The almost sure central limit theorems for certain order statistics of some stationary Gaussian sequences

Abstract. Suppose that \( X_1, X_2, \ldots \) is some stationary zero mean Gaussian sequence with unit variance. Let \( \{k_n\} \) be a certain nondecreasing sequence of positive integers, \( M_n^{(k_n)} \) denote the \( k_n \)th largest maximum of \( X_1, \ldots, X_n \). We aim at proving the almost sure central limit theorems for the suitably normalized sequence \( \{M_n^{(k_n)}\} \) under certain additional assumptions on \( \{k_n\} \) and the covariance function \( r(t) := \text{Cov}(X_1, X_{1+t}) \).

1. Introduction. The almost sure central limit theorem (ASCLT) has become an intensively studied subject in recent time. In the research concerning the ASCLT the following property is investigated. Let \( X_1, X_2, \ldots \) be some r.v.'s, \( f_1, f_2, \ldots, f_k, \ldots \) denote some real-valued measurable functions, defined on \( \mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^k, \ldots \), respectively. We seek conditions under which, for some nondegenerate d.f. \( G \),

\[
\lim_{N \to \infty} \frac{1}{D_N} \sum_{n=1}^{N} d_n I(f_n(X_1, \ldots, X_n) \leq x) = G(x) \quad \text{a.s.}
\]

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for all $x \in C_G$, where: $\{d_n\}$ is some sequence of weights, $D_N = \sum_{n=1}^{N} d_n$, $I$ stands for the indicator function, and $C_G$ denotes the set of continuity points of $G$.

In our investigations, we will restrict ourselves to the case, when the relation above holds with: $d_n = 1/n$, $D_N \sim \log N$, $f_n(X_1, \ldots, X_n) = a_n\left(M_n^{(k_n)} - b_n\right)$, where: $\{k_n\}$ is a certain nondecreasing sequence of positive integers, $M_n^{(k_n)}$ denotes the $k_n$th largest maximum of $X_1, \ldots, X_n$, and $a_n > 0$, $b_n$ are certain normalizing constants.

Let $\Phi$ be the standard normal d.f. The purpose of this paper is to prove that if $X_1, X_2, \ldots$ is a standardized stationary Gaussian sequence, then, under some assumptions on the numerical sequences $\{k_n\}$, $\{u_n\}$ and the covariance function $r(t) := \text{Cov}(X_1, X_{1+t})$, we have for some $\tau$, $0 < \tau < \infty$,

(1) \[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k_n)} \leq u_n\right) = \Phi(\tau) \quad \text{a.s.}
\]

As a direct consequence, we will also show that if:

(2) \[
a_n = \left(\frac{2 \log (n/k_n)}{k_n}\right)^{1/2}, \\
b_n = \left(2 \log (n/k_n)\right)^{1/2} - \frac{\log \log (n/k_n) + \log 4\pi}{2 \left(2 \log (n/k_n)\right)^{1/2}},
\]

then the following strong convergence occurs for all $x \in \mathbb{R}$

(3) \[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(a_n \left(M_n^{(k_n)} - b_n\right) \leq x\right) = \Phi(x) \quad \text{a.s.}
\]

In the case, when $k_n \equiv k$, where $k$ is a fixed positive integer, we will show that, under certain conditions on $\{u_n\}$ and $r(t)$,

(4) \[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \leq u_n\right) = e^{-\sum_{s=0}^{k-1} \frac{x^s}{s!}} \quad \text{a.s.}
\]

for some $\tau$, $0 < \tau < \infty$.

As a direct consequence, we will also show that if:

(5) \[
a_n = (2 \log n)^{1/2}, \quad b_n = (2 \log n)^{1/2} - \frac{\log \log n + \log 4\pi}{2 \left(2 \log n\right)^{1/2}},
\]

then the following strong convergence occurs for all $x \in \mathbb{R}$

(6) \[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(a_n \left(M_n^{(k)} - b_n\right) \leq x\right) = \exp\left(-e^{-x}\right) \sum_{s=0}^{k-1} \frac{(e^{-x})^s}{s!} \quad \text{a.s.}
\]

We should mention here that, in the case of i.i.d. r.v.’s, the ASCLT for the order statistics $M_n^{(k_n)}$ has been proved in Stadtmueller [4], under some extra
assumptions on \( \{k_n\} \). It is also worthwhile to mention that the ASCLT for the ordinary maxima \( M_n = M_n^{(1)} := \max (X_1, \ldots, X_n) \) of some dependent stationary Gaussian sequences has been proved in Csaki and Gonchigdanzan [1] and Dudziński [2].

