ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA

VOL. LIII, 7 SECTIO A 1999

DANIEL GIRELA

Some results on univalent functions with positive Hayman index

Dedicated to Professor Jan Krzyż on the occasion of his 75^{th} *birthday*

ABSTRACT. Let $S(\alpha, 0)$ be the subclass of the familiar class S of normalized univalent functions consisting of functions with Hayman's index α and the unique direction of maximal growth coinciding with the real axis.

In this paper we give an example of $f \in S(\alpha, 0)$ such that $f/k \notin BMOA$, where k is the Koebe function. However, if $\alpha > 0$ and C is the class of close-to-convex functions then $f/k \in BMOA$ for any $f \in S(\alpha, 0) \cap C$. Moreover, if $\alpha > 0$ and $f \in S(\alpha, 0)$ then $k/f \in H^p$ for all $p < \infty$.

1. Introduction and statement of results. Let Δ denote the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. If f is a function which is analytic in Δ and $0 < r < 1$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p}, \qquad 0 < p < \infty,$$

$$M_\infty(r, f) = \max_{|z|=r} |f(z)|.$$

1991 *Mathematics Subject Classification*. 30C55, 30C45.

This research has been supported in part by a grant from "El Ministerio de Educación y Cultura, Spain" (PB97-1081) and by a grant from "La Junta de Andalucía".

For $0 < p \leq \infty$, the Hardy space H^p consists of those functions f, analytic in Δ, for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer to [8] for the theory of Hardy spaces. Let S be the class of functions f analytic and univalent in Δ with $f(0) = 0$, $f'(0) = 1$.

The leading example is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n.$$

Hayman proved in [12] (see also chapter 5 of [9]) that if $f \in S$ then

$$\alpha(f) = \lim_{r \to 1} (1-r)^2 M_\infty(r,f)$$

exists and $0 \leq \alpha(f) \leq 1$. Moreover, $\alpha(f) = 1$ if and only if f is a rotation of the Koebe function.

The number $\alpha(f)$ is called the Hayman index of f. Hayman's regularity theorem [12] asserts that if $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S$ then

$$\alpha(f) = \lim_{n \to \infty} \frac{|a_n|}{n}.$$

If $f \in S$ has the Hayman index $\alpha(f) > 0$, then there is a unique direction of maximal growth, i.e., there is a unique $\theta_0 \in [0, 2\pi)$ such that

$$\lim_{r \to 1} (1-r)^2 |f(re^{i\theta_0})| = \alpha(f).$$

Given $0 < \alpha \leq 1$ and $\theta_0 \in [0, 2\pi)$, we shall denote by $S(\alpha, \theta_0)$ the set of those $f \in S$ with the Hayman index $\alpha(f) = \alpha$ and θ_0 as the direction of maximal growth.

Notice that the elements of $S(\alpha, \theta_0)$ are rotations of those of $S(\alpha, 0)$, hence we shall restrict ourselves to this class. Bazilevich [4], [5] (see also [9, p. 160]) proved that if $\alpha > 0$, $f \in S(\alpha, 0)$ and

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$$

then

$$\sum_{n=1}^{\infty} n \left| \gamma_n - \frac{1}{n} \right|^2 \leq \frac{1}{2} \log \frac{1}{\alpha}.$$

Notice that the left hand side term is $\frac{1}{4}\mathcal{D}(\log(f/k))$, where

$$\mathcal{D}(F) = \frac{1}{\pi}\iint_{\Delta} |F'(z)|^2 \, dx \, dy$$

is the Dirichlet integral of F.

The following well known inequality of Lebedev and Milin [16] (see [9, p. 143], [17, p. 36] or [18, Th. 2.3]) can be used to obtain upper bounds on the integral means of $\exp(F)$ in terms of $\mathcal{D}(F)$.

