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PIOTR WANIURSKI

On zeros of functions in Bergman and Bloch spaces

Abstract. We generalize some necessary conditions for zero sets of A^p functions and Bloch functions obtained in [H] and [GNW], respectively.

1. Introduction. Let A^p, $0 < p < \infty$, denote the Bergman space of functions f analytic in the unit disc $\mathbb{D}$ satisfying

$$\|f\|_p = \left(\frac{1}{\pi} \iint_{\mathbb{D}} |f(z)|^p \, dx\, dy \right)^{1/p} < \infty \, .$$

A function f analytic in $\mathbb{D}$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \, .$$

The space of all Bloch functions will be denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ for which

$$(1 - |z|)|f'(z)| \to 0, \quad \text{as } |z| \to 1.$$

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For $0 < r < 1$, set
$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$
and let us define A^0 as the space of all functions f analytic in $\mathbb{D}$ and such that
$$M_\infty(r,f) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \quad \text{as } r \to 1.$$
The following strict inclusions are well known:
$$\mathcal{B}_0 \subset \mathcal{B} \subset A^0 \subset \bigcap_{0<p<\infty} A^p.$$

If f is an analytic function in $\mathbb{D}$, $f(0) \neq 0$ and $\{z_k\}_{k=1}^\infty$ is the sequence of its zeros repeated according to multiplicity and ordered so that $|z_1| \leq |z_2| \leq |z_3| \ldots$, then $\{z_k\}$ is said to be the sequence of ordered zeros of f.

In 1974 Horowitz [H] obtained the following necessary condition for ordered zeros of A^p functions.

Theorem H. *Assume that $f \in A^p$, $0 < p < \infty$, and $\{z_k\}$ is the ordered zero set of f. Then for all $\varepsilon > 0$,*
$$\sum_{z_k \neq 0} (1-|z_k|)\left(\log\left(\frac{1}{1-|z_k|}\right)\right)^{-1-\varepsilon} < \infty\ .$$

An analogous result for the space A^0 was obtained in [GNW]

Theorem GNW. *If $f \in A^0$ and $\{z_k\}$ is the ordered zero set of f, then for all $\varepsilon > 0$*
$$\sum_{|z_k|>1-\frac{1}{e}} (1-|z_k|)\left(\log\log\left(\frac{1}{1-|z_k|}\right)\right)^{-1-\varepsilon} < \infty\ .$$

These theorems are best possible in the sense that $\varepsilon > 0$ cannot be omitted. More precisely, Horowitz showed that for each $0 < p < \infty$ there is $f \in A^p$ such that the series
$$\sum_{z_k \neq 0} (1-|z_k|)\left(\log\left(\frac{1}{1-|z_k|}\right)\right)^{-1}$$
diverges.

It was also shown in [GNW] that there is a function $f \in \mathcal{B}_0$ for which
$$\sum_{|z_k|>1-\frac{1}{e}} (1-|z_k|)\left(\log\log\left(\frac{1}{1-|z_k|}\right)\right)^{-1} = \infty\ .$$
In this paper we generalize the above stated results and we prove the following theorems.

Theorem 1. *Let h be a nonincreasing function in $[r_0, 1)$ for some $0 < r_0 < 1$, such that $\lim\limits_{r\to 1^-} h(r) = 0$ and*

$$\int_{r_0}^{1} -h'(r)\log\frac{1}{1-r}\,dr < \infty\ . \tag{1}$$

If $f \in A^p$ with $f(0) \neq 0$, and $z_1, z_2, \dots$ are ordered zeros of f then

$$\sum_{|z_k|>r_0} (1-|z_k|)h(|z_k|) < \infty\ .$$

Theorem 3. *Let h be a nonincreasing function in $[r_0, 1)$ for some $1 - \frac{1}{e} < r_0 < 1$, such that $\lim\limits_{r\to 1^-} h(r) = 0$ and*

$$\int_{r_0}^{1} -h'(r)\log\log\frac{1}{1-r}\,dr < \infty\ .$$

If $f \in A^0$ with $f(0) \neq 0$, and $z_1, z_2, \dots$ are ordered zeros of f then

$$\sum_{|z_k|>r_0} (1-|z_k|)h(|z_k|) < \infty\ .$$

2. Necessary conditions for A^p zero sets. For a function f analytic in $\mathbb{D}$, let $n(r, f)$ denote the number of zeros of f in the disc $\{|z| \le r < 1\}$, where each zero is counted according to its multiplicity. We also set

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log r, \quad 0<r<1.$$

Note that if $f(0) \neq 0$, then

$$N(r,f) = \sum_{|z_k|\le r} \log\frac{r}{|z_k|}. \tag{2}$$

Indeed, for $0 < r < 1$ integration by parts gives

$$\begin{aligned} N(r,f) &= \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log r \\ &= [(n(t,f)-n(0,f))\log t]_0^r - \int_0^r \log t\,dn(t,f) + n(0,f)\log r \\ &= n(r,f)\log r - \sum_{|z_k|\le r}\log|z_k| = \sum_{|z_k|\le r}\log\frac{r}{|z_k|}\ . \end{aligned}$$

With this notation the Jensen formula for analytic functions can be written in the following form

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta = \log|f(0)| + N(r,f)\ . \tag{3}$$

For simplicity, if the function f is fixed we will write $n(r)$ and $N(r)$ instead of $n(r,f)$ and $N(r,f)$, respectively.

