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The Denjoy-Wolff theorem for s-condensing mappings

ABSTRACT. In this short note we establish the locally uniform convergence of iterates of fixed-point-free, s-condensing and holomorphic self-maps of the open unit ball in a strictly convex Banach space.

In 1926 Denjoy and Wolff independently and simultaneously proved the theorem that bears their name [8], [34]. Since then there were many modifications and generalizations of this theorem ([1], [5], [6], [13], [15], [17], [18], [20], [23], [27], [31], [32], [33]). The recent one deals with a condensing, fixed-point-free and holomorphic self-map of the open unit ball in a strictly convex Banach space and the convergence of its iterates in the compact-open topology [16]. In this short note we show the locally uniform convergence of the iterates of fixed-point-free, s-condensing and holomorphic self-maps of the open unit ball in a strictly convex Banach space.

1. Preliminaries. All Banach spaces will be complex. If D is a bounded domain in a Banach space $(X, \|\cdot\|)$ then k_D always denotes its Kobayashi distance. We remark in passing that all distances assigned to a convex bounded

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domain D by Schwarz-Pick systems of pseudometrics ([10], [11], [12]) coincide ([9], [14], [22]). Next, the Kobayashi distance k_D is topologically equivalent to the norm metric and this equivalence is even locally uniform ([10], [11], [12]).

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. If D is a bounded convex domain in X and D' is a bounded convex domain in Y, then the family of all holomorphic functions from D to D' is denoted by $H(D, D')$. The compact open topology on $H(D, D')$ is the topology generated by pseudodistances

$$p_K(f,g) = \sup\{\|f(x) - g(x)\|_Y : x \in K\},$$

where $f, g \in H(D, D')$ and K ranges over the compact subsets of D. The topology of locally uniform convergence on $H(D, D')$ is the topology induced by pseudodistances

$$q_{B(a,r)}(f,g) = \sup\{\|f(x) - g(x)\|_Y : x \in B(a,r)\},$$

where $f, g \in H(D, D')$ and $B(a,r) \subset D$ ranges over the open balls in X satisfying $\operatorname{dist}(B(a,r), \partial D) > 0$ and ∂D denotes the boundary of D. If X is a finite-dimensional space, then the topology of locally uniform convergence on $H(D, D')$ is equivalent to the compact open topology. Here we have to mention that the locally uniform convergence of iterates is a necessary claim in many applications ([19], [25], [26], [27], [28], [29]).

Let (D, d) be a metric space. A mapping $f : D \to D$ is said to be d-nonexpansive if

$$d(f(x), f(y)) \leq d(x,y)$$

for all $x, y \in D$. If D is a bounded domain in a Banach space $(X, \|\cdot\|)$ and k_D is the Kobayashi distance in D then each holomorphic $f : D \to D$ is k_D-nonexpansive ([10], [11], [12], [14]).

Let (Y, ρ) be a metric space and let $\emptyset \neq D \subset Y$. We say that $f : D \to D$ is an s-condensing mapping with respect to Kuratowski's measure of noncompactness α_ρ (s-condensing mapping, in short) ([2], [3], [21]), if there exists $s \in [0,1)$ such that

$$\alpha_\rho(f(A)) \leq s\alpha_\rho(A)$$

for each $A \subset D$. The s-condensing mappings are also called set-contractions. The mapping $f : D \to D$ is α_ρ-condensing (condensing, in short) if

$$\alpha_\rho(f(A)) < \alpha_\rho(A)$$

for each $A \subset D$ with $\alpha_\rho(A) > 0$.

Condensing and s-condensing mappings play an important role in the fixed point theory. See, for example [2], [3], [4], [7], [24], [30], and the references mentioned there.

Recently, the following Denjoy-Wolff type theorem has been proved.

Theorem 1.1 [16]. *If B is the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $f : B \to B$ is k_B-nonexpansive, condensing with respect to $\alpha_{\|\cdot\|}$ and fixed-point-free, then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges in the compact-open topology to the constant map taking the value ξ.*

It is also known that in the case of uniformly convex Banach spaces one can prove the locally uniform convergence of the sequence of iterates.

Theorem 1.2 [16]. *Let X be a uniformly convex Banach space with the open unit ball B. Let $f : B \to B$ be a condensing with respect to $\alpha_{\|\cdot\|}$ and k_B-nonexpansive map with no fixed point in B. Then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ.*

In this note we show that Theorem 1.2 is still valid for s-contractions in strictly convex Banach spaces.

2. s-condensing mappings.

Here we recall two basic properties of s-condensing mappings which we need in the proof of the Denjoy-Wolff theorem. The first property is stated in the following

Lemma 2.1. *Let (Y, ρ) be a metric space, $\emptyset \neq D \subset Y$ a bounded set, and $f : D \to D$ an s-condensing mapping with respect to α_ρ. If there exists a sequence $\{A_j\}$ of nonempty subsets of D such that $A_{j+1} \subset f(A_j)$ for $j = 1, 2, \ldots$ and $\{x_j\}$ is a sequence with $x_j \in A_j$ for $j = 1, 2, \ldots$ then $\alpha_\rho(\{x \in D : \exists_j x = x_j\}) = 0$.*

Proof. This is sufficient to observe that

$$\alpha_\rho(\{x \in D : \exists_j x = x_j\}) = \alpha_\rho(\{x \in D : \exists_{j \geq k} x = x_j\}) \\ \leq \alpha_\rho(A_k) \leq s^{k-1} \alpha_\rho(A_1) \underset{k}{\to} 0.$$

□

For the next property we need the definition of a strictly convex domain. We say that a bounded convex open set D in a Banach space X is strictly convex if for every $x, y \in \overline{D}^{\|\cdot\|}$ the open segment

$$(x, y) = \{z \in X : z = \gamma x + (1 - \gamma) y \quad \text{for some } 0 < \gamma < 1\}$$

lies in D.