First Lebedev-Milin inequality. *Let $F(z) = \sum_{j=1}^{\infty} A_j z^j$ be a function which is analytic in Δ with $F(0) = 0$ and let $G(z) = \exp(F(z)) = \sum_{k=0}^{\infty} D_k z^k$, $z \in \Delta$. Then,*

$$(1) \qquad \sum_{k=0}^{\infty} |D_k|^2 \le \exp\left(\sum_{k=1}^{\infty} k|A_k|^2\right).$$

The left hand side of (1) is simply $\|G\|_{H^2}^2$. Hence, (1) can be written as

$$(2) \qquad \|G\|_{H^2}^2 \le \exp\left(\sum_{k=1}^{\infty} k|A_k|^2\right) = \exp(\mathcal{D}(F)).$$

If $0 < \alpha \le 1$, $f \in S(\alpha, 0)$, $0 < p < \infty$ and we take

$$G(z) = \left(\frac{f(z)}{k(z)}\right)^{p/2}, \qquad F(z) = \frac{p}{2}\log\frac{f(z)}{k(z)}$$

then, using Bazilevich's theorem and (2), we obtain that $f/k \in H^p$ and

$$(3) \qquad \left\|\frac{f}{k}\right\|_{H^p} \le \alpha^{-p/2}.$$

This was proved by Hu Ke and Dong Xinhan in [14].

In the same paper Ke and Xinhan gave an example which shows that f/k need not belong to $BMOA$, the space of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\Delta$. We refer to [3] and [11] for the theory of $BMOA$-functions. In section 2 we shall give a new proof of this result. Indeed, we shall show that it can be deduced from a theorem of Hayman and Kennedy [13].

Next we study these questions for certain subclasses of the class S. We recall that a function $f \in S$ is said to be a support point of S if there exists a continuous linear functional J defined on the space of analytic functions in Δ, J nonconstant on S, such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in S\}$. The set of all support points of S will be denoted by $\operatorname{Supp}(S)$. Brickman and Wilken proved in [7] that if $f \in \operatorname{Supp}(S)$ then f is analytic in the closed unit disc $\overline{\Delta}$ except for a pole of order two at one point of the unit circle. From this we easily obtain

Proposition 1.

(i) *If f is a support point of S then it has a positive Hayman index.*

(ii) *If $\alpha > 0$ and $f \in S(\alpha, 0) \cap \mathrm{Supp}(S)$ then f is analytic in $\overline{\Delta}$ except for a pole of order two at 1 and*

$$f/k \in H^{\infty}. \tag{4}$$

Let S^* denote the subset of S consisting of those functions $f \in S$ for which $f(\Delta)$ is starlike with respect to 0.

Hayman proved that if $f \in S$ has a positive Hayman index then $f \notin H^{1/2}$. Using this result, Eenigenburgh and Keogh [10] proved that the rotations of the Koebe function are the only starlike functions with positive Hayman index. Consequently, $S(\alpha, 0) \cap S^* = \emptyset$, if $0 < \alpha < 1$, while, $S(1,0) \cap S^* = \{k\}$, and then we see that the following assertion is trivially true.

(5) If $\alpha > 0$ and $f \in S(\alpha, 0) \cap S^*$ then f satisfies (4).

A function f analytic in Δ is said to be close-to-convex if there exist a real number β and a function $g \in S^*$ such that

$$\mathrm{Re}\, \frac{z f'(z)}{e^{i\beta} g(z)} > 0, \quad z \in \Delta.$$

Every close-to-convex function is univalent (see e.g. Theorem 2.11 of [18]). We shall denote by $\mathcal{C}$ the class of all functions f analytic in Δ with $f(0) = 0$ which are close-to-convex and with $f'(0) = 1$. Clearly, $S^* \subset \mathcal{C} \subset S$. In view of the above, it is natural to ask whether or not (5) is true with $\mathcal{C}$ in the place of S^*. We shall show that the answer to this question is negative. In fact, we can prove the following result.

Theorem 1. *For every $\alpha \in (0,1)$ there exists a function $f \in S(\alpha, 0) \cap \mathcal{C}$ such that f/k is not bounded.*

However, we can also prove the following.

Theorem 2. *If $\alpha > 0$ and $f \in S(\alpha, 0) \cap \mathcal{C}$ then $f/k \in BMOA$.*

The proofs of Theorem 1 and Theorem 2 will be presented in section 3. Finally, in section 4 we shall be dealing with quotients of the form k/f with $f \in S(\alpha, 0)$, $\alpha > 0$.

2. Recalling a theorem of Hayman and Kennedy. Before embarking into the mentioned theorem of Hayman and Kennedy, let us recall that a function F, analytic in Δ is said to be a Bloch function if

$$\sup_{|z|<1} (1-|z|^2)|F'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. We refer to [1] for the theory of Bloch functions. It is clear that if $F \in \mathcal{B}$ then

$$M_\infty(r, F) = \mathrm{O}\left(\log \frac{1}{1-r}\right), \quad \text{as } r \to 1. \tag{6}$$

Also, it is well known that $BMOA \subset \mathcal{B}$ (see e.g. [3]).

