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Disproof of a conjecture on univalent functions

ABSTRACT. We disprove the Gruenberg-Rønning-Ruscheweyh conjecture, namely that $\operatorname{Re}(d*(g/z))(z) > 0$, $|z| < 1$, holds for $g \in \mathcal{S}$, the set of normalized univalent functions in the unit disk $\mathbb{D}$, and d analytic with $|d'(z)| \le \operatorname{Re} d(z)$ in $\mathbb{D}$, $d(0) = 1$. Here $*$ stands for the Hadamard product.

1. Introduction. Let $\mathcal{A}$ denote the space of analytic functions in the unit disc $\mathbb{D}$ (with the topology of local uniform convergence), and write $f \in \mathcal{A}_0$ if $f \in \mathcal{A}$ satisfies $f(0) = 1$. Let $f * g$ be the Hadamard product of $f, g \in \mathcal{A}$, and, as usual, $\mathcal{S}$ the class of univalent functions $f \in \mathcal{A}$, normalized by $f(0) = 0 = f'(0) - 1$. Finally let

$$\mathcal{D} := \{d \in \mathcal{A}_0 : |d'(z)| \le \operatorname{Re} d(z),\ z \in \mathbb{D}\}. \tag{1.1}$$

The following conjecture was made by Gruenberg, Rønning, and Ruscheweyh [5].

Conjecture. *For $d \in \mathcal{D}$ and $f_1, f_2 \in \mathcal{S}$ we have*

$$\operatorname{Re}\left(d(z) *_z \frac{1}{z}\int_0^z (f_1(t) *_t f_2(t))\frac{dt}{t}\right)(z) > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

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Here $*_x$ stands for the Hadamard product with respect to the variable x. Note that choosing $f_2(z) = z(1-z)^{-2}$ (the Koebe function) we arrive at a weaker form of (1.2):

$$\text{(1.3)} \qquad \operatorname{Re}\left(d * \frac{f}{z}\right)(z) > 0, \quad z \in \mathbb{D},\ d \in \mathcal{D},\ f \in \mathcal{S}.$$

In that same paper the truth of (1.2) was established for f_1, f_2 close -to-convex and/or typically real univalent. Also, (1.2) is known to hold in general for a number of special functions in $\mathcal{D}$, including, for instance,

$$d(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \qquad \sum_{k=1}^{\infty}(k+1)|a_k| \le 1,$$

$$d_n(z) = \sum_{k=0}^{n}(k+1)(z/4)^k, \ n \in \mathbb{N}.$$

Other partial verifications, some for (1.3) only, are due to Rønning [7], and others. In this note we disprove (1.2) and (1.3).

Steps towards an overall decision on (1.2) were taken by Fournier and Ruscheweyh [2], and recently by Greiner [4].

Let $\mathcal{D}^+ := \{g \in \mathcal{A}_0 : \forall d \in \mathcal{D},\ \operatorname{Re}(d * g)(z) > 0,\ z \in \mathbb{D}\}$, so that (1.2) and (1.3) take the equivalent forms

$$\text{(1.4)} \qquad \frac{1}{z}\int_0^z (f_1(t) *_t f_2(t))\frac{dt}{t} \in \mathcal{D}^+, \quad f_1, f_2 \in \mathcal{S},$$

and

$$\text{(1.5)} \qquad f/z \in \mathcal{D}^+, \quad f \in \mathcal{S},$$

respectively. A by-product of a general theory developed in [4] was the following simple characterization of $\mathcal{D}^+$, namely

$$\text{(1.6)} \qquad \mathcal{D}^+ = \overline{\text{co}}\left\{\frac{1}{1-xz} + y\frac{xz}{(1-xz)^2} : \ |x|, |y| \le 1\right\},$$

