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The natural operators lifting vector fields to the bundle of affinors

Abstract. All natural operators $T \rightsquigarrow T(T \otimes T^*)$ lifting vector fields X from n-dimensional manifolds M to vector fields $B(X)$ on the bundle of affinors $TM \otimes T^*M$ are described.

1. Introduction. In [3], the second author studied the problem how a 1-form ω on an n-manifold M induces a 1-form $B(\omega)$ on $TM \otimes T^*M$. This problem was reflected in natural operators $B : T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds. It is proved that the set of natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds is a free $C^\infty(\mathbf{R}^n)$-module of dimension $2n$, and there is presented a basis of this module.

In this note we study a similar problem how a vector field X on an n-manifold M induces a vector field $B(X)$ on $TM \otimes T^*M$. This problem is reflected in natural operators $T \rightsquigarrow T(T \otimes T^*)$ over n-manifolds. We prove that the set of natural operators $T \rightsquigarrow T(T \otimes T^*)$ over n-manifolds is a free $C^\infty(\mathbf{R}^n)$-module of dimension $n + 1$. We construct a basis of this module.

We recall that a natural operator $B : T \rightsquigarrow T(T \otimes T^*)$ over n-manifolds is an $\mathcal{M}f_n$-invariant family of regular operators

$$B : \mathcal{X}(M) \to \mathcal{X}(TM \otimes T^*M)$$

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for all n-manifolds M. The invariance means that if vector fields X_1 on M and X_2 on N are φ-related for some local diffeomorphism $\varphi : M \to N$ between n-manifolds then the vector fields $B(X_1)$ and $B(X_2)$ are $T\varphi \otimes T^*\varphi$-related. The regularity means that B transforms smoothly parametrized families of vector fields into smoothly parametrized families of vector fields.

From now on $x^1, \dots, x^n$ are the usual coordinates on $\mathbf{R}^n$ and $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, n$ are the canonical vector fields on $\mathbf{R}^n$.

All manifolds and maps are assumed to be of class C^∞.

2. Examples of natural operators $T \rightsquigarrow T(T \otimes T^*)$.

Example 2.1. Let X be a vector field on an n-manifold M. Let $\mathcal{T} \otimes \mathcal{T}^* X$ be the flow lifting of X to $TM \otimes T^*M$. More precisely, if φ_t is the flow of X, then $T\varphi_t \otimes T^*\varphi_t$ is the flow of $\mathcal{T} \otimes \mathcal{T}^* X$. The correspondence $\mathcal{T} \otimes \mathcal{T}^* : T \rightsquigarrow T(T \otimes T^*)$ given by $X \to \mathcal{T} \otimes \mathcal{T}^* X$ is a natural operator (called the flow operator) in question.

Example 2.2. For $k = 0, \dots, n-1$ we have the canonical vector field L^k on $TM \otimes T^*M$ such that

$$L^k(A) = \frac{d}{dt}\Big|_0 (A + tA^k), \qquad A \in End(T_xM) = T_xM \otimes T^*_xM, \qquad x \in M,$$

where A^k is the k-th power of A ($A^0 = id$). The vector field L^k will be called the k-th Liouville vector field on $TM \otimes T^*M$ (L^1 is the classical Liouville vector field on $TM \otimes T^*M$). The correspondence $L^k : T \rightsquigarrow T(T \otimes T^*)$ is a natural operator in question.

3. The $C^\infty(\mathbf{R}^n)$-module of natural operators $T \rightsquigarrow T(T \otimes T^*)$ over n-manifolds. If $L : V \to V$ is an endomorphism of an n-dimensional vector space V then $a_1(L), \dots, a_n(L)$ denote the coefficient of the characteristic polynomial

$$W_L(\lambda) = \det(\lambda id_V - L) = \lambda^n + a_1(L)\lambda^{n-1} + \dots + a_{n-1}(L)\lambda + a_n(L).$$

Thus for every n-manifold M we have maps $a_1, \dots, a_n : TM \otimes T^*M \to \mathbf{R}$ (as $T_xM \otimes T^*_xM = End(T_xM)$).

The vector space of all natural operators $B : T \rightsquigarrow T(T \otimes T^*)$ over n-manifolds is additionally a module over the algebra $C^\infty(\mathbf{R}^n)$ of smooth maps $\mathbf{R}^n \to \mathbf{R}$. Actually given a smooth map $f : \mathbf{R}^n \to \mathbf{R}$ and a natural operator $B : T \rightsquigarrow T(T \otimes T^*)$ we have natural operator $fB : T \rightsquigarrow T(T \otimes T^*)$ given by

$$(fB)(X) = f(a_1, \dots, a_n)B(X)$$

for any vector field X on an n-manifold M.

