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Some remarks on strong factorization of tent spaces

Abstract. We provide new assertions on factorization of tent spaces.

In this note, we provide new assertions concerning strong factorization of so-called tent spaces. In order to formulate our results we will need some standard definitions ([3, 4, 5]).

Let

$$R_+^{n+1} = \{(x,t) : x \in R^n, t > 0\},$$

$$\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$$

and $B(x,t) = B$ be a ball with center $x \in R^n$.

For $x \in R^n$, let

$$A_\infty(f)(x) = N(f)(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)|,$$

$$A_q(f)(x) = \left(\int_{\Gamma(x)} \frac{|f(y,t)|^q}{t^{n+1}} dydt\right)^{1/q}$$

and

$$C_q(f)(x) = \left(\sup_{x\in B} \frac{1}{|B|} \int_{T(B)} \frac{|f(y,t)|^q}{t} dydt\right)^{1/q},$$

where $T(B)$ is a tent on B in R^n (see [3, 4]).

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Define spaces T_q^p, T_∞^p and T_q^∞ respectively

$$T_q^p = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying} \\ \|f\|_{T_q^p} = \|A_q(f)(x)\|_{L^p(R^n)} < \infty\},$$

$$T_\infty^p = \{f : f \text{ is measurable in } R_+^{n+1} \\ \text{with continuous boundary values on } R^n \\ \text{such that } \|f\|_{T_\infty^p} = \|A_\infty(f)(x)\|_{L^p(R^n)} < \infty\}$$

and

$$T_q^\infty = \{f : f \text{ is measurable in } R_+^{n+1} \\ \text{satisfying } \|f\|_{T_q^\infty} = \|C_q(f)(x)\|_{L^\infty} < \infty\}.$$

One of the main results of [3, 4] asserts that

$$\text{(A)} \qquad T_q^p = T_\infty^p T_q^\infty \text{ for } 0 < p, q < \infty.$$

The mentioned equality was for the first time obtained in [3] for $p > 2$, $q = 2$. Such type strong factorization theorems have numerous applications in the theory of analytic spaces ([2, 4, 6]). We give some results similar in spirit to (A). As we can easily notice mentioned factorizations of T_q^p classes were not considered before for $p = \infty$. In this note we, in particular, intend to give an answer to that natural question. On the other hand T_q^p type classes that were defined above are heavily based on classical L^p spaces in R^n. Our next intention is to replace them by their natural extensions: the well-known L_q^p Lorentz spaces in R^n and to prove, if possible, a result similar to (A) equality.

Let $C(n)^{-1}$ be the volume of the unit ball ([4]) so that $\|P_t^0\|_{L^1(R^n)} = 1$, where $P_t^0(x) = C(n)t^{-n}\chi_{B(0,t)}(x)$ and $\chi_{B(0,t)}(x)$ is the characteristic function of the set $B(0,t)$. For $x \in R^n$, define

$$(P_0^*\mu)(x) = C(n)\int_{\Gamma(x)} \frac{d\mu(y,t)}{t^n}$$

where μ is a positive Borel measure in R_+^{n+1}.

Lemma 1. *Let* $P_0(g)(x,t) = \frac{C(n)}{t^n}\int_{B(x,t)} g(y)dy$, $g \in L^1_{loc}(R^n)$, $S(\mu) = P_0(P_0^*\mu)^{-\tau}$, *where* $0 < \tau \leq 1$ *and* μ *is a positive Borel measure on* R^{n+1}. *Then*

$$\frac{1}{|B|}\int_{T(B)} S\mu(x,t)d\mu(x,t) \leq C\left\|\left(\int_{T(B)\cap\Gamma(y)} \frac{d\mu(x,t)}{t^n}\right)^{1-\tau}\right\|_{L^\infty(B,dy)}.$$

Remark 1: For $\tau = 1$, Lemma 1 was proved in [4].