The following notations will be used throughout the paper:

- \( M_n^{(k_n)} \) – the \( k_n \)-th largest maximum of \( X_1, \ldots, X_n \);
- \( M_n^{(k_n)} \) – the \( k_n \)-th largest maximum among \( X_{m+1}, \ldots, X_n \);
- \( r(t) := \text{Cov} (X_1, X_1+t) \);
- \( \Phi \) – the standard normal d.f.;
- \#A – the cardinality of the set \( A \);
- \( |x| \) – an absolute value of \( x \);
- \( \lfloor x \rfloor \) – the greatest integer less than or equal to \( x \).

Furthermore, \( f(n) \ll g(n) \) and \( f(n) \sim g(n) \) will stand for \( f(n) = O(g(n)) \) and \( f(n)/g(n) \to 1 \), as \( n \to \infty \), respectively.

2. Main results. Our main results are the ASCLTs for certain order statistics of some stationary Gaussian sequences. The first one can be formulated as follows.

**Theorem 1.** Let \( X_1, X_2, \ldots \) be a stationary zero mean Gaussian sequence with unit variance and \( \{k_n\} \) denote a nondecreasing sequence of positive integers, which satisfies:

\[
(7) \quad k_n \to \infty \quad \text{as} \quad n \to \infty, \\
(8) \quad \log k_n \ll (\log n)^{1-\alpha} \quad \text{for some} \quad \alpha > 0, \\
(9) \quad \text{there exists a number} \quad \beta > 1, \quad \text{such that the sequence} \quad \{(\log n)/k_n^\beta\} \\
\quad \text{is nondecreasing for all sufficiently large} \quad n.
\]

Assume in addition that the covariance function \( r(t) := \text{Cov} (X_1, X_1+t) \) fulfils the following condition

\[
(10) \quad \sum_{t=\lfloor n^{1/k_n^\beta} \rfloor}^\infty |r(t)| \ll \frac{1}{n^{k_n-1/k_n^\beta-1+1/k_n^\beta}} \quad \text{for some} \quad \beta \quad \text{satisfying} \quad (9).
\]

Then:

(i) if the numerical sequence \( \{u_n\} \) satisfies:

\[
(11) \quad n \left(1 - \Phi(u_n)\right) \Phi(u_n) \to \infty \quad \text{as} \quad n \to \infty, \\
(12) \quad \frac{k_n - n \left(1 - \Phi(u_n)\right)}{\left\{n \left(1 - \Phi(u_n)\right) \Phi(u_n)\right\}^{1/2}} \to \tau \quad \text{for some} \quad \tau, \quad 0 < \tau < \infty, \quad \text{as} \quad n \to \infty,
\]
then relation (1) holds,

(ii) if the sequences \( \{a_n\}, \{b_n\} \) are such as in (2), then relation (3) holds for all \( x \in \mathbb{R} \).

Our next main result is the ASCLT for the \( k \)-th largest maxima. Here it is.
Theorem 2. Let $X_1, X_2, \ldots$ be a stationary zero mean Gaussian sequence with unit variance and $k$ denote a fixed positive integer. Assume moreover that the covariance function $r(t) := \text{Cov}(X_1, X_1+t)$ fulfills the following condition
\begin{equation}
\sum_{t=\lceil n^{1/\kappa} \rceil}^{\infty} |r(t)| \ll \frac{1}{n^{k-1-1/k^{\beta}-1+1/k^{\beta}}} \text{ for some } \beta > 1.
\end{equation}

Then:
(i) if the numerical sequence $\{u_n\}$ satisfies
\begin{equation}
n(1 - \Phi(u_n)) \to \tau \text{ for some } \tau, \ 0 < \tau < \infty, \text{ as } n \to \infty,
\end{equation}
then relation (4) holds,
(ii) if the sequences $\{a_n\}, \{b_n\}$ are such as in (5), then relation (6) holds for all $x \in \mathbb{R}$.

3. Auxiliary results. In this section, we state and prove three lemmas, which will be used in the proofs of Theorems 1, 2.

Lemma 1. Under the assumptions of Theorem 1, we have that if $m, n$ satisfy $m \leq n/k_n - 1$, then
\begin{equation}
E \left| I \left( M_n^{(k_n)} \leq u_n \right) - I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right| \ll \frac{1}{n^{t}} + \frac{k_n m}{n - k_n}
\end{equation}
for some $\gamma > 0$.

Proof. We have
\begin{equation}
E \left| I \left( M_n^{(k_n)} \leq u_n \right) - I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right| = \left| P \left( M_n^{(k_n)} \leq u_n \right) - P \left( M_{m,n}^{(k_n)} \leq u_n \right) \right|.
\end{equation}
It is clear that $P \left( M_n^{(k_n)} \leq u_n \right) = P(\text{at most } k_n-1 \text{ of } X_1, \ldots, X_n \text{ exceed } u_n)$. Similarly, $P \left( M_{m,n}^{(k_n)} \leq u_n \right) = P(\text{at most } k_n-1 \text{ of } X_{m+1}, \ldots, X_n \text{ exceed } u_n)$.

