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Remarks on Failure of Schauder's Theorem in Noncompact Settings

Abstract. This work is intended as a kind of survey. Our aim is to state some facts concerning various equivalent formulations of Schauder's Fixed Point Theorem. Consequences of failure of the theorem in noncompact settings are also discussed. We do not present many new results, what follows should rather be treated as a material for seminar discussions.

Let C be a nonempty bounded closed and convex subset of a Banach space X. We will deal with continuous mappings $T : C \to C$. If T is a lipschitzian mapping with constant $k > 0$, i.e.

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for} \quad x, y \in C,$$

we write $T \in \mathcal{L}(k)$. If $T \in \mathcal{L}(k)$ for some $k > 0$ we will simply write $T \in \mathcal{L}$. Let us begin with a general version of the classical Brouwer's Theorem [2].

Theorem 1 (L. E. J. Brouwer 1912). *Any nonempty bounded closed and convex subset of a finite-dimensional Banach space X has the fixed point*

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property for continuous mappings, i.e. every continuous mapping $T : C \to C$ *has a fixed point* $x = Tx$.

Eighteen years later when the needs forced the studies of infinite-dimensional spaces J. Schauder [12] published his famous theorem.

Theorem 2 (J. Schauder 1930). *Every nonempty convex and compact subset* C *of a Banach space* X *has the fixed point property for continuous mappings.*

The proof of Schauder's Theorem is based on the following fact. By compactness of C every continuous mapping $T : C \to C$ can be uniformly approximated with a desired accuracy by finite dimensional mappings. More precisely, for any $\varepsilon > 0$ there is a continuous mapping $T_1 : C \to C$ such that $T_1(C) \subset C \cap X_0$, where X_0 is a finite dimensional subspace of X, and for any $x \in C$,

$$\|Tx - T_1x\| < \varepsilon .$$

Using this fact and observing that $T_1 : C \cap X_0 \to C \cap X_0$ has a fixed point by Brouwer's Theorem, a straightforward compactness argument shows that T has a fixed point, too.

One can continue this type of approximate reasoning and observe that every finite dimensional mapping (and thus every compact mapping) can be approximated with a necessary accuracy by lipschitzian mappings. This can be proved by applying Weierstrass-Stone's Theorem to each coordinate of any finite dimensional mapping and approximating it by a polynomial and then utilizing some "technical tricks".

Returning to our situation, for any $\varepsilon > 0$ there is $T_2 \in \mathcal{L}$ such that for any $x \in C$ the inequality $\|Tx - T_2x\| < \varepsilon$ holds. And now, a simple compactness argument gives the following, seemingly weaker, equivalent to Schauder's Theorem.

I. *Every nonempty convex and compact subset* C *of a Banach space* X *has the fixed point property for mappings of class* $\mathcal{L}$.

In other words

II. *If* C *is a nonempty convex and compact subset of a Banach space* X *then every lipschitzian mapping* $T : C \to C$ *has a fixed point.*

And, without repeating standard assumptions on C, we can also state next result which might appear even weaker than the previous one but it is also an equivalent to Schauder's Theorem.

III. *There is $k_0 > 1$ such that every mapping belonging to $\mathcal{L}(k_0)$ has a fixed point (C has the fixed point property for $\mathcal{L}(k_0)$).*

Indeed, this is an immediate consequence of the following fact.

If $T \in \mathcal{L}(k)$ then for any $\lambda \in [0,1]$ the mapping defined by setting $T_\lambda = (1-\lambda)I + \lambda T$ belongs to $\mathcal{L}(1-\lambda+\lambda k)$. Thus for $\lambda \leq (k_0-1)/(k-1)$, $T_\lambda \in \mathcal{L}(k_0)$ and therefore has a fixed point. The conclusion (III implies II) now follows from the fact that T and T_λ have common fixed points, i.e. $\operatorname{Fix} T = \operatorname{Fix} T_\lambda$.