We need the following lemma due to Shapiro and Shields [SS].

Lemma SS. *Let $f \in A^p$, $f(0) \neq 0$. Then*

$$n(r) = \mathrm{O}\left(\frac{1}{1-r}\log\frac{1}{1-r}\right), \quad \textit{as} \quad r \to 1,$$
$$N(r) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \quad \textit{as} \quad r \to 1\,.$$

Now we are ready to prove the first result stated in the introduction.

Proof of Theorem 1. For $0 \le r < 1$ define

$$\varphi(r) = \sum_{|z_n|\le r} (1-|z_n|)\ ,$$

and note that by (2)

$$N(r) = \sum_{|z_n|\le r} \log\frac{r}{|z_n|} = \sum_{|z_n|\le r} \log\frac{1}{|z_n|} + \sum_{|z_n|\le r} \log(1-(1-r))$$
$$\ge \varphi(r) - C(1-r)n(r)\,,$$

for a positive constant C, where the last inequality follows from the equality $\lim\limits_{x\to 0^+} \dfrac{\log(1-x)}{x} = -1$ and $f(0) \neq 0$.

Thus using Lemma SS we obtain

$$\varphi(r) \le N(r) + C(1-r)n(r) \le C\log\frac{1}{1-r}\ . \tag{4}$$

Now note that our assumptions on h insure

$$\lim_{r\to 1^-} h(r)\log\frac{1}{1-r} = 0\ . \tag{5}$$

Indeed,

$$h(r)\log\frac{1}{1-r} = (h(r)-h(1^-))\log\frac{1}{1-r} = \log\frac{1}{1-r}\int_r^1 -h'(t)dt$$
$$\le \int_r^1 -h'(t)\log\frac{1}{1-t}\,dt \to 0, \quad \text{as } r\to 1\,.$$

Integrating by parts and appealing to (4) and (5) give

$$\sum_{|z_k|\ge r_0}(1-|z_k|)h(|z_k|) = \int_{r_0}^1 h(r)d\varphi(r) = [h(r)\varphi(r)]_{r_0}^1 + \int_{r_0}^1 -h'(r)\varphi(r)dr$$
$$\le \lim_{r\to 1^-} Ch(r)\log\frac{1}{1-r} + \int_{r_0}^1 -h'(r)\log\frac{1}{1-r}\,dr$$
$$= \int_{r_0}^1 -h'(r)\log\frac{1}{1-r}\,dr < \infty\,. \qquad \square$$

It is worth noting here that Theorem H is included in the above theorem because $h(r) = \left(\log\frac{1}{1-r}\right)^{-1-\varepsilon}$, $\varepsilon > 0$, satisfies the hypotheses of Theorem 1. Also a necessary condition for A^p zero sets given in [W] can be deduced from Theorem 1. To see this define $\log_1 x = \log x$, $\log_n x = \log(\log_{n-1} x)$, for $n = 2, 3\ldots$ and sufficiently large x. For a given positive integer n let x_n denote the solution of the equation $\log_{n-1} x = 0$.

For $1-\frac{1}{x_n} < r < 1$ we set

$$h(r) = \left(\prod_{i=1}^{n-1}\log_i\frac{1}{1-r}\right)^{-1}\left(\log_n\frac{1}{1-r}\right)^{-1-\varepsilon}.$$

Then the function h is nonincreasing in $(r_n,1)$, where $1-\frac{1}{x_n} < r_n < 1$, and

$$h'(r) = -\frac{\prod_{i=2}^n \log_i\frac{1}{1-r} + \prod_{i=3}^n \log_i\frac{1}{1-r} + \ldots + \prod_{i=n-1}^n \log_i\frac{1}{1-r} + \log_n\frac{1}{1-r} + 1 + \varepsilon}{(1-r)\left(\prod_{i=1}^{n-1}\log_i\frac{1}{1-r}\right)^2\left(\log_n\frac{1}{1-r}\right)^{2+\varepsilon}}.$$