For the given $m, n$, we put:
\begin{equation}
\sum_{(A_1,A_2)} := \sum_{(A_1,A_2): \ A_1 \cup A_2 = \{1,\ldots,n\}, \ #A_1 \leq k_n-1, \ A_2 = \{1,\ldots,n\} \setminus A_1},
\end{equation}
\begin{equation}
\sum_{(B_1,B_2)} := \sum_{(B_1,B_2): \ B_1 \cup B_2 = \{m+1,\ldots,n\}, \ #B_1 \leq k_n-1, \ B_2 = \{m+1,\ldots,n\} \setminus B_1}.
\end{equation}
Hence, by applying (16) and the notations in (17), (18),

\[
E \left| I \left( M_n^{(k_n)} \leq u_n \right) - I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right|
\]

\[
\leq \sum_{(A_1,A_2)} P \left( \bigcap_{a_p \in A_1} \{ X_{a_p} > u_n \} \cap \bigcap_{a_s \in A_2} \{ X_{a_s} \leq u_n \} \right) \prod_{a_p \in A_1} P \left( X_{a_p} > u_n \right) \prod_{a_s \in A_2} P \left( X_{a_s} \leq u_n \right)
\]

\[
- \sum_{(B_1,B_2)} P \left( \bigcap_{b_p \in B_1} \{ X_{b_p} > u_n \} \cap \bigcap_{b_s \in B_2} \{ X_{b_s} \leq u_n \} \right) \prod_{b_p \in B_1} P \left( X_{b_p} > u_n \right) \prod_{b_s \in B_2} P \left( X_{b_s} \leq u_n \right)
\]

\[
+ \left\{ \sum_{(B_1,B_2)} \prod_{b_p \in B_1} P \left( X_{b_p} > u_n \right) \prod_{b_s \in B_2} P \left( X_{b_s} \leq u_n \right) \right\}
\]

\[
=: D_1 + D_2 + D_3.
\]

By (10) and the fact that \( k_n - 1 - 1/k_n^{\beta-1} + 1/k_n^{\beta} \geq 0 \) for any \( n \geq 1 \), we obtain

\[
\sum_{t=1}^{\infty} |r(t)| < \infty.
\]
It follows from (20) and (7) that there exist positive numbers \( \delta, \gamma, n_0 \), such that:

\[
\text{(21)} \quad \sup_{t \geq 1} |r(t)| = \delta < 1,
\]

\[
\text{(22)} \quad 1/k_n^{\beta-1} - 1/k_n^{\beta} < 2/(1 + \delta) - 1 - 2\gamma
\]

for all \( n > n_0 \), if \( \beta \) fulfils (9), (10). Let \( c(n) \) denote the largest integer, such that

\[
\text{(23)} \quad c(n) \left[ n^{1/k_n^{\beta}} \right] + 1 < n.
\]

Thus, we can divide the sequence \( X_1, \ldots, X_n \) into the following \( c(n) + 1 \) blocks:

\[
\left( X_1, \ldots, X_{\left[ n^{1/k_n^{\beta}} \right]} \right), \left( X_{\left[ n^{1/k_n^{\beta}} \right]+1}, \ldots, X_{2\left[ n^{1/k_n^{\beta}} \right]} \right), \ldots,
\]

\[
\left( X_{c(n)\left[ n^{1/k_n^{\beta}} \right]+1}, \ldots, X_n \right).
\]

Since \( X_1, \ldots, X_n \) is a standard normal sequence and (21), (23) hold, then, by applying Theorem 4.2.1 in Leadbetter et al. [3] (the so-called Normal Comparison Lemma), and by using the previously described division of \( \{X_1, \ldots, X_n\} \), as well as the definitions of \( D_1 \) in (19) and \( \sum_{(A_1,A_2)} \) in (17), we have

\[
\text{(24)} \quad D_1 \ll \begin{cases} 
C_1(n) \sum_{d=0}^{c(n)-2} \sum_{i=d\left[ n^{1/k_n^{\beta}} \right]+1}^{(d+1)\left[ n^{1/k_n^{\beta}} \right]} \sum_{j=i+1}^{(d+2)\left[ n^{1/k_n^{\beta}} \right]} |r(j-i)| \exp\left(-\frac{u_n^{2}}{1+\delta}\right) \\
+ C_1(n) \sum_{i=(c(n)-1)\left[ n^{1/k_n^{\beta}} \right]+1}^{\left[ n^{1/k_n^{\beta}} \right]} \sum_{j=i+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^{2}}{1+\delta}\right) \\
+ C_1(n) \sum_{i=c(n)\left[ n^{1/k_n^{\beta}} \right]+1}^{n-1} \sum_{j=i+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^{2}}{1+\delta}\right) \\
+ C_2(n) \sum_{d=0}^{c(n)-2} \sum_{i=d\left[ n^{1/k_n^{\beta}} \right]+1}^{(d+1)\left[ n^{1/k_n^{\beta}} \right]} \sum_{j=(d+2)\left[ n^{1/k_n^{\beta}} \right]+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^{2}}{1+\delta}\right),
\end{cases}
\]
where $C_1(n)$, $C_2(n)$ satisfy:

\begin{align}
C_1(n) &= \sum_{l=0}^{k_n-1} \left\lfloor \frac{n^{1/k_n^a}}{l} \right\rfloor + \sum_{l=0}^{k_n-1} \left( \left\lfloor \frac{n^{1/k_n^a}}{l} \right\rfloor \left( \frac{n^{1/k_n^a}}{k_n - 1 - l} \right) \right), \\
C_2(n) &= \sum_{l=0}^{k_n-1} \left( \left\lfloor \frac{n^{1/k_n^a}}{l} \right\rfloor \left( \frac{n - 2 \left\lfloor \frac{n^{1/k_n^a}}{l} \right\rfloor}{k_n - 1 - l} \right) \right).
\end{align}

Due to the derivation in (24), we have

\begin{align}
D_1 &\ll (c(n) + 1) C_1(n) \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor \sum_{t=1}^{2 \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor - 1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right) \\
&+ (c(n) - 1) C_2(n) \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor \sum_{t=\left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor + 1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right).
\end{align}

Notice that, by the definition of $c(n)$ in (23),

\begin{equation}
(c(n) + 1) \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor \ll n.
\end{equation}

Consequently, due to (27), (28),

\begin{align}
D_1 &\ll nC_1(n) \sum_{t=1}^{2 \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor - 1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right) \\
&+ nC_2(n) \sum_{t=\left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor + 1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right).
\end{align}

In addition, it follows from (25), (26) that:

\begin{align}
C_1(n) &\ll k_n \left( \frac{n^{1/k_n^a}}{k_n - 1} \right)^{k_n-1} = k_n n^{1/k_n^a - 1/k_n^a}, \\
C_2(n) &\ll k_n n^{k_n-1}.
\end{align}

Relations (30), (31) together with derivation (29) imply

\begin{align}
D_1 &\ll k_n n^{1+1/k_n^a - 1/k_n^a} \sum_{t=1}^{2 \left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor - 1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right) \\
&+ k_n n^{k_n} \sum_{t=\left\lfloor \frac{n^{1/k_n^a}}{1} \right\rfloor + 1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1 + \delta}\right).
\end{align}
Recall that, by (10) and (20) (see the reasoning above (20)),
\[
\sum_{t=n^{1/k_n^\alpha}}^{n-1} |r(t)| \ll \frac{1}{n^{k_n-1-1/k_n^\alpha}} \quad \text{and} \quad \sum_{t=1}^{\infty} |r(t)| < \infty.
\]
The relations in (32), (33) yield
\[
D_1 \ll k_n n^{1+1/k_n^\alpha - 1/k_n^\beta} \exp \left( -\frac{u_n^2}{1+\delta} \right).
\]
Since
\[
1 - \Phi (u_n) \sim \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u_n^2}{2} \right) u_n
\]
and, by (11), (12), \( k_n \sim n (1 - \Phi (u_n)) \), we get \( u_n \sim (2 \log (n/k_n))^{1/2} \) and
\[
\exp \left( -\frac{u_n^2}{2} \right) \sim \sqrt{2\pi} \frac{k_n}{n} (2 \log (n/k_n))^{1/2}.
\]
Hence
\[
\exp \left( -\frac{u_n^2}{1+\delta} \right) \ll \frac{(k_n)^{2/(1+\delta)}}{n^{2/(1+\delta)}} (\log (n/k_n))^{1/(1+\delta)}.
\]
From (34), (35), we get
\[
D_1 \ll \frac{(k_n)^{1+2/(1+\delta)} (\log (n/k_n))^{1/(1+\delta)}}{n^{2/(1+\delta)}-1-1/k_n^{\alpha-1}+1/k_n^\beta}.
\]
Furthermore, notice that \( \log (n/k_n) \leq \log n \) and, by (8), \( k_n \ll n^\epsilon \) for any \( \epsilon > 0 \). Therefore
\[
D_1 \ll \frac{(n^\epsilon)^{1+2/(1+\delta)} (\log n)^{1/(1+\delta)}}{n^{2/(1+\delta)}-1-1/k_n^{\alpha-1}+1/k_n^\beta} \quad \text{for any} \ \epsilon > 0.
\]
Since in addition, (22) holds, we have \( 2/(1+\delta) - 1 - 1/k_n^{\alpha-1} + 1/k_n^\beta > 2\gamma \) for any \( n > n_0 \) and some \( \gamma > 0 \). As \( \epsilon \) in (36) may be arbitrary positive number, we can choose \( \epsilon \) satisfying the relation \( \epsilon (1 + 2/(1+\delta)) < \gamma \). Then
\[
(n^\epsilon)^{1+2/(1+\delta)} (\log n)^{1/(1+\delta)} \ll n^\gamma,
\]
and we can write that
\[
D_1 \ll \frac{n^\gamma}{n^{2/\gamma}} = 1/n^\gamma \quad \text{for some} \ \gamma > 0.
\]
In order to estimate the component \( D_2 \) in (19), it is sufficient to apply identical methods to those used in the estimation of \( D_1 \). Therefore, we obtain that
\[
D_2 \ll 1/n^\gamma \quad \text{for some} \ \gamma > 0.
\]
Thus, it remains to bound the term $D_3$ in (19). Let $\tilde{X}_1, \ldots, \tilde{X}_n$ be an i.i.d. standard normal sequence. We denote by $\tilde{M}^{(k_n)}_{m,n}$ the $k_n$th largest maximum of $\tilde{X}_1, \ldots, \tilde{X}_n$ and by $\tilde{M}^{(k_n)}_{m,n}$ the $k_n$th largest maximum among $\tilde{X}_{m+1}, \ldots, \tilde{X}_n$. By the notations in (17), (18) and the definition of the component $D_3$ in (19)