Hayman and Kennedy proved in [13] the following result.

Theorem HK. *Let $\mu(r) \downarrow 0$, as $r \uparrow 1$. Then, there exists $f \in S$ with positive Hayman index and with $\theta_0 = 0$ as direction of maximal growth such that*

$$\limsup_{r\to 1} \frac{\log|f(-r)|}{\mu(r)\left(\log \frac{1}{1-r}\right)^{1/2}} > 0.$$

If we take $\mu(r) = (\log[1/(1-r)])^{-1/8}$ we obtain a function $f \in S(\alpha, 0)$ for some $\alpha > 0$ such that

$$\log|f(-r)| \neq \mathrm{O}\left((\log[1/(1-r)])^{1/4}\right), \quad \text{as } r \to 1.$$

In particular, $|f(-r)| \neq \mathrm{O}(\log[1/(1-r)])$, as $r \to 1$ and then

$$\left|\frac{f(-r)}{k(-r)}\right| \neq \mathrm{O}(\log[1/(1-r)]), \quad \text{as } r \to 1.$$

Then, having in mind the the above comments about Bloch functions, we deduce that f/k is not a Bloch function and, hence, it does not belong to $BMOA$.

3. Close to convex functions with positive Hayman index

Proof of Theorem 1. Take $0 < \alpha < 1$ and let

$$f(z) = \alpha k(z) + \frac{1-\alpha}{2} \log\frac{1+z}{1-z}, \quad z \in \Delta.$$

Then

$$f'(z) = \alpha k'(z) + (1-\alpha)\frac{1}{1-z^2} = \frac{1}{(1-z)^2}\left(\alpha\frac{1+z}{1-z} + (1-\alpha)\frac{1-z}{1+z}\right).$$

It is clear that

$$\operatorname{Re}\left(\alpha\frac{1+z}{1-z} + (1-\alpha)\frac{1-z}{1+z}\right) > 0, \quad z \in \Delta,$$

and then we have that $\operatorname{Re}\{zf'(z)/k(z)\} > 0$, $z \in \Delta$. Hence, f is close-to-convex and then it is easy to see that $f \in \mathcal{C}$. The fact that $f \in S(\alpha, 0)$ then is clear. Also, $|f(-r)| \to \infty$ as $r \to 1$ and hence, f/k is not bounded. This finishes the proof.

Proof of Theorem 2. Suppose that $f \in S(\alpha,0) \cap \mathcal{C}$ with $\alpha > 0$. There exist $\beta \in \mathbb{R}$ and a function $g \in S^*$ such that

$$\text{(7)} \qquad \operatorname{Re}\frac{zf'(z)}{e^{i\beta}g(z)} > 0, \quad z \in \Delta.$$

Set

$$\text{(8)} \qquad P(z) = \frac{zf'(z)}{e^{i\beta}g(z)}, \quad z \in \Delta,$$

and,

$$\text{(9)} \qquad \lambda = P(0), \quad (\text{hence, } \lambda = e^{-i\beta}), \quad \eta = \lambda^{-1} = e^{i\beta}.$$

Let us also define

$$\text{(10)} \qquad H(z) = \lambda\frac{1+\eta^2 z}{1-z}, \quad z \in \Delta.$$

Then H is a conformal mapping from Δ onto the right half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $H(0) = \lambda$. Notice that P is analytic in Δ, $P(\Delta) \subset \mathbb{H}$ and $P(0) = H(0)$. Hence, (see e.g. chapter 2 of [18]) P is subordinate to H, (in short, $P \prec H$).

We have,

$$\text{(11)} \qquad f'(z) = \eta\frac{g(z)}{z}P(z).$$

Since $P \prec H$, it is clear that

$$\text{(12)} \qquad P(r) = O\left(\frac{1}{1-r}\right), \quad \text{as } r \to 1.$$

Krzyz [15] proved that

$$|f'(r)| \sim \alpha k'(r), \quad \text{as } r \to 1. \tag{13}$$

Having in mind that, as we mentioned in section 1, the rotations of the Koebe function are the only starlike with positive Hayman index, and using (11), (12) and (13), we deduce that g is the Koebe function. Hence, we have

$$f'(z) = \eta \frac{1}{(1-z)^2} P(z). \tag{14}$$

Set $F = f/k$. Then, a simple computation gives

$$z^2 F'(z) = \eta z P(z) - (1-z^2) f(z) = f_1(z) - f_2(z), \tag{15}$$

where,

$$f_1(z) = \eta z P(z), \qquad f_2(z) = (1-z^2) f(z), \qquad z \in \Delta. \tag{16}$$

Our next goal is to obtain an upper bound for $M_2(r, F')$. In view of (15), we shall consider $M_2(r, f_1)$ and $M_2(r, f_2)$ separately. The following well known lemma (see e.g. [8, p. 65]) will be used to obtain our estimates.