4. The main result. The main result of this short note is the following classification theorem.

Theorem 1. *The flow operator* $\mathcal{T} \otimes \mathcal{T}^*$ *together with the k-th Liouville operators* L^k *for* $k = 0, \dots, n-1$ *form a basis of the* $C^\infty(\mathbf{R}^n)$*-module of natural operators* $T \rightsquigarrow T(T \otimes T^*)$ *over n-manifolds.*

The proof of Theorem 1 will occupy the rest of this note.

5. The result of J. Dębecki. The vector space $End(\mathbf{R}^n)$ of all endomorphisms of $\mathbf{R}^n$ is a $GL(n)$-space because of the usual (adjoint) action of $GL(n)$ on $End(\mathbf{R}^n)$.

We have the following result of J. Dębecki.

Proposition 1 ([1]). *Any* $GL(n)$*-equivariant map*

$$C : End(\mathbf{R}^n) \to End(\mathbf{R}^n)$$

is of the form

$$C(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \dots, a_n(A))A^k$$

for some uniquely determined maps $f_k : \mathbf{R}^n \to \mathbf{R}$.

6. The vertical type natural operators $B : T \rightsquigarrow T(T \otimes T^*)$ **over** n**-manifolds.** A natural operator $B : T \rightsquigarrow T(T \otimes T^*)$ is of vertical type if $B(X)$ is a vertical vector field for any vector field X on a n-manifold.

Using Proposition 1 we prove the following fact.

Proposition 2. *The* $C^\infty(\mathbf{R}^n)$*-submodule of vertical type natural operators* $B : T \rightsquigarrow T(T \otimes T^*)$ *over n-manifolds is free and n-dimensional. The k-th Liouville operators* L^k *for* $k = 0, \dots, n-1$ *form a basis of this module.*

Proof. Let $B : T \rightsquigarrow T(T \otimes T^*)$ be a vertical type natural operator over n-manifolds. Because of the naturality and the Frobenius theorem this operator is uniquely determined by the restriction of vertical vector field $B(\partial_1)$ to the fiber $End(T_0\mathbf{R}^n) = T_0\mathbf{R}^n \times T_0^*\mathbf{R}^n$.

Using the naturality of B with respect to the homotheties $tid_{\mathbf{R}^n}$ for $t \neq 0$ we see that

$$B(\partial_1)_{|End(T_0\mathbf{R}^n)} = B(t\partial_1)_{|End(T_0\mathbf{R}^n)}$$

for $t \neq 0$. Putting $t \to 0$ we see that

$$B(\partial_1)_{|End(T_0\mathbf{R}^n)} = B(0)_{|End(T_0\mathbf{R}^n)}.$$

Because of the naturality of $B(0)$ with respect to linear automorphisms of $\mathbf{R}^n$ we have a $GL(n)$-equivariant map

$$C : End(T_0\mathbf{R}^n) \to End(T_0\mathbf{R}^n)$$

given by

$$B(0)(A) = \frac{d}{dt}\Big|_0 (A + tC(A))$$

for $A \in End(T_0\mathbf{R}^n)$.

By Proposition 1 we have that

$$C(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \dots, a_n(A))A^k$$

for some uniquely determined maps $f_k : \mathbf{R}^n \to \mathbf{R}$. Then

$$B(\partial_1)(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \dots, a_n(A))L^k(A)$$

for all $A \in End(T_0\mathbf{R}^n)$. That is why $B = \sum_{k=0}^{n-1} f_k L^k$, as well. □

7. Proof of Theorem 1. It is clear that Theorem 1 will be proved after proving the following fact.

Proposition 3. *Let $B : T \rightsquigarrow T(T \otimes T^*)$ be a natural operator over n-manifolds. Then there exists a unique map $f : \mathbf{R}^n \to \mathbf{R}$ such that $B - f\mathcal{T} \otimes \mathcal{T}^*$ is a vertical type operator.*

Let $\pi : T\mathbf{R}^n \otimes T^*\mathbf{R}^n \to \mathbf{R}^n$ be the bundle projection.

Lemma 1. *There exist unique maps $f_k \in C^\infty(\mathbf{R}^n)$ such that*

$$T\pi(B(w^o)(A)) = \sum_{k=0}^{n-1} f_k(a_1(A), \dots, a_n(A))A^k(w)$$

for $A \in End(T_0\mathbf{R}^n) = T_0\mathbf{R}^n \otimes T_0^*\mathbf{R}^n$ and $w \in T_0\mathbf{R}^n$, where w^o is the "constant" vector field on $\mathbf{R}^n$ with $w^o_0 = w$.

Proof. By the invariance of B with respect to the homotheties $tid_{\mathbf{R}^n}$ for $t \neq 0$ we have the homogeneity condition

$$T\pi(B((tw)^o)(A) = tT\pi(B(w^o))(A).$$

Then by the homogeneous function theorem, [2], $T\pi(B(w^o)(A)$ depends linearly on w.