Let us add another version of Schauder's Theorem to our list of its equivalents.

IV. *If a continuous mapping $T : C \to C$ is such that for every $x \in C$, $\|x - Tx\| = d = const$ then $d = 0$.*

It is obvious that Schauder's Theorem implies IV. For the reverse implication, observe that supposing there is a mapping $T : C \to C$ which is fixed point free, we have (by compactness of C) $\min_{x \in C} \|x - Tx\| = d > 0$ and therefore for $T_1 : C \to C$ defined by setting

$$T_1 x = x + d \, \frac{Tx - x}{\|Tx - x\|}$$

the following condition holds $\|x - T_1 x\| = d = \text{const} > 0$.

After the preceding short discussion of equivalent formulations of Schauder's Theorem let us restrict our attention to noncompact sets C.

S. Kakutani [7] was probably the first who in 1943 showed that there are continuous mappings of the unit ball in Hilbert space without fixed points. A stronger result is due to V. Klee [8].

Theorem 3 (V. Klee 1955). *For any nonempty closed convex but noncompact subset C of a Banach space X there exists a continuous mapping $T : C \to C$ which is fixed point free.*

From the topological point of view Klee's result closed one problem. And now the theorems of Schauder and Klee can be mixed in one property.

V. *Any nonempty bounded closed and convex subset C of a Banach space X has the fixed point property for continuous mappings if and only if it is a compact set.*

But there are still problems of qualitative type which remain open. First examples of fixed point free mappings were the nonexpansive mappings

(mappings of class $\mathcal{L}(1)$) or their compact perturbations. One can easily prove that although they do not have fixed points all of them satisfy the condition $\inf\{\|x - Tx\|:\ x \in C\} = 0$.

First examples of mappings which do not satisfy the above condition were given by K. Goebel [3] in 1973. His paper initiated studies of the so-called minimal displacemet problem.

In noncompact settings, given $T : C \to C$, we define minimal displacement d_T of T by the formula $d_T = \inf\{\|x - Tx\|:\ x \in C\}$. The minimal displacement for a given class $\mathcal{F}$ is defined by $\varphi_C(\mathcal{F}) = \sup\{d_T : T \in \mathcal{F}\}$. We try to find the number $\varphi_C(\mathcal{F})$ or at least its estimates.

The class $\mathcal{L}(k)$ of all the lipschitzian mappings with constant k is of special interest and has been investigated. We abbreviate $\varphi_C(\mathcal{L}(k))$ to $\varphi_C(k)$. One can easily prove that $\varphi_C(k) \leq (1 - 1/k)\, r(C)$, where $r(C)$ is the Chebyshev radius of C.

It is known [3] that there are sets C for which the above estimate is sharp and there are also sets for which it is not. Several questions posed in Goebel's paper still remain open. Some further results in solving minimal displacement problem for the unit ball B_H in Hilbert space were obtained but unfortunately the progress is very slow. We still do not know the function $\varphi_{B_H}(k)$. Fundamental in this theory is the following result of Sternfeld and Lin [10].

Theorem 4 (Y. Sternfeld, P. K. Lin 1985). *For any noncompact bounded and convex subset C of a Banach space X there is a lipschitzian mapping $T : C \to C$ for which $d_T = \inf\{\|x - Tx\|:\ x \in C\} > 0$.*

Observe now that using standard approximation argument presented in the first part of our paper one can find such a mapping in each class $\mathcal{L}(k)$, $k > 1$. Hence, the above theorem has its following equivalent formulation.

VI. *If C is noncompact then $\varphi_C(k) > 0$ for $k > 1$.*

We can also get an analogue of IV.