where $\overline{\text{co}}$ denotes the closed convex hull. We shall give a quick proof of this fact in Sect.2. Furthermore, the structure of $\mathcal{D}^+$ as exhibited in (1.6) raised stronger doubts concerning the truth of (1.4) and (1.5). And indeed, heavy computational work produced a function $f \in \mathcal{S}$ for which f/z was actually separated from the compact and convex set $\mathcal{D}^+$ by an explicitly constructed linear functional. Once this was established, the data thus obtained were used to produce a concrete function $p \in \mathcal{D}$, in fact a polynomial of degree 3, which fails to have the property described in (1.3), namely

$$\begin{aligned} p(z) = 1 &+ \left(-\frac{161}{400} + \frac{69i}{250}\right) z + \left(-\frac{237}{2500} - \frac{277i}{1000}\right) z^2 \\ &+ \left(\frac{9}{400} + \frac{949i}{10000}\right) z^3. \end{aligned} \tag{1.7}$$

The actual counterexample to (1.3), as derived in Sect. 2, lacks rigidity as the verification of $p \in \mathcal{D}$ is just numerical (not really questionable, though). And also the question remained open whether the conjecture might survive if one further restricts the functions in $\mathcal{D}$ to those with real Taylor coefficients only. That even this is not true can be shown by the following more sophisticated argument.

As a convex and compact set $\mathcal{D}$ contains its extreme points and, of course, it would be sufficient to prove the conjectures for the extreme points only. In [2] the question of a characterization of these extreme points was raised. It was shown that $d \in \mathcal{D}$ is necessarily an extreme point of $\mathcal{D}$ if d is analytic in $\overline{\mathbb{D}}$ and satisfies $\gamma_d(z) = 0, |z| = 1$, where

$$\gamma_d(z) := \operatorname{Re} d(z) - |d'(z)|. \tag{1.8}$$

Special examples are the functions

$$\frac{1 + m_n x z^n}{1 - m_n x z^n} \in \mathcal{D}, \quad m_n = \sqrt{n^2+1} - n, \ n \in \mathrm{N}, \ |x| = 1, \tag{1,9}$$

(for which (1.2) and (1.3) are only partially verified so far). Kühnau [6] gave a negative answer to (a slightly transformed form of) the question raised in [2], namely whether the functions (1.9) are the only ones with this property. Later, Fournier and Ruscheweyh [3] obtained a fairly complete picture of the set of those functions. This latter information, together with a criterion of Bshouty and Hengartner [1], which concerns the form of continuous linear functionals over $\mathcal{A}$ that are maximized by the Koebe function over $\mathcal{S}$, can be used to show the existence of functions in $\mathcal{D}$, even with real coefficients, contradicting (1.3). We give the details in Sect. 3.

In spite of the fact that (1.2) and (1.3) are false we believe that they can still be a useful source of valid estimates in $\mathcal{S}$, if considered for specific members in $\mathcal{D}$. It is still a fascinating open problem to characterize (independently) the set of functions $d \in \mathcal{A}_0$ satisfying $\operatorname{Re}\left(d * \frac{f}{z}\right)(z) > 0$, $z \in \mathbb{D}$, $f \in \mathcal{S}$.

The authors wish to thank Richard Fournier for the countless discussions on the subject.

2. The first counterexample

2.1. The structure of $\mathcal{D}^+$. We introduce the following notation: for $\mathcal{V} \subset \mathcal{A}_0$ set

$$\mathcal{V}^+ := \{w \in \mathcal{A}_0 : \mathrm{Re}(v * w)(z) > 0,\ z \in \mathbb{D},\ v \in \mathcal{V}\}, \tag{2.1}$$

and $\mathcal{V}^{++} := (\mathcal{V}^+)^+$. Note that this concept is closely related to the one used in the duality theory for Hadamard products (cf. [8]). Let

$$\mathcal{V}_0 := \left\{ \frac{1}{1-xz} + y\,\frac{xz}{(1-xz)^2} : |x|, |y| \le 1 \right\}.$$

It is then easily verified that

$$\mathcal{D} = \mathcal{V}_0^+, \quad \mathcal{D}^+ = \mathcal{V}_0^{++}.$$

A set $\mathcal{V} \subset \mathcal{A}_0$ is said to be *complete* if $f \in \mathcal{V}$ implies $f_x \in \mathcal{V}$ for all $|x| \le 1$, where $f_x(z) := f(xz), z \in \mathbb{D}$.

