and let $\mathfrak{X}(M)$ denote the set of all differentiable vector fields on M. We say that M admits an *almost contact structure* if there exists a tensor field ϕ of type $(1,1)$ on M, a vector field ξ and a 1-form ω such that

(i) $\phi^2 = -id + \omega \otimes \xi$ i.e.
$\phi(\phi(X)) = -X + \omega(X)$, for $X \in \mathfrak{X}(M)$

(ii) $\omega(\xi) = 1$.

One can prove that for a given almost contact structure (ϕ, ξ, ω) on M there always exists a Riemannian metric g on M such that

(iii) $g(\phi(X), \phi(Y)) = g(X,Y) - \omega(X)\omega(Y)$, for $X, Y \in \mathfrak{X}(M)$.

We call the quadruple (ϕ, ξ, ω, g) a *metric almost contact structure* on M.

1991 *Mathematics Subject Classification.* 53C25.
Key words and phrases. Almost contact structure, contact structure, Sasakian manifold.

Definition 1.1. A manifold M of dimension $2n+1$ is said to be a contact manifold if there exists a 1-form ω on M such that

$$\omega \wedge (d\omega)^n \neq 0 .$$

The above form ω is said to be a contact form on M .

It is well known that a contact manifold (M,ω) admits the existence of a metric almost contact structure (ϕ,ξ,ω,g) such that

$$d\omega(X,Y) = g(X,\phi(Y)) \quad \text{for} \quad X,Y \in \mathfrak{X}(M).$$

Moreover, we say that the almost contact structure (ϕ,ξ,ω) on M is *normal* if

$$\{\phi,\phi\}(X,Y) + 2d\omega(X,Y)\xi = 0 \quad \text{for} \quad X,Y \in \mathfrak{X}(M),$$

where $\{\phi,\phi\}$ denotes the Nijenhuis tensor for ϕ.

Definition 1.2. A manifold M is called a Sasakian manifold if there exists a normal contact structure on M .

A necessary condition for the existence of a Sasakian structure on a manifold M is given in the following

Theorem 1.1. (cf. [BG]) *If a manifold M is Sasakian then its first Betti number $b_1(M)$ is even.*

Examples of compact, normal, almost contact manifolds which are not Sasakian manifolds are well known. Here we construct a contact manifold being five-dimensional and not Sasakian. The basic idea of our construction is a modification of that given in the Ph.D. thesis [H] of the first author.

Remark 1.1. It is easy to find such an example in three dimensional cases; one of them is the three-dimensional torus T^3. However, the three-dimensional contact manifolds are much simpler and differ qualitatively from the higher dimensional ones.

2. The example. Let us consider the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Let us note that $A_1, A_2 \in \mathrm{SL}(3,\mathbb{Z})$, $A_1A_2 = A_2A_1$ and that we have

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 0 \\ ta & 1 & 0 \\ tb + \frac{1}{2}t(t-1)ac & tc & 1 \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

Thus

$$A_1^t = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t(t-1) & t & 1 \end{pmatrix}, \qquad A_2^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

Each of the matrices A_1 and A_2 can be considered as a diffeomorphism of the three-dimensional torus T^3 onto itself. In the product $\mathbb{R}^2 \times T^3$ we introduce an equivalence relation $\sim$ by

$$(t_1, t_2, x) \sim (t_1 + 1, t_2, A_1x)$$
$$(t_1, t_2, x) \sim (t_1, t_2 + 1, A_2x),$$

where $(t_1, t_2) \in \mathbb{R}^2$ and $x \in T^3$. In the standard manner the quotient $\mathbb{R}^2 \times T^3/\sim$ is furnished with the structure of a real five-dimensional orientable manifold. We also obtain a fibre bundle with a typical fibre T^3 over the base T^2. This manifold is denoted by $T^5_{A_1,A_2}$ and called a toral bundle of type $(2,3)$, cf. [HM].