Lemma 1. *If* $1 < p < \infty$, *then*

$$\int_0^{2\pi} \left| \frac{1}{1 - re^{i\theta}} \right|^p d\theta = \mathrm{O}\left(\frac{1}{(1-r)^{p-1}} \right), \quad \text{as } r \to 1.$$

We start working with f_1. Since $P \prec H$, we have $M_2(r, P) \leq M_2(r, H)$ $(0 < r < 1)$ (see Theorem 2.1 of [18]). Then, using Lemma 1, we easily obtain

$$M_2(r, f_1) = \mathrm{O}\left(\frac{1}{(1-r)^{1/2}} \right), \quad \text{as } r \to 1. \tag{17}$$

In order to obtain an upper bound for $M_2(r, f_2)$ we shall start working with f_2'. A simple computation gives

$$f_2'(z) = -2z f(z) + (1-z^2) f'(z) = -2z f(z) + \eta \frac{1+z}{1-z} P(z). \tag{18}$$

Since $f \in S$, we have (see [2], Theorem 5.1 of [18] or chapter 7 of [9])

$$M_2(r, f) = \mathrm{O}\left(\frac{1}{(1-r)^{3/2}} \right), \quad \text{as } r \to 1. \tag{19}$$

We shall make use of Baernstein $\star$-function to estimate the integral means of $\{(1+z)/(1-z)\}\, P(z)$.

Recall [2] (see also [9, Chapt. 7]) that if u is a real-valued function defined in Δ such that, for $0 < r < 1$, the function $\theta \to u(re^{i\theta})$ is integrable on $[-\pi, \pi]$, the function $u^\star$ is defined on $\{z \in \mathbb{C} : 0 < |z| < 1, \operatorname{Im} z \geq 0\}$ by

$$u^\star(re^{i\theta}) = \sup_{|E|=2\theta} \int_E u(re^{it})\, dt, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi,$$

where the supremum is taken over all measurable subsets E of $[-\pi, \pi]$ whose Lebesgue measure, denoted by $|E|$, is equal to 2θ.

Set

$$\begin{aligned} &u_1(z) = \log\left|\frac{1+z}{1-z}\right|, \qquad u_2(z) = \log|P(z)|, \\ &u(z) = \log\left|\frac{1+z}{1-z}P(z)\right|, \ z \in \Delta. \end{aligned} \tag{20}$$

The functions u_1, u_2 and u are harmonic in Δ. Since $P \prec H$, we have also $u_2 \prec \log|H|$ and then

$$u_2^\star \leq (\log|H|)^\star \tag{21}$$

(see Lemma 7 in p. 230 of [9]). Now,

$$\log|H(z)| = \log|1 + \eta^2 z| + \log\frac{1}{|1-z|}. \tag{22}$$

Since $|\eta| = 1$, we see that, for $0 < r < 1$, the functions $t \to \log|1 + \eta^2 re^{it}|$ and $t \to \log|1 + re^{it}|$ are equidistributed on $[-\pi, \pi]$ and then

$$\left(\log|1 + \eta^2 re^{i\theta}|\right)^\star = \left(\log|1 + re^{i\theta}|\right)^\star, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi. \tag{23}$$

Using (21), (22), Lemma 6 in p. 230 of [9] and having in mind that, for $0 < r < 1$, both the functions $\theta \to \log|1 + re^{i\theta}|$ and $\theta \to \log\frac{1}{|1-re^{i\theta}|}$ are symmetric functions of θ on $[-\pi, \pi]$, decreasing on $[0, \pi]$, we obtain that, for $0 < r < 1$ and $0 \leq \theta \leq \pi$,

$$\begin{aligned} u_2^\star(re^{i\theta}) &\leq \left(\log|1 + \eta^2 re^{i\theta}|\right)^\star + \left(\log\frac{1}{|1-re^{i\theta}|}\right)^\star \\ &= \left(\log|1 + re^{i\theta}|\right)^\star + \left(\log\frac{1}{|1-re^{i\theta}|}\right)^\star \\ &= \left(\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)^\star. \end{aligned} \tag{24}$$