$$
D_3 = P\left(\tilde{M}^{(k_n)}_{m,n} \leq u_n\right) - P\left(\tilde{M}^{(k_n)}_{n} \leq u_n\right) - P\left(\tilde{M}^{(k_n)}_{n} \neq \tilde{M}^{(k_n)}_{m,n}\right).
$$

As $m \leq n/k_n - 1$ and $k_n \ll n^\gamma$ for any $\gamma > 0$, it follows from Lemma 1 in Stadtmueller [4] that

$$
P\left(\tilde{M}^{(k_n)}_{n} \neq \tilde{M}^{(k_n)}_{m,n}\right) \ll k_n m/(n - k_n).
$$

Thus, due to (39),

$$
D_3 \ll k_n m/(n - k_n).
$$

Relations (19), (37), (38) and (40) imply the desired result in (15). \hfill \Box

**Lemma 2.** Under the assumptions of Theorem 1, we have that if $m, n$ satisfy $m \leq n/k_n - 1$, then

$$
Cov\left( I\left(M^{(k_n)}_m \leq u_m\right), I\left(M^{(k_n)}_{m,n} \leq u_n\right) \right) \ll \frac{1}{n^\gamma}
$$

for some $\gamma > 0$.

**Proof.** Let $X_1, X_2, \ldots$ be a standardized stationary Gaussian sequence, \{$k_n\}$, \{$r(t)\}$, \{$u_n\}$ satisfy (7)–(12), respectively, and $m \leq n/k_n - 1$. Clearly

$$
Cov\left( I\left(M^{(k_n)}_m \leq u_m\right), I\left(M^{(k_n)}_{m,n} \leq u_n\right) \right) = P\left(M^{(k_n)}_m \leq u_m, M^{(k_n)}_{m,n} \leq u_n\right) - P\left(M^{(k_n)}_m \leq u_m\right)P\left(M^{(k_n)}_{m,n} \leq u_n\right).
$$

For the given $m, n$, we set

$$
\sum_{(A_1,A_2,B_1,B_2)} := \sum_{(A_1,A_2,B_1,B_2): \ A_1 \cup A_2 = \{1, \ldots, m\}, \ A_1 \neq k_n - 1, \ B_1 = \{1, \ldots, m\} \setminus A_1, \ B_1 \cup B_2 = \{m+1, \ldots, n\} \setminus B_1}.
$$

Since

$$
P\left(M^{(k_n)}_m \leq u_m\right) = P(\text{at most } k_n - 1 \text{ of } X_1, \ldots, X_m \text{ exceed } u_m)$$

and

$$
P\left(M^{(k_n)}_{m,n} \leq u_n\right) = P(\text{at most } k_n - 1 \text{ of } X_{m+1}, \ldots, X_n \text{ exceed } u_n),$$

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then, by relation (42) and the notation in (43), we can write that

\[
\left| \text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \right| \leq \sum_{(A_1, A_2, B_1, B_2)} F(A_1, A_2, B_1, B_2),
\]

where

\[
F(A_1, A_2, B_1, B_2) := \left| P \left( \bigcap_{a_p \in A_1} \{X_{a_p} > u_m\} \cap \bigcap_{a_s \in A_2} \{X_{a_s} \leq u_m\} \cap \bigcap_{b_p \in B_1} \{X_{b_p} > u_n\} \cap \bigcap_{b_s \in B_2} \{X_{b_s} \leq u_n\} \right) \right|.
\]