So, we can define a map $C : End(T_0\mathbf{R}^n) \to End(T_0\mathbf{R}^n)$ by

$$C(A)(w) = T\pi(B(w^o)(A))$$

for all $A \in End(T_0\mathbf{R}^n)$ and $w \in T_0\mathbf{R}^n$.

Because of the naturality of B with respect to linear automorphisms of $\mathbf{R}^n$, C is $GL(n)$-equivariant. Then applying Proposition 1 we end the proof. □

Lemma 2. *Let $B : T \rightsquigarrow T(T \otimes T^*)$ be as in Lemma 1. Let $f_0, \dots, f_{n-1}$ be the maps from Lemma 1. Then $f_j = 0$ for $j = 1, \dots, n-1$.*

Proof. Consider $j = 1, \dots, n-1$. Let $b = (b_1, \dots, b_n) \in \mathbf{R}^n$. Let $A \in End(T_0\mathbf{R}^n)$ be such that $A(\partial_i(0)) = \partial_{i+1}(0)$ for $i = 1, \dots, n-1$ and $A(\partial_n(0)) = -b_n\partial_1(0) - \dots - b_1\partial_n(0)$. Then $a_i(A) = b_i$ for $i = 1, \dots, n$.

Let $\varphi_t = (x^1, \dots, x^{j+1} + tx^{j+1} + \dots, \dots, x^n)$ be the flow of $\partial_{j+1} + x^{j+1}\partial_{j+1}$ near $0 \in \mathbf{R}^n$.

Since $T_0\varphi_1 \circ A \circ T_0\varphi_1^{-1} \neq A$ (as the left hand side evaluated at $\partial_j(0)$ is equal to $2\partial_{j+1}(0)$ and the right hand side evaluated in the same vector $\partial_j(0)$ is equal to $\partial_{j+1}(0)$), we have

$$\mathcal{T} \otimes \mathcal{T}^*(x^{j+1}\partial_{j+1})(A) \neq 0. \tag{1}$$

Using the Zajtz theorem [4], since $(\partial_{j+1} + x^{j+1}\partial_{j+1})(0) = \partial_{j+1}(0) \neq 0$, we find a diffeomorphism $\eta : \mathbf{R} \to \mathbf{R}$ such that

$$j_0^1\psi = id \tag{2}$$

and

$$\psi_*\partial_{j+1} = \partial_{j+1} + x^{j+1}\partial_{j+1} \tag{3}$$

near $0 \in \mathbf{R}^n$, where $\psi(x^1, \dots, x^n) = (x^1, \dots, x^j, \eta(x^{j+1}), \dots, x^n)$.

Clearly ψ preserves ∂_1. Because of (2), ψ preserves A. Then ψ preserves $B(\partial_1)(A)$.

Because of (2), ψ preserves any vertical vector tangent to $T\mathbf{R}^n \otimes T^*\mathbf{R}^n$ at A. Moreover, ψ preserves all ∂_l for $l = 1, \dots, n$ with $l \neq j+1$. By (3), ψ sends $\mathcal{T} \otimes \mathcal{T}^*(\partial_{j+1})(A)$ into $\mathcal{T} \otimes \mathcal{T}^*(\partial_{j+1} + x^{j+1}\partial_{j+1})(A)$. Then ψ sends

$$B(\partial_1)(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \dots, a_n(A))\mathcal{T} \otimes \mathcal{T}^*(\partial_{k+1})(A) + \text{some vertical vector}$$

into $B(\partial_1)(A) + f_j(b)\mathcal{T} \otimes \mathcal{T}^*(x^{j+1}\partial_{j+1})(A)$.

Then because of (1), we have $f_j(b) = 0$, as well. □

Proof of Proposition 3. Because of Lemmas 1 and 2 we have

$$B(\partial_1)(A) = f_0(a_1(A), \dots, a_n(A))\mathcal{T} \otimes \mathcal{T}^*(\partial_1)(A) + \text{some vertical vector}$$

for any $A \in End(T_0\mathbf{R}^n)$. Since B is determined by $B(\partial_1)$ over 0, the proof of Proposition 3 is complete. □

References

[1] Dębecki, J., *Natural transformations of affinors into functions and affinors*, Suppl. Rend. Circolo Mat. Palermo **30(II)** (1993), 101–112.

[2] Kolář, I., Michor, P. W. and Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[3] Mikulski, W. M., *Liftings of* 1*-forms to the bundle of affinors*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **65** (2001), 109–113.

[4] Zajtz, A., *On the order of natural differential operators and liftings*, Ann. Polon. Math. **49** (1988), 169–175.

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