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Asymptotic Normal Structure, Semi-Opial Property and Fixed Points

Abstract. We apply some of Banach space properties depending on the metric behaviour of weakly convergent sequences to obtain the existence of fixed points for nonexpansive mappings.

Introduction. In this paper we deal with such properties of Banach spaces as normal structure, weak normal structure, asymptotic normal structure, weak asymptotic normal structure and different kinds of Opial's conditions. Our aim is to concentrate on applications of the above mentioned properties to the fixed point theory of nonexpansive mappings.

1. Notations, definitions and basic facts. Throughout the paper, $(X, \|\cdot\|)$ denotes a Banach space. The convex closure of a subset C of X is denoted by $\operatorname{conv} C$. Also $B(x,r)$ always denotes the closed ball centered at x with radius $r > 0$.

For $x \in X$ and a bounded sequence $\{x_n\}$ the *asymptotic radius of* $\{x_n\}$ *at* x [3] is the number

$$r(x, \{x_n\}) = \overline{\lim_{n\to\infty}} \|x - x_n\|.$$

Now for a nonempty closed subset C of X the *asymptotic radius of* $\{x_n\}$ *in* C [3] is the number

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

The *asymptotic center of* $\{x_n\}$ *in* C [3] is the set

$$Ac(C,\{x_n\}) = \{x \in C : r(x,\{x_n\}) = r(C,\{x_n\})\}.$$

When C is fixed we will simply write $Ac(\{x_n\})$.

The space X is said to have *normal structure* (*weak normal structure*) [2] if for each bounded (weakly compact) and convex subset C of X consisting of more than one point there is a point $x \in C$ such that

$$\sup\{\|y-x\| : \ y \in C\} < \operatorname{diam} C.$$

We denote this property briefly by NS (w-NS).

We will say that X has *asymptotic normal structure* (*weak asymptotic normal structure*) [1], ANS (w-ANS) for short, if for each bounded (weakly compact) and convex subset C of X consisting of more than one point and each sequence $\{x_n\}$ in C satisfying $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ there is a point $x \in C$ such that $\underline{\lim}_{n\to\infty}\|x_n - x\| < \operatorname{diam} C$.

It follows, directly from the definition, that if X has ANS, then for any $\{x_n\}$ such that $\operatorname{diam}\{x_n\} > 0$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ we get

$$\underline{\lim}_{n\to\infty}\|x_n - x\| < \operatorname{diam}\{x_n\}$$

for some $x \in \operatorname{conv}\{x_n\}$.

Moreover, this statement is equivalent to the original definition of ANS. Similarly, if we add an assumption that $\operatorname{conv}\{x_n\}$ is weakly compact, then we obtain an equivalent definition of w-ANS.

A Banach space X is said to satisfy *Opial's condition* (*nonstrict Opial's condition*) [11] if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for $y \neq x$

$$\underline{\lim}_{n\to\infty}\|x_n - x\| < \underline{\lim}_{n\to\infty}\|x_n - y\|,$$

or respectively

$$\underline{\lim}_{n\to\infty}\|x_n - x\| \le \underline{\lim}_{n\to\infty}\|x_n - y\|.$$

We say that Banach space X has *semi-Opial's property* (SO) if for any bounded nonconstant sequence with $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ there exists a subsequence $\{x_{n_i}\}$ weakly convergent to x and such that

$$\lim_{i\to\infty}\|x - x_{n_i}\| < \operatorname{diam}\{x_n\}.$$

We say that Banach space X has *weak semi-Opial's property* (w-SO) [7] if for any bounded nonconstant sequence $\{x_n\}$ with the weakly compact convex hull and with $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ there exists a subsequence $\{x_{n_i}\}$ weakly convergent to x and such that $\lim_{i\to\infty}\|x - x_{n_i}\| < \operatorname{diam}\{x_n\}$.

It is known that the w-NS implies the w-ANS but there exist spaces with w-ANS lacking the w-NS [1].