By (7) and (10) (see the reasoning above (20)), \( |r(t)| \to 0 \) as \( t \to \infty \). Hence, there exist positive numbers \( \delta, \gamma, n_1 \), such that:

\[
\sup_{t \geq 1} |r(t)| = \delta < 1,
\]

(45)

\[
1/k_n^{\beta - 1} - 1/k_n^{\beta} < 1/(1 + \delta) - 1/2 - \gamma
\]

(46)

for all \( n > n_1 \), if \( \beta \) fulfils (9), (10).

Let \( c(m) \) denote the largest integer, such that

\[
c(m) \left[ m^{1/k_m^{\beta}} \right] + 1 < m.
\]

Thus, we can divide the sequence \( X_1, \ldots, X_m, X_{m+1}, \ldots, X_n \) into the blocks:

\[
\left( X_1, \ldots, X_{\left[ m^{1/k_m^{\beta}} \right]} \right), \left( X_{\left[ m^{1/k_m^{\beta}} \right]} + 1, \ldots, X_{2\left[ m^{1/k_m^{\beta}} \right]} \right), \ldots,
\]

\[
\left( X_{c(m)\left[ m^{1/k_m^{\beta}} \right]} + 1, \ldots, X_m \right), \left( X_{m+1}, \ldots, X_{\left[ n^{1/k_n^{\beta}} \right]} \right),
\]

\[
\left( X_{\left[ n^{1/k_n^{\beta}} \right]} + 1, \ldots, X_n \right).
\]

By using such a division, as well as Theorem 4.2.1 in Leadbetter et al. [3], the relation in (44) and the definition of \( \sum_{(A_1, A_2, B_1, B_2)} \) in (43), we obtain
It follows from the derivation in (47) that

\[
\left| \text{Cov} \left( I \left( M_{m}^{(k_{m})} \leq u_{m} \right) , I \left( M_{m,n}^{(k_{n})} \leq u_{n} \right) \right) \right| \\
\leq \left\{ \begin{array}{c}
C_{1} (m, n) \sum_{d=0}^{c(m)-1} \sum_{i=d \left[ m^{1/k_{m}} \right] +1}^{m+ \left[ n^{1/k_{n}} \right] +1} \sum_{j=m+1}^{m+ \left[ n^{1/k_{n}} \right] +1} |r (j-i)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right) \\
+ C_{1} (m, n) \sum_{i=c(m)}^{m} \sum_{j=m+1}^{m+ \left[ n^{1/k_{n}} \right] +1} |r (j-i)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right) \\
+ C_{2} (m, n) \sum_{d=0}^{c(m)-1} \sum_{i=d \left[ m^{1/k_{m}} \right] +1}^{m+ \left[ n^{1/k_{n}} \right] +1} \sum_{j=m+1}^{m+ \left[ n^{1/k_{n}} \right] +1} |r (j-i)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right) \\
+ C_{2} (m, n) \sum_{i=c(m)}^{m} \sum_{j=m+1}^{m+ \left[ n^{1/k_{n}} \right] +1} |r (j-i)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right) \end{array} \right. 
\]

where \( C_{1} (m, n), C_{2} (m, n) \) satisfy:

\[
C_{1} (m, n) = \sum_{l_{1}=0}^{k_{m}-1} \sum_{l_{2}=0}^{k_{n}-1} \left( \left[ m^{1/k_{m}} \right] \left[ n^{1/k_{n}} \right] \right), \\
C_{2} (m, n) = \sum_{l_{1}=0}^{k_{m}-1} \sum_{l_{2}=0}^{k_{n}-1} \left( \left[ m^{1/k_{m}} \right] \left[ n^{1/k_{n}} \right] \right) \left( n - m - \left[ n^{1/k_{n}} \right] \right). 
\]

It follows from the derivation in (47) that

\[
\left| \text{Cov} \left( I \left( M_{m}^{(k_{m})} \leq u_{m} \right) , I \left( M_{m,n}^{(k_{n})} \leq u_{n} \right) \right) \right| \\
\leq (c (m) + 1) C_{1} (m, n) \left[ m^{1/k_{m}} \right] \sum_{t=1}^{m+ \left[ n^{1/k_{n}} \right] -1} |r (t)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right) \\
+ (c (m) + 1) C_{2} (m, n) \left[ m^{1/k_{m}} \right] \sum_{t=\left[ n^{1/k_{n}} \right] +1}^{n-1} |r (t)| \exp \left( -\frac{u_{m}^2 + u_{n}^2}{2 \left( 1 + \delta \right)} \right). 
\]
In addition, by the definition of \( c(m) \) (see the relation below (46)),
\[
(c(m) + 1) \left[ m^{1/k_m^\beta} \right] \ll m.
\]