Note that $\mathcal{V}_0$ is compact and complete, so that (1.6) follows immediately from the following general theorem.

Theorem 1. *Let $\mathcal{V} \subset \mathcal{A}_0$ be compact and complete. Then $\mathcal{V}^{++} = \overline{\mathrm{co}}\,\mathcal{V}$.*

Proof. We may assume that $\mathcal{V}$ is not empty.
1) First, we prove the assertion for $\mathcal{V}$ convex. Then we need to show that $\mathcal{V} = \mathcal{V}^{++}$. That $\mathcal{V}$ is a subset of $\mathcal{V}^{++}$ is obvious. To show that $\mathcal{V}$ includes $\mathcal{V}^{++}$ we assume that there exists $h \in \mathcal{V}^{++} \setminus \mathcal{V}$. Then by a standard separation theorem in the locally convex topological vector space $\mathcal{A}$ we find a continuous linear functional λ on $\mathcal{A}$ satisfying

$$\mathrm{Re}\,\lambda(h) < \alpha < \min_{v \in \mathcal{V}} \mathrm{Re}\,\lambda(v),$$

for some $\alpha \in \mathbb{R}$. In fact, we may assume $\alpha = 0$, since, otherwise, we can replace λ by the functional $f \mapsto \lambda(f) - \alpha f(0)$. By Toeplitz' representation theorem there exists a function g, analytic in $\overline{\mathbb{D}}$, such that $\lambda(f) = (g * f)(1)$ holds for all $f \in \mathcal{A}$. Since $v_0 \equiv 1$ belongs to $\mathcal{V}$, we have

$$0 < \mathrm{Re}\,\lambda(v_0) = \mathrm{Re}(g * v_0)(1) = \mathrm{Re}\,g(0),$$

which implies that

$$w(z) := \frac{g(z) - i\,\mathrm{Im}\,g(0)}{\mathrm{Re}\,g(0)}$$

is well-defined and is in $\mathcal{A}_0$. Using the completeness of $\mathcal{V}$ we find that $\mathrm{Re}(w * v)(z) > 0$, for all $z \in \mathbb{D}$ and $v \in \mathcal{V}$, which implies $w \in \mathcal{V}^+$.

On the other hand, $\mathrm{Re}(w*h)(1) < 0$ and hence $h \notin \mathcal{V}^{++}$, a contradiction. Therefore $\mathcal{V} \supset \mathcal{V}^{++}$, as asserted.

2) Now consider the general case for $\mathcal{V}$. We show that $\mathcal{V}^{+} = (\overline{\mathrm{co}}\,\mathcal{V})^{+}$. Clearly $\mathcal{V}^{+} \supset (\overline{\mathrm{co}}\,\mathcal{V})^{+}$. Let $w \in \mathcal{V}^{+}$, so that $\mathrm{Re}(v * w) > 0$ in $\mathbb{D}$ for each $v \in \mathcal{V}$. Since the inequality is invariant under convex combinations, and both $\mathcal{V}$ and $\overline{\mathrm{co}}\,\mathcal{V}$ are compact, it is clear that the same inequality holds for each $v \in \overline{\mathrm{co}}\,\mathcal{V}$ Hence $w \in (\overline{\mathrm{co}}\,\mathcal{V})^{+}$.

1) and 2) together give $\mathcal{V}^{++} = (\overline{\mathrm{co}}\,\mathcal{V})^{++} = \overline{\mathrm{co}}\,\mathcal{V}$, as asserted. ■

Corresponding representations of $\overline{\mathrm{co}}\,\mathcal{V}$ in terms of Hadamard duality are valid with the right half plane in (2.1) replaced by an arbitrary convex set Ω with $1 \in \Omega$. For details and related results see [4].