Obviously the w-SO property implies the w-ANS but it is still an open question whether the w-ANS implies the w-SO property. However, we have the following

Proposition. *If the Banach space X has the w-ANS and satisfies nonstrict Opial's condition, then it has the w-SO property.*

Proof. Let $\{x_n\}$ be a sequence of points in X such that $x_n - x_{n+1} \to 0$, $\operatorname{diam}\{x_n\} > 0$ and $\operatorname{conv}\{x_n\}$ is weakly compact. By the above mentioned equivalent definition of the w-ANS there exists a subsequence $\{x_{n_i}\}$ and $\bar{x} \in \operatorname{conv}\{x_n\}$ satisfying

$$\lim_{i\to\infty}\|x_{n_i} - \bar{x}\| < \operatorname{diam}\{x_n\}.$$

We may assume without loss of generality that $\{x_{n_i}\}$ is weakly convergent to an x. Now, by nonstrict the Opial condition, we have

$$\lim_{i\to\infty}\|x_{n_i} - x\| \le \lim_{i\to\infty}\|x_{n_i} - \bar{x}\| < \operatorname{diam}\{x_n\}.$$

The following conditions imply the w-SO property [7] of X:

(1) X has Opial's condition.
(2) X has uniformly normal structure [10].
(3) X is nearly uniformly convex [6].
(4) $X = X_\beta$, where $1 < \beta < 2$, $X_\beta = (l^2, |\cdot|_\beta)$, and, for $x \in l^2$, $|x|_\beta = \max(\|x\|_2, \beta\|x\|_\infty)$ [1], [5], [8].
(5) X is the James quasi-reflexive space [13].

It is worth to remark here that the proposition gives only a sufficient condition for the w-SO property. There exist spaces with the w-SO property but without the nonstrict Opial condition.

For example $L^p([0,2\pi])$ spaces with $1 < p < \infty$, have uniformly normal structure, and thus (see point (2)) they are the w-SO, but if $p \ne 2$ they do not satisfy the nonstrict Opial condition [11].

A mapping $T : C \to X$, $C \subset X$, is called *nonexpansive* if

$$\|T(x) - T(y)\| \le \|x - y\| \quad \text{for all} \quad x, y \in C.$$

It is said that C is T*-invariant* whenever $T(C) \subset C$.

2. Fixed point theorems. In this section we consider the fixed point problem for the sum of two convex weakly compact subsets of a Banach space. At the beginning we recall some known theorems. First of them, which was proved in [12], is the following.

Theorem A. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and* $C_1, C_2, ..., C_k$, $k \in \mathbb{N}$, *be nonempty weakly compact convex subsets of* X. *Let* $C = \bigcup_{i=1}^{k} C_i$ *be contractible in the strong topology and let* $T : C \to C$ *be nonexpansive. Then* T *has a fixed point in* C.

The next one is due to T. Kuczumow, S. Reich and A. Stachura [9].

Theorem B. *Let* $(X, \|\cdot\|)$ *be a Banach space and* $C_1, C_2, ..., C_k$, $k \in \mathbb{N}$, *be nonempty weakly compact convex subsets of* X *with normal structure. If* $C_1, C_2, ..., C_k$ *satisfy the following condition*

$$C_i \cap C_j \neq \emptyset \Leftrightarrow |i-j| \leq 1$$

for all $1 \leq i, j \leq k$, *then* $C = \bigcup_{i=1}^{k} C_i$ *has the fixed point property for nonexpansive mappings.*

We want to apply the w-SO property of a Banach space to obtain a similar result for the sum of two sets. Considering the sum of only two sets is connected with the difficulties to find a suitable approximate fixed point sequence for nonexpansive mapping. Unfortunately, the Furi-Martelli theorem [4], which is used in the proof of Theorem B, does not give any useful information about approximate fixed point sequence.