Lemma 2.2 [16]. *Let D be a strictly convex bounded domain in a Banach space $(X, \|\cdot\|)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D which converge to $\xi \in \partial D$ and to $\eta \in \overline{D}$, respectively. If $\sup\{k_D(x_n, y_n) : n = 1, 2, ...\} = c < \infty$ then $\xi = \eta$.*

3. The Denjoy-Wolff Theorem for s-condensing mappings. In this section we prove two main theorems. The first one is the following.

Theorem 3.1. *If B is the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $f : B \to B$ is a holomorphic, s-condensing with respect to $\alpha_{\|\cdot\|}$, and fixed-point-free mapping, then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ.*

Proof. By Theorem 1.1 there exists $\xi \in \partial B$ such that $\lim_n f^n(0) = \xi$. Then, by Lemma 2.1, for each sequence $\{x_j\}$ with $\|x_j\| \leq r < 1$ for $j = 1, 2, ...$, and every strictly increasing sequence of natural numbers $\{n_j\}$ we have

$$\alpha_{\|\cdot\|}(\{x \in B : \exists_j x = f^{n_j}(x_j)\}) = \lim_j \alpha_{\|\cdot\|}(A_j) = 0,$$

where $A_j = T^{n_j}(B)$ for $j = 1, 2, ...$. We also have

$$\sup_j k_B(f^{n_j}(0), f^{n_j}(x_j)) \leq \sup_j k_B(0, x_j) \leq \operatorname{arg\,tanh} r < \infty.$$

Hence for an arbitrary convergent subsequence $\{f^{n_{j_m}}(x_{j_m})\}$ we get $\lim_m f^{n_{j_m}}(x_{j_m}) = \xi$ by Lemma 2.2. It implies that $\lim_j f^{n_j}(x_j) = \xi$ and therefore the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ.

□

Remark 3.1. Since the proofs of Lemmas 2.1 and 2.2 and Theorems 1.1 and 3.1 have a strictly metric character, we can extend Theorem 3.1 to the case of k_B-nonexpansive mappings.

In many applications in place of sequences of iterates of mappings we have to deal with one-parameter continuous semigroups ([19], [25], [26], [27], [28], [29]) and therefore we need an analog of Theorem 3.1. We begin with the definition of a one-parameter continuous s-condensing semigroup of mappings.

Definition 3.1. Let (X, ρ) be a metric space and let $\emptyset \neq D \subset X$ be a bounded subset of X. A family $S = \{S_t\}_{t \geq 0}$ of selfmappings of D is called

a one-parameter continuous s-condensing with respect to α_ρ semigroup of mappings if it satisfies the following properties:
(i) $S_0 = \mathrm{Id}$,
(ii) $S_{t+s} = S_t \circ S_s$,
(iii) $S_t : D \longrightarrow D$ is s -condensing with respect to α_ρ,
(iv) $[0,\infty) \ni t \longrightarrow S_t x$ is continuous for every $x \in D$.

Now we are ready to state the basic theorem for such a semigroup.

Theorem 3.2. *Let B be the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $S = \{S_t\}_{t\geq 0}$ a one parameter continuous s-condensing with respect to $\alpha_{\|\cdot\|}$ semigroup of k_B-nonexpansive mappings on B. If for some $t_0 > 0$ the mapping S_{t_0} is fixed point free, then there is a point $\xi \in \partial B$ such that S converges locally uniformly to ξ, as t tends to infinity.*

Proof. Since S_{t_0} has no fixed point in B, Theorem 1.1 says that there is a point $\xi \in \partial B$ such that $S_{t_0,n} = S_{t_0}^n$ converges to ξ locally uniformly on B. Fix $\widetilde{z} \in B$. By continuity of the semigroup $S = \{S_t\}_{t\geq 0}$ the set

$$C = \{z_s \in B : z_s = S_s(\widetilde{z}), 0 \leq s \leq t_0\}$$

is a compact subset of B. Hence for each $\epsilon > 0$ one can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\sup_{0\leq s\leq t_0} \|S_{nt_0+s}(\widetilde{z}) - \xi\| &= \sup_{0\leq s\leq t_0} \|S_{t_0,n}(z_s) - \xi\| \\ &= \sup_{z\in C} \left\|S_{t_0}^n(z) - \xi\right\| < \epsilon\end{aligned}$$

for all $n \geq n_0$. This implies that $\|S_t(\widetilde{z}) - \xi\| < \epsilon$ for each $t \geq n_0 t_0$. Now we repeat arguments from the proof of the previous theorem to get the locally uniform convergence of S to ξ. □

Remark 3.2. In the case of the open Hilbert ball it is worth comparing Theorem 3.2 with Proposition 4.3 in [19]. For the convenience of the reader we recall this proposition.

Proposition 3.3 [16]. *Let B be the open unit ball in a Hilbert space H. Let $\mathcal{G}$ be a class of k_B nonexpansive mappings on B which satisfies the Denjoy-Wolff property and let $S = \{S_t\}_{t\geq 0}$ be a one parameter continuous semigroup of k_B-nonexpansive mappings on B which has no common fixed point in B. If $S_{t_0} \in \mathcal{G}$ for at least one $t_0 > 0$, then there is a point $\xi \in \partial B$ such that S converges to ξ, as t tends to infinity, uniformly on each compact subset of B.*

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