2.2. The counterexample. We numerically verify that p in (1.7) is a member of $\mathcal{D}$. Figure 1 shows the graph of $\gamma_p(e^{it})$, which indicates that this function has two local minima, close to $t = 0.0$ and to $t = 2.7$. A numerical search for these minima with the **FindMinimum** utility in the software package **Mathematica 3.0** yields, with 16 digits of precision,

$$\gamma_p(e^{0.03942138484132064\, i}) = 0.0001244567844780886...,$$
$$\gamma_p(e^{2.72322228407483\, i}) = 0.006637906373948358.... .$$

Hence we may assume that $p \in \mathcal{D}$.

The Pick function $f_0(z) := 1 + \frac{2(1-z)}{z}\left(1 - z - \sqrt{1 - z + z^2}\right)$ maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus [-1, -7 + 4\sqrt{3}]$, and $f_1(z) := e^{it/2}\,\frac{1+f_0(z)}{1-e^{it}f_0(z)}$, $t := 1/10$, maps $\mathbb{D}$ onto the right half-plane minus a small circular slit emerging from the origin, tangentially to the positive real axis. Hence $f_2 := f_1^2$ is a univalent function (slit-mapping) in $\mathbb{D}$.

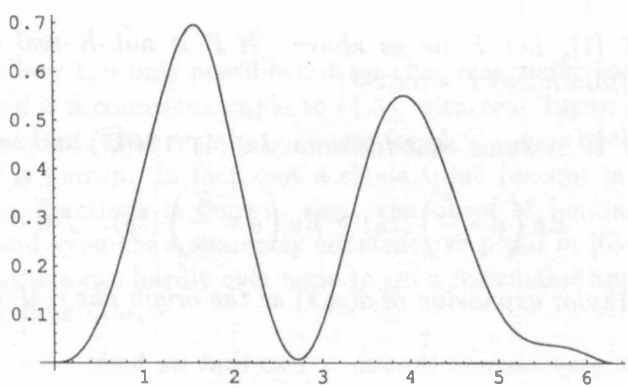

Figure 1

After a renormalization we obtain

$$f_3(z) := [f_2(z) - f_2(0)]/f_2'(0) \in \mathcal{S},$$

and a direct calculation yields the Taylor expansion

$$f_3(z) = z + \frac{3+3e^{it}}{8} z^2 + \frac{27+9e^{it}+2e^{2it}}{16} z^3 + \frac{162+270e^{it}+75e^{2it}+5e^{3it}}{128} z^4 + \cdots .$$

This gives

$$\operatorname{Re}\left(p * \frac{f_3}{z}\right)(1) = \frac{274482 - 276546\cos t + 1707\cos 2t + 1125\cos 3t}{1280000} + \frac{6466\sin t - 5371\sin 2t - 949\sin 3t}{256000} = 0.00112858...,$$

a contradiction to (1.3).

3. The second counterexample. One of the two basic tools for the proof of the existence of 'real' counterexamples to (1.3) is a result of Bshouty and Hengartner [1], which we describe (and adapt) first. Let F be a real valued functional on $\mathcal{A}$ which admits a Gâteaux derivative, i.e. for each $f \in \mathcal{A}$ there exists a continuous linear functional L_f on $\mathcal{A}$ such that

$$F(f + \varepsilon\varphi + o(\varepsilon)) = F(f) + \varepsilon \operatorname{Re} L_f(\varepsilon) + o(\varepsilon), \quad \varepsilon \to 0, \ \varphi \in \mathcal{A}.$$

F is said to be K-real if $\operatorname{Re} L_K(izq(z)) = 0$, where $K(z) := z(1-z)^{-2}$ is the Koebe function, and q is an arbitrary member of $\mathcal{H}(\overline{\mathbb{D}})$, with $q(0) = 0$ and with all Taylor coefficients about the origin real (we denote this class of functions by $\mathcal{Q}$).

