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Parallelograms inscribed in a curve having a circle as $\frac{\pi}{2}$-isoptic

Abstract. Jean-Marc Richard observed in [7] that maximal perimeter of a parallelogram inscribed in a given ellipse can be realized by a parallelogram with one vertex at any prescribed point of ellipse. Alain Connes and Don Zagier gave in [4] probably the most elementary proof of this property of ellipse. Another proof can be found in [1]. In this note we prove that closed, convex curves having circles as $\frac{\pi}{2}$-isoptics have the similar property.

1. Introduction. Let C be a closed and strictly convex curve. We fix an interior point of C as an origin of a coordinate system. Denote $e^{it} = (\cos t, \sin t)$, $ie^{it} = (-\sin t, \cos t)$. The function $p : \mathbb{R} \to \mathbb{R}$

$$p(t) = \sup_{z\in C}\langle z, e^{it}\rangle$$

is called the support function of C. For a strictly convex curve p is differentiable. We assume that the function p is of class C^2 and the curvature of C is positive. We have the following equation of C in terms of its support function

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}. \tag{1.1}$$

Then $\|z\| = \sqrt{p^2(t) + \dot{p}^2(t)}$ and $R(t) = p(t) + \ddot{p}(t)$ is a radius of curvature of C at t.

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The α-isoptic of C consists of those points in the plane from which the curve is seen under the fixed angle α (for the geometric properties of isoptics see [2], [3], [5], [6], [8]). Suppose that $\frac{\pi}{2}$-isoptic of C is a circle of radius r with the center in the origin of a coordinate system. Then

$$p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = r^2, \tag{1.2}$$

and

$$p^2(t+\pi) + p^2\left(t + \frac{\pi}{2}\right) = r^2, \tag{1.3}$$

so $p(t) = p(t+\pi)$ and the center of the circle is a center of symmetry of C. The curve (1.1) has a circle with the center in the origin of a coordinate system as an $\frac{\pi}{2}$-isoptic if and only if (1.2) holds good.

Example 1.1. Let C be an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then

$$p(t) = \sqrt{a^2\cos^2 t + b^2\sin^2 t},$$

$$z(t) = (x(t), y(t)) = \sqrt{a^2\cos^2 t + b^2\sin^2 t}e^{it} + \frac{\sin t\cos t(b^2-a^2)}{\sqrt{a^2\cos^2 t + b^2\sin^2 t}}ie^{it}$$

is its equation in terms of a support function and $p^2(t)+p^2\left(t+\frac{\pi}{2}\right) = a^2+b^2$.

2. Extremal property of the perimeter of inscribed parallelograms. Assume that a curve C given by (1.1) has a circle with a center in an origin of a coordinate system as an $\frac{\pi}{2}$-isoptic. Then we have (1.2) and

$$p(t)\dot{p}(t) + p\left(t+\frac{\pi}{2}\right)\dot{p}\left(t+\frac{\pi}{2}\right) = 0. \tag{2.1}$$

Fix t and consider inscribed parallelogram with $z(t)$ as one of the vertices. There exists α such that $z(t+\alpha)$, $-z(t)$, $-z(t+\alpha)$ are its remaining vertices and

$$d_t(\alpha) = |z(t+\alpha) - z(t)| + |z(t+\alpha) + z(t)| \tag{2.2}$$

is a half of a perimeter of parallelogram.

Theorem 2.1. *Let C be a strictly convex curve having a circle with a center in an origin of a coordinate system as an $\frac{\pi}{2}$-isoptic and let $d_t(\alpha)$ be the function given by* (2.2). *Then*

(i) $d_t'\left(\frac{\pi}{2}\right) = 0$, *where prime denotes the derivative with respect to α,*
(ii) $d\left(\frac{\pi}{2}\right) = d_t\left(\frac{\pi}{2}\right)$ *does not depend on t.*

Proof. We have

$$e^{i(t+\alpha)} = \cos\alpha e^{it} + \sin\alpha ie^{it},$$
$$ie^{i(t+\alpha)} = -\sin\alpha e^{it} + \cos\alpha ie^{it},$$
$$z(t+\alpha) = (p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha)e^{it} + (p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha)ie^{it}.$$

Let

$$\begin{aligned} A &= p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha - p(t), \\ B &= p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha - \dot{p}(t), \\ C &= p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha + p(t), \\ D &= p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha + \dot{p}(t). \end{aligned}$$