Proof. Let $h(y) = P_0^*\mu(y)$, $y \in R^n$. Modifying proofs in [4] we have

$$\int_{T(B)} S\mu(x,t)d\mu(x,t) \le C(n)\int_{T(B)}\int_{B(x,t)} h(y)^{-\tau}\frac{d\mu(x,t)}{t^n}dy$$
$$\le C\int_{R^n} h(y)^{-\tau}\int_{T(B)\cap\Gamma(y)}\frac{d\mu(x,t)}{t^n}dy$$
$$\le C|B|\sup_{y\in R^n}\left(\int_{T(B)\cap\Gamma(y)}\frac{d\mu(x,t)}{t^n}\right)^{1-\tau}.$$

The proof is complete. □

Let X, Y and Z be quasinormed subspaces of the class of all measurable functions in R^n. For $0 < \alpha \le 1$, we say $X \overset{\alpha}{\subset} YZ$, if for any $u \in X$, there exist $w \in Y$, $v \in Z$ such that $u = w \cdot v^\alpha$.

Let $T_q^{\infty,\infty}$ be the class of measurable functions f satisfying

$$\|f\|_{T_q^{\infty,\infty}} = \left\|\left(\int_{\Gamma(y)}\frac{|f|^q}{t^{n+1}}dxdt\right)^{1/q}\right\|_{L^\infty(R^n)} < \infty.$$

Theorem 1. *Let $0 < q < \infty$, $p > 0$ and $0 < \alpha = s/q \le 1$. Then $T_q^{\infty,\infty} \overset{\alpha}{\subset} T_\infty^p T_q^\infty$.*

Remark 2: If we replace $T_q^{\infty\infty}$ classes in Theorem 2 with larger T_q^p classes, then for $s = q$ Theorem 1 is known (see [3, 4]).

Proof. We will modify the proof of [3, 4]. As the proof in [4] (p. 316), we have

$$(*)\qquad \left(\int_X |f|^{-s}d\nu\right)^{-1/s} \le \left(\int_X |f|^r d\nu\right)^{1/r}, \quad r,s>0,$$

where ν is a measure in R^n. Let us put $d\nu = P_t^0(x)dx$, $f = A_q(u)$ in $(*)$. Then we have

$$V = (P_0(A_q(u))^r)^{1/r} \ge C(P_0(A_q(u))^{-s})^{-1/s},$$

that is, $V^{-s} \le C(P_0(A_q(u))^{-s})$.

Let $d\mu(x,t) = u(x,t)^q\frac{dxdt}{t}$. Then $(A_q(u))^q = CP_0^*\mu$ and

$$V^{-s} \le CP_0(P_0^*\mu)^{-s/q} = P_0(P_0^*\mu)^{-\alpha},$$

where $0 < \alpha = s/q \leq 1$. Since $\omega^q = \frac{u^q}{V^s}$, from Lemma 1 we have

$$\begin{aligned}\int_{T(B)} \omega(x,t)^q \frac{dxdt}{t} &\leq C \int_{T(B)} V^{-s} d\mu(x,t) \\ &\leq C \int_{T(B)} S_\alpha(\mu) d\mu \\ &\leq C|B| \left\| \int_{T(B)\cap\Gamma(y)} \frac{|u(x,t)|^q dxdt}{t^{n+1}} \right\|^{1-s/q}_{L^\infty(B,dy)} \\ &\leq C|B| \|u\|^{q-s}_{T_q^{\infty,\infty}},\end{aligned}$$

which proves that for $u \in T_q^{\infty,\infty}$ and $\omega^q = \frac{u^q}{V^s}$, we have $\omega \in T_q^\infty$.

For $u \in T_q^{\infty,\infty}$, let $V = (P_0(A_q(u))^r)^{1/r}$. Then (see [3, 4]) $NP_0(f) \leq CM(f)$ and hence $N(V) \leq C(M(A_q(u))^r)^{1/r}$, $p > r$, where $M(f)$ is the Hardy–Littlewood maximal function. Thus $V \in T_\infty^p$ for every p. Indeed M is a bounded operator from $L^p(R^n)$ into $L^p(R^n)$, $p > 1$. Hence $V \in T_\infty^p$ for every $p > 0$. One the other hand if $\omega = (\frac{u^q}{V^s})^{1/q}$, then we can show that for $u \in T_q^{\infty,\infty}$ and $V \in T_\infty^p$, $p > 0$

$$\left(\frac{1}{|B|} \int_{T(B)} \omega(x,t)^q \frac{dxdt}{t} \right)^{1/q} \leq C\|u\|^{1-s/q}_{T_q^{\infty,\infty}} \quad \text{for } s \leq q.$$

The proof is complete. □

We now turn to another extension of (A). The following facts from the theory of Lorentz classes $L^{p,q}(R^n)$ are needed (see [1, 7]).