VII. *If C is noncompact then in each class $\mathcal{L}(k)$, $k > 1$, there is T such that for any $x \in C$, $\|x - Tx\| = const > 0$.*

To prove it we assign to a lipschitzian map T with $d_T > 0$ the mapping T_1 defined by putting

$$T_1 x = x + d_T \frac{Tx - x}{\|Tx - x\|}.$$

Of course T_1 is lipschitzian and $d_{T_1} = d_T = \|x - T_1 x\|$ for any $x \in C$. Having such a mapping in the class $\mathcal{L}$ we can find another mapping in every class $\mathcal{L}(k)$ by taking a linear combination of T_1 and the identity (see III).

Mappings with positive minimal displacement and in particular with constant displacement behave in a special way.

Let $T : C \to C$, $T \in \mathcal{L}(k)$, be such that $\|x - Tx\| = d_T$ for all $x \in C$. Given $x \in C$ consider three points: Tx, T^2x and $u = \frac{1}{2}(Tx + T^2x)$. We have

$$\begin{aligned} d_T = \|Tu - u\| &= \left\| \frac{1}{2}Tu + \frac{1}{2}Tu - \frac{1}{2}Tx - \frac{1}{2}T^2x \right\| \\ &\le \frac{1}{2}k\|u - x\| + \frac{1}{2}k\|u - Tx\| \\ &\le \frac{1}{4}k\|x - Tx\| + \frac{1}{4}k\|x - T^2x\| + \frac{1}{4}k\|Tx - T^2x\| \\ &= \frac{1}{4}k\|x - T^2x\| + \frac{1}{2}kd_T . \end{aligned}$$

Thus

$$\|x - T^2x\| \ge 2\left(\frac{2}{k} - 1\right) d_T . \tag{1}$$

Therefore, although T^2 is not necessarily a mapping with constant displacement, the following inequality holds

$$d_{T^2} \ge 2\left(\frac{2}{k} - 1\right) d_T . \tag{2}$$

Observe that for k close to 1 the coefficient on the right side of the inequality is close to 2. As an obvious consequence of the above consideration we can state our next proposition.

VIII. *If C is noncompact then for any $\varepsilon > 0$ there is a continuous mapping $T : C \to C$ such that $d_T > 0$ and $d_{T^2} \ge (2 - \varepsilon)d_T$.*

Of course as such a mapping T we can take any lipschitzian mapping with constant sufficiently close to 1 which displaces any point $x \in C$ by the same fixed distance.

On the other hand, the above consideration leads to the following equivalent to Schauder's Theorem.

IX. *If C is compact then for any $\varepsilon \in (0,2]$ and any continuous mapping $T : C \to C$ there is $x \in C$ such that*

$$\|x - T^2x\| \leq (2-\varepsilon)\|x - Tx\|.$$

(The inequality sign $\leq$ can be replaced by strict inequality sign $<$ or by equality sign).

In particular, for $\varepsilon = 2$ we have the following, seemingly weaker, equivalent to Schauder's Theorem.

X. *If C is compact then any continuous mapping $T : C \to C$ has a periodic point with period 2 (i.e. T^2 has a fixed point).*

With such observations studies of so called rotative mappings were initiated by K. Goebel and M. Koter [5, 9] and one chapter in [4] is also devoted to such mappings.

Let a positive integer n and a real number $a < n$ be given. A continuous mapping $T : C \to C$ is said to be (a, n)-rotative if for each $x \in C$,

$$\|x - T^n x\| \leq a\|x - Tx\|. \tag{3}$$

Observe that the notion of rotativeness is especially justified when we consider mappings with constant displacement. For such mappings T we have $\|x - T^n x\| \leq n\|x - Tx\|$ and the inequality (3) with $a < n$ is resulted only by a certain "turning" of a sequence of successive iterations $x, Tx, T^2x, \ldots, T^n x, \ldots$. If we do not suppose that T has constant displacement this inequality may also be caused by a certain "shortening" of the distances between consecutive terms of this sequence and the significance of the notion "rotative" is not that clear. In both cases $(n,0)$-rotativeness means n-periodicity ($T^n = \mathrm{Id}$).

The following theorem is basic in the theory of rotative mappings [6].