Multiplication in $\mathbb{R}^5$ is defined as follows

$$\begin{aligned}(t_1, t_2, x_1, x_2, x_3) * (t_1', t_2', x_1', x_2', x_3') &= \Big(t_1 + t_1', t_2 + t_2', x_2 + x_2', x_2 + x_2' + t_1x_1', x_3 + x_3' \\ &\quad + \left(t_2 + \frac{1}{2}t_1^2 - \frac{1}{2}t_1\right)x_1' + t_1x_2'\Big).\end{aligned}$$

The pair $(\mathbb{R}^5, *)$ forms a Lie group denoted by $\mathbb{G}^{2,3}$. For a uniform discrete subgroup $\Gamma = \{(t_1, t_2, x_1, x_2, x_3) : t_1, t_2, x_1, x_2, x_3 \in \mathbb{Z}\}$ one can see that the compact orientable manifold $T^5_{A_1,A_2}$ and the homogeneous space $\Gamma \backslash \mathbb{G}^{2,3}$ are diffeomorphic, cf. [HM], [C].

It is easy to observe that the forms

$$\widetilde{\alpha}^1 = dt_1, \widetilde{\alpha}^2 = dt_2, \widetilde{\alpha}^3 = dx_1, \widetilde{\alpha}^4 = -t_1dx_1 + dx_2,$$

$$\widetilde{\alpha}^5 = \left(\frac{1}{2}t_1^2 + \frac{1}{2}t_1 - t_2\right)dx_1 - t_1dx_2 + dx_3$$

create a basis for left-invariant 1-forms on $\mathbb{G}^{2,3}$ and give at the same time five globally defined, linearly independent 1-forms $\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5$ on $T^5_{A_1,A_2}$.

If we put $\omega = \alpha^4 + \alpha^5$ then we have $\omega \wedge (d\omega)^2 \neq 0$ which means that ω determines a contact structure on the manifold $T^5_{A_1,A_2}$. We are going to show that there are no Sasakian structures on $T^5_{A_1,A_2}$. For this purpose we shall determine the first homology group with the coefficient in $\mathbb{Z}$ of our manifold $T^5_{A_1,A_2}$. Since the fundamental group $\pi(T^5_{A_1,A_2})$ of the toral bundle $T^5_{A_1,A_2} = \Gamma \setminus \mathbb{G}^{2,3}$ is isomorphic with the group Γ, cf. [HM], [H], [C] then by the Hurewicz theorem, cf. [BT], we get

$$H_1(T^5_{A_1,A_2}, \mathbb{Z}) = \Gamma/[\Gamma,\Gamma],$$

where $[\Gamma,\Gamma]$ denotes the commutator subgroup of Γ. Note that the group Γ has five generators a, b, c, d, e acting on $\mathbb{R}^5$ as follows:

$$\begin{aligned}
a &: (t_1, t_2, x_1, x_2, x_3) \mapsto (t_1 + 1, t_2, x_1, x_2 + x_1, x_3 + x_2) \\
b &: (t_1, t_2, x_1, x_2, x_3) \mapsto (t_1, t_2 + 1, x_1, x_2, x_3 + x_1) \\
c &: (t_1, t_2, x_1, x_2, x_3) \mapsto (t_1, t_2, x_1 + 1, x_2, x_3) \\
d &: (t_1, t_2, x_1, x_2, x_3) \mapsto (t_1, t_2, x_1, x_2 + 1, x_3) \\
e &: (t_1, t_2, x_1, x_2, x_3) \mapsto (t_1, t_2, x_1, x_2, x_3 + 1).
\end{aligned}$$

Therefore we have the following relations

$$ad = eda, \qquad ac = dca, \qquad bc = ecb.$$

Moreover, the remaining pairs commute. Examining the commutator of the group Γ we see that $\Gamma/[\Gamma,\Gamma]$ is isomorphic with $\mathbb{Z}^3$. This means that the first Betti number $b_1(T^5_{A_1,A_2})$ is equal to 3. Summarizing the above considerations we have

Theorem 2.1. *The toral bundle $T^5_{A_1,A_2}$ is a five-dimensional, compact, contact manifold which is not a Sasakian manifold.*

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received December 9, 1996