For $q, p \in (1, \infty)$, the Hardy–Littlewood maximal operator is extended in $L^{p,q}(R^n)$ (see [5, 1, 7]) and we have

$$\|M(f)\|_{L^{p,q}} \leq C\|f\|_{L^{p,q}}, \tag{1}$$

and

$$\|M(f)\|_{L^{p,\infty}} \leq C\|f\|_{L^{p,\infty}}. \tag{2}$$

Let f be a measurable function in R_+^{n+1}. Define

$$\|f\|_{LT_q^{p,s}} = \|A_q(f)\|_{L^{p,s}(R^n)},$$

$$\|f\|_{LT_\infty^{p,s}} = \|N(f)\|_{L^{p,s}}.$$

For $0 < p < \infty$, the spaces $LT_q^{p,s}$ and $LT_\infty^{p,s}$ are defined by

$$LT_q^{p,s} = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying } \|f\|_{LT_q^{p,s}} < \infty\},$$

$$LT_\infty^{p,s} = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying } \|f\|_{LT_\infty^{p,s}} < \infty\}.$$

Theorem 2. *Let $s \leq p \leq q < \infty$. Then $LT_q^{p,s} = LT_\infty^{p,s} T_q^\infty$.*

Remark 3: For $p = s$, this was obtained in [3, 4] before and it coincides with (A).

Proof. We again use some ideas from [3, 4]. Note first, if $\|A_q(f)\|_{L^{p,s}(R^n)} < \infty$, putting $V = (P_0(A_q(f))^r)^{1/r}$ as in the previous case we have $NV \le C(M(A_q(f)^r))^{1/r}$, $p, s > r$, where M is the Maximal Hardy–Littlewood operator. By (1) we have

$$\|NV\|_{L^{p,s}} \le C\|A_q(f)\|_{L^{p,s}(R^n)} < \infty \quad \text{for } p, s > 0$$

since

$$\||f|^r\|_{L^{p,s}} = \|f\|_{L^{rp,rs}}, \quad p, s, r > 0.$$

The proof of the fact that $\omega = \frac{u}{V} \in T_q^\infty$ follows from the same arguments as in [4]. Let us show the reverse with the same restriction on parameters. Let $\omega \in T_q^\infty$, $V \in LT_\infty^{p,s}$. We will show that

$$\left\| \left(\int_{\Gamma(y)} \frac{|\omega V|^q}{t^{n+s}} dxdt \right)^{1/q} \right\|_{L^{p,s}(R^n)} < \infty.$$

By Hölder inequality for Lorentz classes (see [5]), the following estimate holds:

$$D = \left\| \left(\int_{\Gamma(y)} \frac{|\omega(x,t)V(x,t)|^q}{t^{n+s}} dxdt \right)^{1/q} \right\|_{L^{p,s}(R^n)}$$

$$\le C\|NV\|_{L^{\frac{p_1\tau}{q},\frac{s_1\tau}{q}}} \left\| \int_{\Gamma(y)} \frac{V^{q-\tau}\omega^q}{t^{n+s}} \right\|_{L^{\frac{p_2}{q},\frac{s_2}{q}}} = AB,$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$. Choosing τ such that $\frac{\tau p_1}{q} = p$, $\frac{\tau s_1}{q} = s$, then $\frac{p_2}{q} = \frac{s_2}{q} = 1$ and $B \le C\|\omega\|_{T_q^\infty}\|NV\|_{L^{q-\tau}}$ which follows (A). Hence $D \le \|NV\|_{L^{p,s}}\|NV\|_{L^{q-\tau,q-\tau}}$. Note that $\tau = \frac{qs}{s_1} = \frac{pq}{p_1}$, $q - \tau = q(1 - \frac{p}{p_1}) = p = q - qp(\frac{1}{p} - \frac{1}{q})$. Hence using known embeddings for Lorentz classes (see [1, 7]) we have $D \le \|NV\|_{L^{p,s}(R^n)}$ for $s \le p$. The proof is complete. □

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