Theorem 5 (K. Goebel, M. Koter 1981). *For $a < n$ any nonexpansive and (n,a)-rotative selfmapping $T : C \to C$ of a closed and convex set C has a fixed point.*

Note that in this theorem no special geometrical structure is assumed on C. We even do not require boundedness of C. As there exist examples of bounded sets C which are fixed point free for nonexpansive mappings we can write the following remark.

XI. *There exists a convex closed and bounded set C and a mapping $T : C \to C$, $T \in \mathcal{L}(1)$ which is not (n,a)-rotative for any n and $a < n$.*

Theorem 5 has some further extensions. Given n and a with $a < n$ set $\gamma_n(a) = \sup\{k$: any k-lipschitzian and (n,a)-rotative mapping $T : C \to C$ has a fixed point$\}$. We abstract here from the set C and the space in which it is included and we have the following [5].

Theorem 6 (K. Goebel, M. Koter 1981). *For any n and $a < n$, $\gamma_n(a) > 1$.*

In the definition of $\gamma_n(a)$ we can restrict ourselves either to a given set C or to subsets of a given space or to certain subclasses of class $\mathcal{L}(k)$.

For example, we can consider only classes of mappings with constant displacement. In such a class our definition is the following:

$$\gamma_n^{\text{const}}(a) = \sup\Big\{k : \text{the only } k\text{-lipschitzian and } (n,a)\text{-rotative mapping } T : C \to C \text{ with constant displacement is } T = Id\Big\}$$

$$= \inf\Big\{k : \text{there is } k\text{-lipschitzian and } (n,a)\text{-rotative mapping } T : C \to C \text{ with constant positive displacement}\Big\}.$$

Of course we have $\gamma_n(a) \le \gamma_n^{\text{const}}(a)$. Not much is known about the values of $\gamma_n(a)$. First estimate $\gamma_2(a) \ge \frac{1}{2}\left((2-a) + \sqrt{(2-a)^2 + a^2}\right)$ proved by K. Goebel and M. Koter in [5] is not sharp [4]. It is not even known whether $\gamma_2(0) < +\infty$ or whether there exists a set C and a fixed point free k-lipschitzian mapping $T : C \to C$ such that $T^2 = \text{Id}$.

Let us observe that substituting $a = 2(2/k - 1)$ into (1) or (2) we can now, after our modification, write $\gamma_2^{\text{const}}(a) \ge 4/(a+2)$.

By the above remarks the propositions VIII, IX and X can be modified .

XII. *If C is noncompact then for any $\varepsilon > 0$ there is a continuous mapping $T : C \to C$ such that $d_T > 0$ and $d_{T^n} \ge (n-\varepsilon)d_T$.*

Once again as such the mappings there have to be k-lipschitzian mappings (with constant or not displacement) and with k sufficiently close to 1.

We continue in this fashion to get, for any $\varepsilon \in (0,n]$, the following equivalent to Schauder's Theorem.

XIII. *If C is compact then for any $\varepsilon > 0$ and for any continuous mapping $T : C \to C$ there exists $x \in C$ such that $\|x - T^n x\| \leq (n - \varepsilon)\|x - Tx\|$.*

In particular, for $\varepsilon = n$ we get another equivalent formulation.

XIV. *If C is compact then every continuous mapping $T : C \to C$ has periodic point with period n (i.e. an x such that $T^n x = x$).*

We conclude our discussion with some constructive examples of lipschitzian mappings with constant displacement.