Then

$$d_t(\alpha) = \sqrt{A^2+B^2} + \sqrt{C^2+D^2}$$

and

$$\begin{aligned} &d_t'(\alpha) = (p(t+\alpha) + \ddot{p}(t+\alpha)) \\ &\times \left(\frac{\dot{p}(t+\alpha) + p(t)\sin\alpha - \dot{p}(t)\cos\alpha}{\sqrt{A^2+B^2}} + \frac{\dot{p}(t+\alpha) - p(t)\sin\alpha + \dot{p}(t)\cos\alpha}{\sqrt{C^2+D^2}} \right). \end{aligned}$$

Putting $\alpha = \frac{\pi}{2}$, we get

$$\begin{aligned} d_t'\left(\frac{\pi}{2}\right) = R\left(t+\frac{\pi}{2}\right) \Bigg(&\frac{\dot{p}(t+\frac{\pi}{2}) + p(t)}{\sqrt{\left(p(t)+\dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) - \dot{p}(t)\right)^2}} \\ &+ \frac{\dot{p}(t+\frac{\pi}{2}) - p(t)}{\sqrt{\left(p(t)-\dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) + \dot{p}(t)\right)^2}} \Bigg). \end{aligned}$$

From (2.1) we have

$$\dot{p}\left(t+\frac{\pi}{2}\right) = -\frac{p(t)\dot{p}(t)}{p(t+\frac{\pi}{2})},$$

and since

$$\begin{aligned} &\left(p\left(t+\frac{\pi}{2}\right) - \dot{p}(t)\right)\left(p\left(t+\frac{\pi}{2}\right) + \dot{p}(t)\right) \\ &\qquad = p^2\left(t+\frac{\pi}{2}\right) - \dot{p}^2(t) = r^2 - (p^2(t) + \dot{p}^2(t)) \\ &\qquad = r^2 - \|z(t)\|^2 > 0, \end{aligned}$$

we obtain

$$\operatorname{sgn}\left(p\left(t+\frac{\pi}{2}\right) - \dot{p}(t)\right) = \operatorname{sgn}\left(p\left(t+\frac{\pi}{2}\right) + \dot{p}(t)\right).$$

Hence

$$\begin{aligned} &\frac{\dot{p}(t+\frac{\pi}{2}) + p(t)}{\sqrt{\left(p(t)+\dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) - \dot{p}(t)\right)^2}} \\ &\qquad\qquad + \frac{\dot{p}(t+\frac{\pi}{2}) - p(t)}{\sqrt{\left(p(t)-\dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) + \dot{p}(t)\right)^2}} \end{aligned}$$

$$= \frac{-\frac{p(t)\dot p(t)}{p(t+\frac{\pi}{2})} + p(t)}{\sqrt{\left(p(t) - \frac{p(t)\dot p(t)}{p(t+\frac{\pi}{2})}\right)^2 + \left(p(t+\frac{\pi}{2}) - \dot p(t)\right)^2}} + \frac{-\frac{p(t)\dot p(t)}{p(t+\frac{\pi}{2})} - p(t)}{\sqrt{\left(p(t) + \frac{p(t)\dot p(t)}{p(t+\frac{\pi}{2})}\right)^2 + \left(p(t+\frac{\pi}{2}) + \dot p(t)\right)^2}}$$

$$= \frac{p(t)}{p(t+\frac{\pi}{2})}\left(\frac{(p(t+\frac{\pi}{2}) - \dot p(t))p(t+\frac{\pi}{2})}{|p(t+\frac{\pi}{2}) - \dot p(t)|\sqrt{p^2(t) + p^2(t+\frac{\pi}{2})}} - \frac{(p(t+\frac{\pi}{2}) + \dot p(t))p(t+\frac{\pi}{2})}{|p(t+\frac{\pi}{2}) + \dot p(t)|\sqrt{p^2(t) + p^2(t+\frac{\pi}{2})}}\right) = 0,$$

which proves the first part of Theorem 2.1.