Theorem. *Let* $(X, \|\cdot\|)$ *be a Banach space satisfying the weak semi-Opial condition and let* A_1, A_2 *be nonempty weakly compact convex subsets of* X *such that their intersection* $A_1 \cap A_2$ *is nonempty. Then every nonexpansive mapping* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *has a fixed point in* $A_1 \cup A_2$.

Proof. Let $T : A_1 \cup A_2 \to A_1 \cup A_2$ be nonexpansive. Denote by $\mathcal{M}$ a family of all nonempty and T-invariant subsets of $A_1 \cup A_2$ such that
(i) every such subset has the form $B_1 \cup B_2$, where $B_1 \subset A_1$, $B_2 \subset A_2$,
(ii) if B_k, $k = 1, 2$, is nonempty, then it is convex and weakly compact,
(iii) if both B_1, B_2 are nonempty sets, then $B_1 \cap B_2 \neq \emptyset$.

Of course such a family is ordered by set inclusion. Thus by Zorn's Lemma there exists at least one set $C_1 \cup C_2$ which is minimal and T-invariant. By a reasoning similar to that in [5, pages 35-36], we can restrict our considerations to the case in which all the members of the family $\mathcal{M}$ are separable.

Thus $C_1 \cup C_2$ is also separable. Moreover (analogously as in Lemma 3.3 [5]) we observe that $\operatorname{conv}(T(C_1 \cup C_2) \cap C_1) \cup \operatorname{conv}(T(C_1 \cup C_2) \cap C_2) = C_1 \cup C_2$.

Now we can consider two cases.

Case 1. The minimal set $C_1 \cup C_2$ is convex. In such a situation we can apply the Baillon-Schöneberg theorem [1] to get a fixed point of T.

Case 2. The minimal set $C_1 \cup C_2$ is not convex. We will show that it is impossible. Fix z in $C_1 \cap C_2$. By a standard argument (see, e.g.,[5]), we find a sequence $\{x_n\}$ of points in $C_1 \cup C_2$ such that

$$x_n = \frac{1}{n} z + \frac{n-1}{n} T(x_n).$$

Since T is nonexpansive, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= \left\| \frac{1}{n(n+1)}(z - T(x_n)) + \frac{n}{n+1}(T(x_n) - T(x_{n+1})) \right\| \\ &\leq \frac{1}{n(n+1)} \|z - T(x_n)\| + \frac{n}{n+1} \|x_n - x_{n+1}\|. \end{aligned}$$

Therefore

$$\|x_n - x_{n+1}\| \leq \frac{1}{n} \|z - T(x_n)\|$$

and

$$x_n - x_{n+1} \underset{n \to \infty}{\longrightarrow} 0.$$

Now, by the semi-Opial condition, there is $\bar{x}$ in $\operatorname{conv}\{x_n\}$ and a subsequence $\{x_{n_i}\}$ satisfying $\lim_{i \to \infty} \|x_{n_i} - \bar{x}\| < \operatorname{diam}\{x_n\}$. By separability of $C_1 \cup C_2$ (taking a subsequence if necessary) the limit $\lim_{i \to \infty} \|x_{n_i} - y\|$ exists for any $y \in C_1 \cup C_2$. Moreover, we can assume that $x_{n_i} \in C_1$ for $i \in N$ (in case $x_{n_i} \in C_2$ for $i \in N$ the proof is analogous). Here we have two possibilities: either there are $x, y \in C_1$ such that

$$(*) \qquad \lim_{i \to \infty} \|x_{n_i} - x\| < \lim_{i \to \infty} \|x_{n_i} - y\|,$$

or for each $y \in C_1$ there exists a constant c such that

$$\lim_{i \to \infty} \|x_{n_i} - y\| = c.$$

In both situations we will find disjoint nonempty sets $B_1 \subset C_1$, $B_2 \subset C_2$ such that