Since $u = u_1 + u_2$, (24) and Lemma 6 in [9, p. 230] imply

(25)
$$u^*(z) \le u_1^*(z) + u_2^*(z) = 2\left(\log\left|\frac{1+z}{1-z}\right|\right)^* = \left(\log\left|\frac{1+z}{1-z}\right|^2\right)^*, \quad \text{for all } z.$$

Having in mind the definition of u and using Proposition 3 of [2] (see also Lemma 5 in p. 218 of [9]), we see that (25) implies

$$M_2\left(r, \frac{1+z}{1-z}P(z)\right) \le M_2\left(r, \left(\frac{1+z}{1-z}\right)^2\right), \quad 0 < r < 1,$$

which, using Lemma 1, gives

(26)
$$M_2\left(r, \frac{1+z}{1-z}P(z)\right) = O\left(\frac{1}{(1-r)^{3/2}}\right), \quad \text{as } r \to 1.$$

Now, (18), (19) and (26) imply

$$M_2(r, f_2') = O\left(\frac{1}{(1-r)^{3/2}}\right), \quad \text{as } r \to 1,$$

which, using Theorem 5.5 in p. 80 of [8], gives

$$M_2(r, f_2) = O\left(\frac{1}{(1-r)^{1/2}}\right), \quad \text{as } r \to 1.$$

This, with (15) and (17), implies

$$M_2(r, F') = O\left(\frac{1}{(1-r)^{1/2}}\right), \quad \text{as } r \to 1,$$

and consequently (see [8, Th. 5.4]) F belongs to the mean Lipschitz space $\Lambda^2_{1/2}$. Since $\Lambda^2_{1/2} \subset BMOA$, (see [6]), this proves that $F \in BMOA$ which ends the proof of Theorem 2.

4. Some results on functions of the form k/f with $f \in S(\alpha, 0)$, $\alpha > 0$. Let $0 < \alpha \le 1$, $f \in S(\alpha, 0)$ and $0 < p < \infty$. If we argue as in the proof of (3) but taking

$$G(z) = \left(\frac{k(z)}{f(z)}\right)^{p/2}, \qquad F(z) = \frac{p}{2}\log\frac{k(z)}{f(z)} = -\frac{p}{2}\log\frac{f(z)}{k(z)}$$

then, using Bazilevich's theorem and the “first Lebedev-Milin inequality” we obtain the following

Proposition 2. *Let $0 < \alpha \leq 1$ and $f \in S(\alpha, 0)$. Then $k/f \in H^p$ for all $p \in (0, \infty)$ and*

$$(27) \qquad \|k/f\|_{H^p} \leq \alpha^{-p/2}, \quad 0 < p < \infty.$$

We do not know whether Proposition 2 can be improved. However, we shall make some remarks. First, let us state the following

Lemma 2. *If f and g are two elements of S then f/g is a normal function.*

We recall that a function f which is meromorphic in Δ is a normal function if and only if

$$\sup_{z\in\Delta}(1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} < \infty.$$

We refer to [1] and chapter 9 of [18] for the theory of normal functions. Lemma 2 is an extension of the well known result that if $f \in S$ then it is normal (see Lemma 9.3 in p. 262 of [18]). The proof of Lemma 2 reduces to an elementary calculation involving the spherical derivative of f/g and will be omitted.

We do not know whether or not if $f \in S(\alpha, 0)$ with $\alpha > 0$ then the function k/f does belong to H^∞ or at least to $BMOA$. However, we should remark that, if they exist, the counterexamples must be essentially distinct from those given in the previous sections to show that f/k may not be bounded or may not belong to $BMOA$. Indeed, to prove that these examples worked we studied the behaviour of the function f in a direction distinct from that of maximal growth. The following lemma shows that this would not work in the case of k/f.

Lemma 3. *Let $0 < \alpha \leq 1$ and $f \in S(\alpha, 0)$. Then the following two conditions are equivalent.*

(i) *k/f is not bounded;*

(ii) *there exists a sequence $\{z_n\} \subset \Delta$ such that*

$$\lim_{n\to\infty} z_n = 1, \quad \text{and} \quad \lim_{n\to\infty}\left|\frac{k(z_n)}{f(z_n)}\right| = \infty.$$

Proof. The implication (ii) $\Longrightarrow$ (i) is trivial. To prove the reverse implication, take $f \in S(\alpha, 0)$ with $\alpha > 0$ and suppose that k/f is not bounded.