Let

$$h(t) = d_t\left(\frac{\pi}{2}\right) = \sqrt{\left(p\left(t\right) + \dot p\left(t+\frac{\pi}{2}\right)\right)^2 + \left(\dot p(t) - p\left(t+\frac{\pi}{2}\right)\right)^2} + \sqrt{\left(p(t) - \dot p\left(t+\frac{\pi}{2}\right)\right)^2 + \left(\dot p(t) + p\left(t+\frac{\pi}{2}\right)\right)^2}.$$

Then

$$\dot h(t) = \frac{R(t)(\dot p(t) - p(t+\frac{\pi}{2})) + R(t+\frac{\pi}{2})(p(t) + \dot p(t+\frac{\pi}{2}))}{\sqrt{\left(p(t) + \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) - p(t+\frac{\pi}{2})\right)^2}} + \frac{R(t+\frac{\pi}{2})(\dot p(t+\frac{\pi}{2}) - p(t)) + R(t)(\dot p(t) + p(t+\frac{\pi}{2}))}{\sqrt{\left(p(t) - \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) + p(t+\frac{\pi}{2})\right)^2}}$$

$$= R\left(t+\frac{\pi}{2}\right)\left(\frac{p(t) + \dot p(t+\frac{\pi}{2})}{\sqrt{\left(p(t) + \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) - p(t+\frac{\pi}{2})\right)^2}} - \frac{p(t) - \dot p(t+\frac{\pi}{2})}{\sqrt{\left(p(t) - \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) + p(t+\frac{\pi}{2})\right)^2}}\right)$$
$$+ R(t)\left(\frac{\dot p(t) - p(t+\frac{\pi}{2})}{\sqrt{\left(p(t) + \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) - p(t+\frac{\pi}{2})\right)^2}} + \frac{\dot p(t) + p(t+\frac{\pi}{2})}{\sqrt{\left(p(t) - \dot p(t+\frac{\pi}{2})\right)^2 + \left(\dot p(t) + p(t+\frac{\pi}{2})\right)^2}}\right).$$

Since the first summand is equal to zero for each t and the second summand is equal to the first at $t + \frac{\pi}{2}$, they are equal to zero. $\square$

3. The converse theorem. In this section we shall prove the converse of Theorem 2.1. For this purpose we define the function $d(t) = d_t(\frac{\pi}{2})$.

Theorem 3.1. *Let C be a closed and strictly convex curve of class C^2 with positive curvature having a center of symmetry. Suppose that an origin of a coordinate system is in the center of C and $d'_t(\frac{\pi}{2}) = 0$. Then $\dot{d}(t) = 0$ and $\frac{\pi}{2}$-isoptic of C is a circle.*

Proof. The equality $d'_t(\frac{\pi}{2}) = 0$ is equivalent to

$$\tag{3.1} \begin{aligned} &\frac{\dot{p}(t+\frac{\pi}{2}) + p(t)}{\sqrt{\left(p(t) + \dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) - \dot{p}(t)\right)^2}} \\ &\qquad = \frac{p(t) - \dot{p}(t+\frac{\pi}{2})}{\sqrt{\left(p(t) - \dot{p}(t+\frac{\pi}{2})\right)^2 + \left(p(t+\frac{\pi}{2}) + \dot{p}(t)\right)^2}}. \end{aligned}$$

The equality (3.1) for $t + \frac{\pi}{2}$ gives

$$\tag{3.2} \begin{aligned} &\frac{\dot{p}(t) + p(t+\frac{\pi}{2})}{\sqrt{\left(p(t+\frac{\pi}{2}) + \dot{p}(t)\right)^2 + \left(p(t) - \dot{p}(t+\frac{\pi}{2})\right)^2}} \\ &\qquad = \frac{p(t+\frac{\pi}{2}) - \dot{p}(t)}{\sqrt{\left(p(t+\frac{\pi}{2}) - \dot{p}(t)\right)^2 + \left(p(t) + \dot{p}(t+\frac{\pi}{2})\right)^2}}. \end{aligned}$$

From (3.1) and (3.2) we get

$$\frac{\dot{p}(t+\frac{\pi}{2}) + p(t)}{p(t+\frac{\pi}{2}) - \dot{p}(t)} = \frac{p(t) - \dot{p}(t+\frac{\pi}{2})}{\dot{p}(t) + p(t+\frac{\pi}{2})},$$

or equivalently

$$p(t)\dot{p}(t) + p\left(t + \frac{\pi}{2}\right) + p\left(t + \frac{\pi}{2}\right)\dot{p}\left(t + \frac{\pi}{2}\right) = 0,$$

which gives

$$p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = \text{const.} \qquad \square$$

Example 3.1 ([5]). Let $p(t) = \cos(\frac{\pi}{4} + k\sin(2t))$. For k sufficiently small $p(t)$ is a support function of a closed and strictly convex curve having a circle as $\frac{\pi}{2}$-isoptic and different from an ellipse.