Relations (50), (51) yield
\[
\left| \text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \right|
\]
\[
\ll mC_1(m, n) \sum_{t=1}^{m+\left\lceil n^{1/k_n^\beta} \right\rceil -1} |r(t)| \exp \left( -\frac{u_m^2 + u_n^2}{2(1 + \delta)} \right)
\]
\[
+ mC_2(m, n) \sum_{t=\left\lceil n^{1/k_n^\beta} \right\rceil +1}^{n-1} |r(t)| \exp \left( -\frac{u_m^2 + u_n^2}{2(1 + \delta)} \right).
\]

Moreover, observe that, due to (48), (49):
\[
C_1(m, n) \ll k_mk_n \left( m^{1/k_m^\beta} \right)^{k_m-1} \left( n^{1/k_n^\beta} \right)^{k_n-1}
\]
\[
= k_mk_n \left( m^{1/k_m^\beta - 1/k_m^\beta - 1/k_n^\beta + 1/k_n^\beta} \right)^{k_n-1},
\]
\[
C_2(m, n) \ll k_mk_n \left( m^{1/k_m^\beta} \right)^{k_m-1} \left( n^{1/k_n^\beta} \right)^{k_n-1} = k_mk_n \left( m^{1/k_m^\beta - 1/k_m^\beta - 1/k_n^\beta + 1/k_n^\beta} \right)^{k_n-1}.
\]

Relations (53), (54) together with derivation (52) imply
\[
\left| \text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \right|
\]
\[
\ll k_mk_n \sum_{t=\left\lceil n^{1/k_n^\beta} \right\rceil +1}^{n-1} |r(t)| \exp \left( -\frac{u_m^2 + u_n^2}{2(1 + \delta)} \right).
\]

By assumption (10) and relation (20) (see the reasoning above (20))
\[
\sum_{t=\left\lceil n^{1/k_n^\beta} \right\rceil +1}^{n-1} |r(t)| \ll \frac{1}{n^{k_n-1/k_n^\beta + 1/k_n^\beta}} \quad \text{and} \quad \sum_{t=1}^{\infty} |r(t)| < \infty.
\]

This and (55) yield
\[
\left| \text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \right|
\]
\[
\ll k_mk_n \left( m^{1/k_m^\beta - 1/k_m^\beta - 1/k_n^\beta + 1/k_n^\beta} \right)^{k_n-1} \exp \left( -\frac{u_m^2 + u_n^2}{2(1 + \delta)} \right).
\]
Moreover, it follows from (35) that
\[
\exp \left( -\frac{\epsilon^2 + \epsilon^2}{2(1 + \delta)} \right) \leq \frac{(k_m)^{1/(1+\delta)}}{m^{1/(1+\delta)}} \frac{(\log (m/k_m))^{1/2(1+\delta)}}{n^{1/(1+\delta)}} \left( \log (n/k_m) \right)^{1/2(1+\delta)}.
\]

As in addition, the sequence \( \{k_n\} \) is nondecreasing, we have \( k_m \leq k_n \). This, relation (57) and the fact that
\[
(\log (m/k_m))^{1/2(1+\delta)} (\log (n/k_n))^{1/2(1+\delta)} \leq (\log m)^{1/2(1+\delta)} (\log n)^{1/2(1+\delta)}
\]
yield
\[
\exp \left( -\frac{\epsilon^2 + \epsilon^2}{2(1 + \delta)} \right) \leq \frac{(k_n)^{2/(1+\delta)}}{m^{1/(1+\delta)} n^{1/(1+\delta)}}.
\]

Relations (56), (58) imply
\[
\text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \leq \frac{m^{1+1/k_m - 1/k_m^\beta - 1/(1+\delta)}}{n^{1/(1+\delta)-1/k_m^\beta - 1/k_m^\beta + 1/k_m}} \left( k_n \right)^{2+2/(1+\delta)} (\log n)^{1/(1+\delta)}
\]
\[
\leq \frac{\left( k_n \right)^{2+2/(1+\delta)} (\log n)^{1/(1+\delta)}}{n^{1/(1+\delta)-1/k_m^\beta - 1/k_m^\beta + 1/(1+\delta)-1/2-1/k_m^\beta + 1/k_m^\beta}}.
\]

Notice that, by (46), we obtain
\[
1/(1 + \delta) - 1/2 - 1/k_m^\beta - 1/k_m^\beta + 1/(1 + \delta) - 1/2 - 1/k_m^\beta - 1/k_m^\beta > 2\gamma
\]
for all \( m, n > n_1 \) and some \( \gamma > 0 \).