Example 1. [4] Let us consider Banach space of continuous functions $C[0,1]$ and take $C = \{x \in C[0,1] : 0 = x(0) \leq x(t) \leq x(1) = 1, t \in [0,1]\}$. Let $e(t) \equiv t$ on $[0,1]$. Select $\alpha \in C$, $\alpha \neq e$, and define $T_\alpha : C \to C$ by setting $(T_\alpha x)(t) = \alpha(x(t)) = (\alpha \circ x)(t)$. The mapping T_α inherits the behaviour of α, i.e. if $\alpha(t) - \alpha(s)| \leq k|t - s|$ for $t, s \in [0,1]$ then $\|T_\alpha x - T_\alpha y\| \leq k\|x - y\|$ for $x, y \in C$. Since any function $x \in C$ takes on all the values from the interval $[0,1]$, T_α has the following property:

$$\begin{aligned}\|T_\alpha x - x\| &= \max\{|\alpha(x(t)) - x(t)| : t \in [0,1]\} \\ &= \max\{|\alpha(t) - t| : t \in [0,1]\} = \|\alpha - e\| = d_{T_\alpha} = \text{const}.\end{aligned}$$

Thus T_α has a constant positive displacement. Moreover, $T_\alpha^n = T_{\alpha^n}$ where α^n denotes the n-fold iteration of α. Thereby all the iterates of T_α have also constant displacement $d_{T_\alpha^n} = \|e - \alpha^n\|$.

Now consider the special case where $\alpha(t) = \min\{kt, 1\}$. Thus we have $d_{T_\alpha} = 1 - 1/k$ and $d_{T_\alpha^2} = 1 - 1/k^2 = (1 + 1/k)\, d_{T_\alpha}$. Comparing this with the definition of $\gamma_2^{\text{const}}(a)$ and setting $a = 1 + 1/k$ we get

$$\gamma_2^{\text{const}}(a) \leq \begin{cases} +\infty & \text{for } a \leq 1, \\ 1/(a-1) & \text{for } 1 < a < 2. \end{cases}$$

We know only this estimate which is probably not sharp.

Second example is connected with the result proved by B. Nowak [11] for a certain class of spaces and then generalized onto all Banach spaces by Benyamini and Sternfeld [1].

Theorem 7 (B. Nowak 1979, Benyamini and Sternfeld 1983). *For any infinite dimensional Banach space X there is a lipschitzian retraction of the unit ball onto the unit sphere.*

It means that there exists a mapping $R : B \to S$ in a class $\mathcal{L}(k)$ such that $Tx = x$ for all $x \in S$. This theorem shows how different is finite dimensional case of compact balls from the infinite dimensional one of noncompact balls. The fact that in finite dimensional case S is not a retract of the ball is equivalent to Brouwer's Theorem (Theorem 1). Theorem 7 can be proved as a consequence of the result of Sternfeld and Lim (Theorem 4) [10].

Example 2. Assume that $R: B \to S$ is a lipschitzian retraction with constant k ($R \in \mathcal{L}(k)$). Given $l > k$ define T by setting $Tx = x - Rx/l$. T is a homeomorphism of B onto $(1 - 1/l)\,B$. Moreover, $T \in \mathcal{L}(1 + k/l)$. For $l = k$ the mapping T defined by the formula $Tx = x - Rx/k$ belongs to class $\mathcal{L}(2)$ and maps B onto $(1 - 1/k)\,B$ although not necessarily homeomorphically. In both cases $d_T = 1/l = \text{const}$. To obtain examples of mappings described in Proposition VIII and Proposition XII one can take mappings from this example with sufficiently large l.

The problem is how to find at least one constructive example of the retraction R. Let us end this article by giving such an example.

Example 3. Let $X = C[0,1]$. For $x \in B$, define A by setting

$$(Ax)(t) = |x(t) + 1 - 2(1 - \|x\|)t| - 1 + 2(1 - \|x\|)t\,.$$

Observe first that for any $x \in S$ we have $Ax = x$. It is easy to prove that $A \in \mathcal{L}(5)$. A little bit more difficult is to prove that $\inf\{\|Ax\| : x \in B\} > 0$ and to find its value. The retraction $R : B \to S$ can be obtained by setting $Rx = Ax/\|Ax\|$. Thus the retraction is defined constructively, all the detailed computation being left to the reader.

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