Example 3.2. Let $p(t) = \sqrt{a\sin^2 3t + b\cos^2 9t + c}$, for positive a, b, c. For c sufficiently big $p(t) + \ddot{p}(t) > 0$ for each t and $p^2(t) + p^2(t + \frac{\pi}{2}) = a + b + 2c$ so $p(t)$ is a support function of a closed and strictly convex curve having a circle as $\frac{\pi}{2}$-isoptic. This curve cannot be an ellipse because an origin of a coordinate system is its center of symmetry and $p(t)$ is a periodic function with a period $\frac{\pi}{3}$. Hence its curvature function is also periodic with the same period and this curve has more then four vertices. More generally we can take $p(t) = \sqrt{a\sin^2 mt + b\cos^2 mnt + c}$, where m and m are odd integers and a, b, c are positive.

Remark 3.1. Let C be an ellipse. We fix a diameter PP' and consider an ellipse C' with focuses at P and P' which is tangent to C. Then points Q and Q' of tangency give a diameter such that a perimeter of parallelogram $PQP'Q'$ is maximal. The common tangent of C and C' at Q (resp. Q') makes equal angels with the sides PQ and $P'Q$ (resp. PQ' and $P'Q'$). This means that for parallelogram of maximal perimeter a tangent at any vertex makes equal angles with adjoining sides. This is a part of a more general fact. Let C be any closed and convex curve given in an arbitrary parametrization $z = z(t)$ of class C^1. Fix the points $z(t_1)$ and $z(t_2)$. Let $z(t_0)$ be such a point that the perimeter of the triangle $z(t_1)z(t_2)z(t_0)$ is maximal. Then the tangent at t_0 makes equal angels with the sides $z(t_0)z(t_1)$ and $z(t_0)z(t_2)$. Indeed,

$$\begin{aligned}
&\frac{d}{dt}(|z(t)-z(t_1)| + |z(t)-z(t_2)|)\\
&\qquad = \frac{\langle z(t)-z(t_1), \dot z(t)\rangle}{|z(t)-z(t_1)|} + \frac{\langle z(t)-z(t_2), \dot z(t)\rangle}{|z(t)-z(t_2)|}\\
&\qquad = \frac{|z(t)-z(t_1)||\dot z(t)|\cos\measuredangle(z(t)-z(t_1), \dot z(t))}{|z(t)-z(t_1)|}\\
&\qquad\quad + \frac{|z(t)-z(t_2)||\dot z(t)|\cos\measuredangle(z(t)-z(t_2), \dot z(t))}{|z(t)-z(t_2)|}\\
&\qquad = |\dot z(t)|(\cos\measuredangle(z(t)-z(t_1), \dot z(t)) + \cos\measuredangle(z(t)-z(t_2), \dot z(t))).
\end{aligned}$$

For $t = t_0$ we obtain

$$\cos\measuredangle(z(t_0)-z(t_1), \dot z(t_0)) + \cos\measuredangle(z(t_0)-z(t_2), \dot z(t_0))) = 0.$$

References

[1] Berger, M., *Geometrie, Vol. 2*, Nathan, Paris, 1990.

[2] Cieślak, W., Miernowski, A. and Mozgawa, W., *Isoptics of a strictly convex curve*, Global Differential Geometry and Global Analysis, 1990 (Berlin), Lecture Notes in Math., 1481, Springer, Berlin, 1991, 28–35.

[3] Cieślak, W., Miernowski, A. and Mozgawa, W., *Isoptics of a strictly convex curve II*, Rend. Sem. Mat. Univ. Padova **96** (1996), 37–49.

[4] Connes, A., Zagier, D., *A property of parallelograms inscribed in ellipses*, Amer. Math. Monthly **114** (2007), 909–914.

[5] Green, J. W., *Sets subtending a constant angle on a circle*, Duke Math. J. **17** (1950), 263–267.

[6] Matsuura, S., *On nonconvex curves of constant angle*, Functional analysis and related topics, 1991 (Kyoto), Lecture Notes in Math., 1540, Springer, Berlin, 1993, 251–268.

[7] Richard, J-M., *Safe domain and elementary geometry*, Eur. J. Phys. **25** (2004), 835–844.

[8] Wunderlich, W., *Kurven mit isoptischem Kreis*, Aequationes Math. **6** (1971), 71–78.

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