$$T(B_1) \subset B_2, \; T(B_2) \subset B_1.$$

Indeed, if $(*)$ holds, then let us take the asymptotic center $Ac(\{x_{n_i}\})$ of the sequence $\{x_{n_i}\}$ with respect to $C_1 \cup C_2$. The fact $x_n - T(x_n) \to 0$ implies

$$T(Ac(\{x_{n_i}\})) \subset Ac(\{x_{n_i}\}).$$

Moreover, from the inequality $(*)$ we get

$$Ac(\{x_{n_i}\}) \neq C_1 \cup C_2.$$

If

$$Ac(\{x_{n_i}\}) = B_1 \cup B_2,\ \emptyset \neq B_1 \subset C_1,\ \emptyset \neq B_2 \subset C_2,$$

then, by minimality of $C_1 \cup C_2$, we get

$$B_1 \cap B_2 = \emptyset$$

and

$$T(B_1) \subset B_2,\ T(B_2) \subset B_1.$$

Suppose now that the condition $(*)$ is not fulfilled. Thus for any $y \in C_1 \cup C_2$ we have

$$(**) \qquad c = \lim_{i\to\infty} \|x_{n_i} - y\| < \operatorname{diam}\{x_n\} = d.$$

Consider the family $\left(B\left(y, \frac{c+d}{2}\right) \cap C_1\right) \cup \left(B\left(y, \frac{c+d}{2}\right) \cap C_2\right)$, $y \in C_1 \cup C_2$, and observe that it has the finite intersection property.
Indeed, let $y_1, y_2, ..., y_k$ be any points in $C_1 \cup C_2$. Then, by $(**)$, for any $l = 1, 2, ..., k$ there is i_l such that

$$\|x_{n_i} - y_l\| < (c+d)/2 \quad \text{for} \quad i > i_l.$$

Therefore x_{n_j} belongs to each set $\left(B\left(y_l, \frac{c+d}{2}\right) \cap C_1\right) \cup \left(B\left(y_l, \frac{c+d}{2}\right) \cap C_2\right)$ where j is the greatest of the numbers $i_1, i_2, ..., i_k$. Moreover the set $B\left(y, \frac{c+d}{2}\right) \cap C_k$, $k = 1, 2$, are weakly compact and convex. Hence the intersection of all the sets of our family is nonempty.

Let

$$D = \bigcap_{y \in C_1 \cup C_2} \left(B\left(y, \frac{c+d}{2}\right) \cap C_1\right) \cup \left(B\left(y, \frac{c+d}{2}\right) \cap C_2\right).$$

We will show that D is T-invariant. Let us take $y \in C_1 \cup C_2$, (e.g. $y \in C_1$) and recall that $\operatorname{conv}(T(C_1 \cup C_2) \cap C_1) = C_1$. Thus we can approximate y by a convex combination of elements $T(y_i)$ from $T(C_1 \cup C_2)$. For $w \in D$

and $\varepsilon > 0$ we find $\sum_{i=1}^{m} \alpha_i T(y_i)$ such that $\|w - \sum_{i=1}^{m} \alpha_i T(y_i)\| < \varepsilon$ and therefore

$$\|y - T(w)\| \leq \|\sum_{i=1}^{m} \alpha_i T(y_i) - T(w)\| + \|y - \sum_{i=1}^{m} \alpha_i T(y_i)\|$$
$$\leq \sum_{i=1}^{m} \alpha_i \frac{c+d}{2} + \varepsilon = \frac{c+d}{2} + \varepsilon.$$

Hence D is $T-$invariant. We will show that $D \neq C_1 \cup C_2$. Indeed, for $w \in D$ and $y \in C_1 \cup C_2$, we have $\|w - y\| \leq (c+d)/2 < \operatorname{diam}\{x_n\}$. Next, there are x_{n_1}, x_{n_2} such that $\|x_{n_1} - x_{n_2}\| > (c+d)/2$ and therefore at least one of them does not belong to D. Let $D = B_1 \cup B_2$. Because of the minimality of $C_1 \cup C_2$ the sets B_1, B_2 are disjoint and $T(B_1) \subset B_2$, $T(B_2) \subset B_1$, which we claimed.