Lemma 1 [1]. *Let F be as above. If F is not K-real then K cannot maximize (minimize) F over $\mathcal{S}$.*

Corollary 1. *Assume that for some $d \in \mathcal{A}_0 \cap \mathcal{H}(\overline{\mathbb{D}})$ and some $z_0 \in \partial\mathbb{D}$*

$$\operatorname{Re}\left(d * \frac{f}{z}\right)(z_0) \geq \operatorname{Re}\left(d * \frac{K}{z}\right)(z_0), \quad f \in \mathcal{S}.$$

Then the Taylor expansion of $d(z_0 z)$ at the origin has real coefficients only.

Proof. We may assume that $z_0 = 1$ so that we have

$$\operatorname{Re}(zd * f)(1) > \operatorname{Re}(zd * K)(1), \ f \in \mathcal{S}, \ z \in \mathbb{D}.$$

Then the functional $F(f) := \mathrm{Re}(zd * f)(1), f \in \mathcal{A}$, fulfils the assumptions of Lemma 1 (with $L_f(\varphi) = F(\varphi)$), and takes its minimum over $\mathcal{S}$ at K. Hence it has to be K-real. This means that $\mathrm{Re}(zd * izq)(1) = 0$, $\forall q \in \mathcal{Q}$. Choosing $q = z^n, n \in \mathbb{N}$, we readily deduce that for this to be true a necessary condition is that all the Taylor coefficients of d are real. ■

The following lemma is stated in [3] in a slightly different form and is immediately derived from those results.

Lemma 2. *Let B be a finite Blaschke product. Then there exists a unique function $d \in \mathcal{H}(\overline{\mathbb{D}}) \cap \mathcal{D}$ such that $h := d'/B \in \mathcal{H}(\overline{\mathbb{D}})$ with $h(z) \neq 0$ for $z \in \overline{\mathbb{D}}, h(0) > 0$, and $|h(z)| = \mathrm{Re}\, d(z)$ on $\partial\mathbb{D}$.*

Note that if B has real Taylor coefficients, then the corresponding d must have the same property. This follows from the uniqueness of the representation given in Lemma 2, because $d(z)$ and $\overline{d(\overline{z})}$ belong to the same B.

Now we can construct the counterexample to (1.3): Choose a Blaschke product with real coefficients having at least 2 zeros in $\mathbb{D}$ that is nonvanishing at the origin. Then the corresponding d of Lemma 2 has real coefficients as well, and satisfies $d'(0) \neq 0$. Furthermore we note that, by the argument principle, there are (at least) three points $z_j \in \partial\mathbb{D}, j = 1, 2, 3$, such that $z_j d'(z_j) < 0$. Our construction implies that at least one of the functions $d(z_j z)$ has not only real Taylor coefficients, say $d(z_1 z)$. Now assume that conjecture (1.3) were true for d. Then, for all $f \in \mathcal{S}$,

$$\mathrm{Re}\left(d * \frac{f}{z}\right)(z_1) > 0 = \mathrm{Re}\, d(z_1) - |d'(z_1)|$$
$$= \mathrm{Re}\, d(z_1) + z_1 d'(z_1) = \mathrm{Re}\left(d * \frac{K}{z}\right)(z_1),$$

which, by Corollary 1, is only possible if $d(z_1 z)$ has real coefficients, a contradiction. Hence d is a counterexample to (1.3), with real Taylor coefficients.

It should be noted, however, that - except for (1.9) - none of the functions d of Lemma 2 is known. In fact, not a single value (except in the origin) of any of these functions is known, since the proof of Lemma 2 is nonconstructive, and even the apparently constructive proof in [6] is of such a complexity that one can hardly ever hope to get a reasonable approximation to any of these functions.

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