Moreover,

$$
\begin{aligned}
&\int_{r_n}^{1} -h'(r)\log\frac{1}{1-r}\,dr \\
&= \int_{r_n}^{1} \frac{\prod_{i=2}^{n}\log_i\frac{1}{1-r} + \cdots + \prod_{i=n-1}^{n}\log_i\frac{1}{1-r} + \log_n\frac{1}{1-r} + 1 + \varepsilon}{(1-r)\left(\prod_{i=1}^{n-1}\log_i\frac{1}{1-r}\right)^2\left(\log_n\frac{1}{1-r}\right)^{2+\varepsilon}}\log\frac{1}{1-r}\,dr \\
&< \int_{r_n}^{1} \frac{n\prod_{i=1}^{n}\log_i\frac{1}{1-r}}{(1-r)\left(\prod_{i=1}^{n-1}\log_i\frac{1}{1-r}\right)^2\left(\log_n\frac{1}{1-r}\right)^{2+\varepsilon}}\,dr \\
&= \int_{r_n}^{1} \frac{n}{(1-r)\prod_{i=1}^{n-1}\log_i\frac{1}{1-r}\left(\log_n\frac{1}{1-r}\right)^{1+\varepsilon}}\,dr < \infty\,.
\end{aligned}
$$

Consequently, if $\{z_n\}$ are ordered zeros of $f \in A^p$ then

$$
\sum_{|z_k|>r_n} \frac{1-|z_k|}{\log\left(\frac{1}{1-|z_k|}\right)\ldots\log_{n-1}\left(\frac{1}{1-|z_k|}\right)\left(\log_n\left(\frac{1}{1-|z_k|}\right)\right)^{1+\varepsilon}} < \infty\,.
$$

The next theorem shows that the assumption (1) is essential.

Theorem 2. *Let $h \in C^1$ be a nonincreasing function in $[r_0, 1)$ for some $0 < r_0 < 1$, such that $\lim_{r\to 1^-} h(r) = 0$ and*

$$
\int_{r_0}^{1} -h'(r)\log\frac{1}{1-r}\,dr = \infty.
$$

Then there exists a function $f \in A^p$, $f(0) \neq 0$, whose zeros $z_1, z_2, \ldots$ satisfy

$$
\sum_{|z_k|>r_0} (1-|z_k|)h(|z_k|) = \infty\,.
$$

Proof. Let $\mu > 1,\ \beta > 2,\ \beta \in \mathbb{N}$. We set

$$
f(z) = \prod_{k=1}^{\infty}(1-\mu z^{\beta^k}), \qquad z \in \mathbb{D}\,.
$$

Horowitz [H] showed that $f \in A^p$ for some μ and β.

We first show that the zeros $\{z_k\}$ of the function f satisfy the condition

$$
\varphi(r) = \sum_{|z_k|\le r} (1-|z_k|) \ge C\log\frac{1}{1-r}\,.
$$

The zeros of f lie on the circles $|z| = \left(\frac{1}{\mu}\right)^{\frac{1}{\beta^i}}$, $i = 1, 2, \dots$. Hence

$$|z_k| = \left(\frac{1}{\mu}\right)^{\frac{1}{\beta}} = r_1 \quad \text{for } 1 \le k \le \beta\,,$$

$$|z_k| = \left(\frac{1}{\mu}\right)^{\frac{1}{\beta^2}} = r_2 \quad \text{for } \beta + 1 \le k \le \beta + \beta^2\,,$$

and

$$|z_k| = \left(\frac{1}{\mu}\right)^{\frac{1}{\beta^n}} = r_n \quad \text{for } N_{n-1} < k \le N_n\,, \tag{6}$$

where

$$N_n = \beta + \beta^2 + \cdots + \beta^n = \frac{\beta(\beta^n - 1)}{\beta - 1}\,, \quad n = 1, 2, \dots.$$

Thus we have

$$\sum_{|z_k| \le r_n} \log \frac{1}{|z_k|} = \beta \log \frac{1}{r_1} + \beta^2 \log \frac{1}{r_2} + \cdots + \beta^n \log \frac{1}{r_n}$$
$$= \beta \log \mu^{1/\beta} + \beta^2 \log \mu^{1/\beta^2} + \cdots + \beta^n \log \mu^{1/\beta^n} = n \log \mu\,.$$

Since for $x \in [r_1, 1)$

$$C \log \frac{1}{x} < 1 - x\,,$$

with a positive constant C, we get

$$\varphi(r_n) = \sum_{|z_k| \le r_n} (1 - |z_k|) \ge Cn \log \mu\,.$$

It follows from (6) that

$$\frac{1}{1 - r_n} < \frac{\beta^n}{C \log \mu}\,, \quad n = 1, 2, \dots,$$

and consequently,

$$n > \frac{1}{\log \beta} \log \frac{1}{1 - r_n} + \log(C \log \mu)\,.$$

This implies

$$\varphi(r_n) \ge C \log \frac{1}{1 - r_n}\,, \quad n = 1, 2, \dots.$$

If $r_n < r < r_{n+1}$, then

$$\varphi(r) \geq \varphi(r_n) > Cn \log \mu > C(n+1) > C \log \frac{1}{1-r_{n+1}} > C \log \frac{1}{1-r} .$$