If $|\xi| = 1$ and $\xi \neq 1$, we have $\limsup_{\substack{z\to\xi \\ z\in\Delta}} |k(z)| < \infty$ and, by the Koebe 1/4-Theorem $\liminf_{\substack{z\to\xi \\ z\in\Delta}} |f(z)| \geq 1/4$. Then, clearly

$$\limsup_{\substack{z\to\xi \\ z\in\Delta}} |k(z)/f(z)| < \infty, \quad \text{if } |\xi| = 1, \xi \neq 1.$$

Consequently, since k/f is not bounded,

$$\limsup_{\substack{z\to 1 \\ z\in\Delta}} |k(z)/f(z)| = \infty. \tag{28}$$

This is equivalent to (ii).

Remark. We should observe that if $f \in S(\alpha, 0)$ is such that k/f is not bounded then, since by Lemma 2 the function k/f is normal, ∞ cannot be an asymptotic value of k/f at 1, i.e. there is no curve Γ in Δ ending at 1 such that $\lim_{\substack{z\to 1 \\ z\in\Gamma}} |\{k(z)/f(z)\}| = \infty$. Otherwise k/f would have the non-tangential limit ∞ at 1 (see Theorem 9.3 of [18]) contrarily to the fact k/f is bounded along the radius ending at 1. Consequently, the "lim sup" in (28) is attained in "discrete way".

References

[1] Anderson, J. M., J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.

[2] Baernstein, A., *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.

[3] Baernstein II, A., *Analytic functions of bounded mean oscillation*, Aspects of contemporary complex analysis, Academic Press, London, 1980, pp. 3–36.

[4] Bazilevic, I. E., *Coefficient dispersion of univalent functions*, Math. Sbornik **68 (110)** (1965), 549–560 (Russian); English transl. in Amer. Math. Soc. Transl. (2) (1968) **71**, 168–180.

[5] ———, *On a univalence criterion for regular functions and the dispersion of their coefficients*, Math. Sbornik **74 (116)** (1967), 133–146 (Russian); English transl. in Math. USSR Sb. **3 (1967)**, 123–137.

[6] Bourdon, P., J. Shapiro and W. Sledd, *Fourier series, mean Lipschitz spaces and bounded mean oscillation*, Analysis at Urbana 1, Proceedings of the Special Year in Modern Analysis at the University of Illinois, 1986-87 (E. R. Berkson, N. T. Peck and J. Uhl, eds.), London Math. Soc. Lecture Note Series 137, Cambridge Univ. Press, 1989, pp. 81–110.

[7] Brickman, L. and D. Wilken, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc. **42, No. 2** (1974), 523–528.

[8] Duren, P. L., *Theory of H^p Spaces*, Academic Press, New York, 1970.

[9] ———, *Univalent Functions*, Springer-Verlag, New York, 1983.

[10] Eenigenburg, P. J. and F. R. Keogh, *The Hardy class of some univalent functions and their derivatives*, Michigan Math. J. **17** (1970), 335–346.
[11] Garnett, J. B., *Bounded analytic functions*, Academic Press, New York, 1981.
[12] Hayman, W. K., *The asymptotic behavior of p-valent functions*, Proc. London Math. Soc. **5** (1955), 257–284.
[13] ______ and P. B. Kennedy, *On the growth of multivalent functions*, J. London Math. Soc. **33** (1958), 333–341.
[14] Ke, H. and D. Xinhan, *The asymptotic behavior of univalent functions*, Proc. Amer. Math. Soc. **100, No. 1** (1987), 75–81.
[15] Krzyz, J., *On the maximum modulus of univalent functions*, Bull. Acad. Polon. Sci. **3** (1955), 203–206.
[16] N. A. Lebedev, I. M. Milin, *An inequality*, (Russian), Vestnik Leningrad. Univ. **20,** (1965), no. 19, 157–158.
[17] Milin, I. M., *Univalent functions and orthonormal systems*, vol. 49, Amer. Math. Soc. Transl. of Math. Monographs, Providence, R. I., 1977.
[18] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck und Ruprecht, Göttingen, 1975.

Análisis Matemático, Facultad de Ciencias
Universidad de Málaga
29071 Málaga, Spain
e-mail: girela@anamat.cie.uma.es

received October 7, 1998