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A Property of Polynomials on Complex Algebraic Sets

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. We show that if a polynomial F does not vanish on an algebraic set V then there exists a polynomial G such that $FG = 1$ on V and $\deg G \leq (\deg F)(\deg G)$.

1. Main result. If a complex polynomial does not vanish at any point then it is constant. The aim of this note is to extend this property to polynomial functions on algebraic sets. Our main result is as follows.

Theorem. *Let $F : \mathbb{C}^n \to \mathbb{C}$ be a polynomial and let $V \subset \mathbb{C}^n$ be a non-empty algebraic set. Suppose that $F(z) \neq 0$ for all $z \in V$. Then there exists a polynomial $G : \mathbb{C}^n \to \mathbb{C}$ such that $F(z)G(z) = 1$ for all $z \in V$ and* $\deg(FG) \leq (\deg F)(\deg V)$.

Our theorem can be considered as a version of the effective Nullstellensatz (see [2], [5]). Its proof is given in Section 2 of this paper. We will use some notions and properties of algebraic sets. Our main reference is [3].

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Recall that for every non-empty algebraic set $V \subset \mathbb{C}^n$ we have $\deg V = \sum_{i=1}^{k} \deg V_i$ where $V = V_1 \cup \ldots \cup V_k$ is the decomposition of V into irreducible components (see [3], p. 419).

For every $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we put $|z| = \max\{|z_1|, \ldots, |z_n|\}$. We say that an inequality holds for $|z| \gg 1$ if it holds for large $|z|$. If $F : \mathbb{C}^n \to \mathbb{C}$ is a non-zero polynomial then its degree $\deg F$ is equal to the smallest number $q > 0$ such that $|F(z)| \le c_q |z|^q$ with a constant $c_q > 0$ for $|z| \gg 1$ ([4], Lemma 1.1).

Corollary. *Suppose that a polynomial $F : \mathbb{C}^n \to \mathbb{C}$ does not vanish on a nonalgebraic set $V \subset \mathbb{C}^n$. Then there is a constant $c > 0$ such that*

$$|F(z)| \le c|z|^{-(\deg F)(\deg V - 1)} \quad \textit{for } |z| \gg 1 \textit{ and } z \in V.$$

Proof of Corollary. By our theorem there is a polynomial $G : \mathbb{C}^n \to \mathbb{C}$ such that $F(z)G(z) = 1$ for $z \in V$ and $\deg G \le (\deg F)(\deg V - 1)$. Let $z \in V$. Then we get

$$|F(z)| = |G(z)|^{-1} \ge c|z|^{-\deg G} \ge c|z|^{-(\deg F)(\deg V - 1)}$$

for $||z| \gg 1$ with a constant $c > 0$.

The estimate of our theorem is optimal.

Example. Let $F(x, y) = x$ for $(x, y) \in \mathbb{C}^2$ and let $V = \{(x, y) \in \mathbb{C}^2 : xy^{d-1} - 1 = 0\}$ where $d > 1$ is an integer. According to the theorem there is a polynomial $G : \mathbb{C}^n \to \mathbb{C}$ such that $FG = 1$ on V and $\deg G \le d - 1$ (one may take $G(x, y) = y^{d-1}$). On the other hand, for every polynomial G of degree N such that $FG = 1$ on V we have $N \ge d - 1$. Indeed, if $FG = 1$ on V then $|x| \ge c \max(|x|, |y|)^{-N}$ for $\max(|x|, |y|) \ge 1$ and $(x, y) \in V$ with a constant $c > 0$ because $|G(x, y)| \le c_1 \max(|x|, |y|)^N$ for $\max(|x|, |y|) \gg 1$ with $c_1 > 0$. Consequently $|y|^{N-(d-1)} \ge c$ for $|y| \gg 1$ and we get $N \ge d - 1$.

2. Proof. Let us suppose that V is irreducible of dimension k. The cases $k = 0$ and $k = n$ being trivial we assume that $0 < k < n$. According to Sadullaev's theorem ([3], p. 389) we may choose new coordinates (X, Y) in $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ such that

(i) the projection $\pi : V \to \mathbb{C}^k$ given by $\pi(x, y) = x$ is finite of degree $d = \deg V$,

(ii) there is a constant $c > 0$ such that $|y| \le c|x|$ for $(x, y) \in V$ and $\max(|x|, |y|) \gg 1$.

Let $P(X,T) = T^d + P_1(x)T^{d-1} + \dots + P_d(X) \in \mathbb{C}[X,T]$ be the characteristic polynomial of $F|_V$ with respect to π (see [3], p. 307 and [1], p. 138). Then for every $x \in \mathbb{C}^k$ we have

(iii) $P(x,T) = (T - F(x,y^{(1)}) \dots (T - F(x,y^{(d)}),$

where $(x, y^{(j)}) \in \pi^{-1}(x)$ for $j = 1, \dots, d$ and every element $z = (x,y) \in \pi^{-1}(x)$ appears in the sequence $(x,y^{(1)}), \dots, (x,y^{(d)})$ with multiplicity $\mathrm{mult}_z(\pi)$. By (ii) there exists a constant $c_F > 0$ such that $|F(x,y)| \le c_F|x|^{\deg F}$ for $(x,y) \in V$ and $\max(|x|,|y|) \gg 1$.

Let $s_j(T_1, \dots, T_d)$ be the j-th elementary symmetric function. Then

$$\begin{aligned}|P_j(x)| &= |s_j(F(x,y^{(1)}), \dots, F(x,y^{(d)})| \\ &\le s_j(|F(x,y^{(1)})|, \dots, |F(x,y^{(d)})|) \\ &\le s_j(c_F|x|^{\deg F}, \dots, c_F|x|^{\deg F}) = \binom{d}{j} c_F^j |x|^{j \deg F}\end{aligned}$$

for $|x| \gg 1$. Therefore

(iv) $\deg P_j \le j \deg F$ for $j = 1, \dots, d$.

Moreover,

(v) $P_d \equiv c \neq 0$.

Indeed, for $x \in \mathbb{C}^k$ we have $P_d = (-1)^d F(x,y^{(1)}, \dots, F(x,y^{(d)}) \neq 0$ because the polynomial F does not vanish on V. Then by (v) we get

$$F(x,y)(F(x,y)^{d-1} + p_1(x)F(x,y)^{d-2} + \dots + P_{d-1}(x)) = -c$$

for $(x,y) \in V$.

Let $G = -c^{-1}(F^{d-1} + P_1F^{d-2} + \dots + P_{d-1})$. Obviously $FG = 1$ on V and by (iv) we get easily the estimate $\deg G \le (d-1) \deg F$ that is

$$\deg(FG) \le d(\deg F) = (\deg F)(\deg V).$$

Our theorem is proved for polynomials on irreducible sets.

Let V be any algebraic set and let $V = V_1 \cup \dots \cup V_k$ be the decomposition of V into irreducible components V_i. Then by what we have just proved there are polynomials G_i such that $FG_i = 1$ on V_i and $\deg(FG_i) \le (\deg F)(\deg V_i)$. Obviously $\prod_{i=1}^k (1 - FG_i) = 0$ on V and it suffices to take

$$G = \sum_i G_i - F \sum_{i<j} G_iG_j + \dots + (-1)^{k+1} F^{k-1}(G_1, \dots, G_k).$$

This completes the proof of our main result.

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