Having such the sets B_1 and B_2, we can apply the Baillon-Schöneberg theorem [1] to the mapping $T^2 : B_1 \to B_1$ to get a fixed point $b_1 \in B_1$, $(T^2(b_1) = b_1)$.

For $t > 0$ we define the set

$$A(t) = \{x \in C_1 \cup C_2 : \|x - b_1\|^2 + \|x - T(b_1)\|^2 \leq t^2\}$$

and put $A_k(t) = A(t) \cap C_k$, $k = 1, 2$. Let

$$t_0 = \min\{t : A(t) \neq \emptyset\}, \quad t_1 = \min\{t : A(t) = C_1 \cup C_2\}.$$

Of course such t_0, t_1 exist and $t_0 \neq 0$. Moreover, for $t \geq t_0$ we have $T(A(t)) \subset A(t)$.

Let us also observe that $A(t_0) \neq C_1 \cup C_2$ and hence $t_0 < t_1$. Indeed, if $A(t_0) = C_1 \cup C_2$ then for all $x \in C_1 \cup C_2$ we have the equality

$$\|x - b_1\|^2 + \|x - T(b_1)\|^2 = t_0^2.$$

Let us take $z \in C_1 \cap C_2$ and for example $z \neq b_1$. Then $(z + b_1)/2$ belongs to C_1 and in view of convexity of $\|\cdot\|^2$ we reach the following contradiction

$$t_0^2 = \|\frac{z+b_1}{2} - b_1\|^2 + \|\frac{z+b_1}{2} - T(b_1)\|^2$$
$$\leq \frac{1}{4}\|z - b_1\|^2 + \frac{1}{2}\|z - T(b_1)\|^2 + \frac{1}{2}\|b_1 - T(b_1)\|^2 < \frac{1}{2}t_0^2 + \frac{1}{2}t_0^2 = t_0^2.$$

Since $A_k(t_0) \neq \emptyset$, $k = 1, 2$, and both sets are weakly compact and convex, by minimality of $C_1 \cup C_2$, we have $A_1(t_0) \cap A_2(t_0) = \emptyset$ and

$$T(A_1(t_0)) \subset A_2(t_0), \quad T(A_2(t_0)) \subset A_1(t_0).$$

One can also easily prove that

$$T(A_1(t)) \subset A_2(t), \quad T(A_2(t)) \subset A_1(t) \quad \text{for} \quad t_0 \le t < t_1.$$

To get

$$\overline{\bigcup_{t_0 \le t < t_1} A_k(t)} = A_k(t_1) = C_k \quad \text{for} \quad k = 1,\ 2$$

it is sufficient to observe that for fixed $y_0 \in A_k(t_0)$ and for any $x \in C_k$, $x \neq y_0$, $0 < \alpha_n < 1$ and $\alpha_n \to 1$, we have

$$\alpha_n x + (1 - \alpha_n) y_0 \to x \quad \text{as} \quad n \to \infty$$

and

$$\begin{aligned} &\|\alpha_n x + (1-\alpha_n)y - b_1\|^2 + \|\alpha_n x + (1-\alpha_n)y - T(b_1)\|^2 \\ &\le \alpha_n \left(\|x - b_1\|^2 + \|x - T(b_1)\|^2\right) + (1-\alpha_n)\left(\|y - b_1\|^2 + \|y - T(b_1)\|^2\right) \\ &\le \alpha_n t_1^2 + (1-\alpha_n) t_0^2 < t_1^2. \end{aligned}$$

The above considerations guarantee us the following inclusion

$$T(A_1(t_1) \cap A_2(t_1)) \subset A_1(t_1) \cap A_2(t_1),$$

which contradicts the minimality of $C_1 \cup C_2$ and completes the proof.

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