Putting $r^* = \max[r_0, r_1]$ and integrating by parts gives

$$\sum_{|z_k| \geq r^*} (1-|z_k|)h(|z_k|) = \int_{r^*}^1 h(r)d\varphi(r) \geq -h(r^*)\varphi(r^*) + \int_{r^*}^1 -h'(r)\varphi(r)dr$$

$$\geq -h(r^*)\varphi(r^*) + C\int_{r^*}^1 -h'(r) \log \frac{1}{1-r} dr = \infty . \quad \square$$

Applying Theorem 2 to the function

$$h(r) = \left(\prod_{i=1}^n \log_i \frac{1}{1-r} \right)^{-1} ,$$

we obtain Remark 1 in [W].

3. Zeros of A^0 functions.

Proof of Theorem 3. It follows from the definition of the space A^0 and from the Jensen formula (3) that zeros of $f \in A^0$ satisfy

$$(7) \qquad N(r) = \mathrm{O}\left(\log\log \frac{1}{1-r} \right) , \quad r \to 1 .$$

This in turn implies

$$n(r) = \mathrm{O}\left(\frac{1}{1-r} \log\log \frac{1}{1-r} \right) \quad \text{and} \quad \varphi(r) = \mathrm{O}\left(\frac{1}{1-r} \log\log \frac{1}{1-r} \right) .$$

Now the claim follows by the same method as in the proof of Theorem 1. $\square$

Theorem 3 is best possible in the following sense.

Theorem 4. *Let $h \in C^1$ be a nonincreasing function in $[r_0, 1)$ for some $0 < r_0 < 1$, such that $\lim_{r \to 1^-} h(r) = 0$ and*

$$\int_{r_0}^1 -h'(r) \log\log \frac{1}{1-r}\, dr = \infty .$$

Then there exists a function $f \in \mathcal{B}_0$, $f(0) \neq 0$, whose zeros $z_1, z_2, \ldots$ satisfy

$$\sum_{|z_k|>r_0} (1-|z_k|)h(|z_k|) = \infty\,.$$

Proof. It was shown in [GNW] that there exists a function $f \in \mathcal{B}_0$ satisfying the inequality

$$N(r,f) \ge \beta \log\log \frac{1}{1-r}\ , \quad r_0 < r < 1\,,$$

for a positive constant β and $r_0 \in (0,1)$.

Integrating by parts twice we get

$$\begin{aligned}
\sum_{|z_n|\ge r_0} (1-|z_n|)h(|z_n|) &= \int_{r_0}^1 (1-r)h(r)dn(r) \\
&\ge \mathrm{O}(1) + \int_{r_0}^1 (h(r) - (1-r)h'(r))n(r)\,dr \\
&\ge \mathrm{O}(1) + \int_{r_0}^1 rh(r)\frac{n(r)}{r}\,dr \\
&= \mathrm{O}(1) + \int_{r_0}^1 rh(r)\,dN(r) \\
&\ge \mathrm{O}(1) + \int_{r_0}^1 (-h(r) - rh'(r))N(r)\,dr \\
&= \mathrm{O}(1) + \int_{r_0}^1 -h(r)N(r)\,dr + \int_{r_0}^1 -rh'(r)N(r)\,dr\,.
\end{aligned}$$

Since f is a Bloch function, $f \in A^0$, and by (7)

$$N(r) = \mathrm{O}\left(\log\log\frac{1}{1-r}\right)\ , \quad r \to 1\,.$$

This implies

$$\int_{r_0}^1 h(r)N(r)\,dr < \infty\,.$$

Hence

$$\begin{aligned}
\sum_{|z_n|\ge r_0} (1-|z_n|)h(|z_n|) &\ge \mathrm{O}(1) + \int_{r_0}^1 -r_0h'(r)N(r)\,dr \\
&\ge \mathrm{O}(1) + \beta r_0 \int_{r_0}^1 -h'(r)\log\log\frac{1}{1-r}\,dr = \infty\,. \quad \square
\end{aligned}$$

References

[GNW] Girela, D., M. Nowak and P. Waniurski, *On the zeros of Bloch functions*, Math. Proc. Cambridge Philos. Soc. **139** (2000), 117-128.

[H] Horowitz, Ch., *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693–710.

[SS] Shapiro, H.S., A.L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229.

[W] Waniurski, P., *On zeros of Bloch functions and related spaces of analytic functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **54** (2000), 149–158.

Instytut Matematyki UMCS
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: peter@golem.umcs.lublin.pl

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