Due to (59) and (60) and the fact that, by assumption (8), \( k_n \ll n^\epsilon \) for any \( \epsilon > 0 \), we have
\[
\text{Cov} \left( I \left( M_m^{(k_m)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \leq \frac{\left( k_n \right)^{2+2/(1+\delta)} (\log n)^{1/(1+\delta)}}{n^{2\gamma}}
\]
\[
\leq \frac{\left( n^\epsilon \right)^{2+2/(1+\delta)} (\log n)^{1/(1+\delta)}}{n^{2\gamma}}
\]
for some \( \gamma > 0 \) and any \( \epsilon > 0 \). As \( \epsilon \) in (61) may be arbitrary positive number, we can choose \( \epsilon > 0 \). Then
\[(n^\gamma)^{2+2/(1+\delta)} (\log n)^{1/(1+\delta)} \ll n^\gamma \text{ and} \]

\[\left| \text{Cov} \left( I \left( M_n^{(k_n)} \leq u_m \right), I \left( M_n^{(k_n)} \leq u_n \right) \right) \right| \ll \frac{n^\gamma}{n^{2\delta}} = \frac{1}{n^\gamma} \]

for some \( \gamma > 0 \), which is the result in (41), we wished to prove. \( \square \)

The following property will be also needed in our further considerations.

**Lemma 3.** Under the assumptions of Theorem 1, we have

\[(62) \lim_{n \to \infty} P \left( M_n^{(k_n)} \leq u_n \right) = \Phi (\tau), \]

where \( \tau \) satisfies (12).

**Proof.** The relation in (62) follows immediately from Theorem 4.2.1 (the Normal Comparison Lemma) and Theorem 2.5.2 in Leadbetter et al. [3]. \( \square \)

4. **Proofs of main results.** In this section, we give the proofs of Theorems 1, 2. As we mentioned earlier, the results stated in Lemmas 1–3 are important ingredients of these proofs.

**Proof of Theorem 1 (i).** First, we will show that

\[(63) \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left\{ I \left( M_n^{(k_n)} \leq u_n \right) - P \left( M_n^{(k_n)} \leq u_n \right) \right\} = 0 \quad \text{a.s.} \]

By Lemma 3.1 in Csaki and Gonchigdanzan [1], in order to prove (63), it is sufficient to show that the following property occurs for some \( \varepsilon > 0 \)

\[(64) \text{Var} \left( \sum_{n=1}^{N} \frac{1}{n} I \left( M_n^{(k_n)} \leq u_n \right) \right) \ll (\log N)^2 (\log \log N)^{-1+\varepsilon}. \]

We have

\[(65) \text{Var} \left( \sum_{n=1}^{N} \frac{1}{n} I \left( M_n^{(k_n)} \leq u_n \right) \right) \leq \sum_{n=1}^{N} \frac{1}{n^2} \text{Var} \left( I \left( M_n^{(k_n)} \leq u_n \right) \right) \]

\[+ 2 \sum_{1 \leq m < n \leq N} \frac{1}{mn} \left| \text{Cov} \left( I \left( M_n^{(k_m)} \leq u_m \right), I \left( M_n^{(k_n)} \leq u_n \right) \right) \right| =: \sum_1 + \sum_2. \]

It is clear that

\[(66) \sum_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \]
Our purpose now is to estimate $\sum_2$ in (65). Observe that

$$\left| \text{Cov} \left( I \left( M_{m}^{(k_m)} \leq u_m \right), I \left( M_{n}^{(k_n)} \leq u_n \right) \right) \right| \ll E \left| I \left( M_{m}^{(k_n)} \leq u_n \right) - I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right| + \left| \text{Cov} \left( I \left( M_{m}^{(k_n)} \leq u_m \right), I \left( M_{m,n}^{(k_n)} \leq u_n \right) \right) \right|.$$ 

Thus, by (67) and Lemmas 1, 2, we have that if $m, n$ satisfy $m \leq \frac{n}{k_n} - 1$, then

$$\left| \text{Cov} \left( I \left( M_{m}^{(k_n)} \leq u_m \right), I \left( M_{n}^{(k_n)} \leq u_n \right) \right) \right| \ll \frac{1}{n^\gamma} + \frac{k_n m}{n - k_n}$$

for some $\gamma > 0$. Consequently

$$\sum_2 \ll \sum_{1 \leq m < n \leq N \atop m \leq n/k_n - 1} \frac{1}{mn \ n^\gamma} + \sum_{1 \leq m < n \leq N, \ m \leq n/k_n - 1} \frac{k_n m}{mn \ n - k_n} + \sum_{1 \leq m < n \leq N, \ m > n/k_n - 1} \frac{1}{mn} \tag{67}$$

$$=: G_1 + G_2 + G_3.$$

Notice that

$$G_1 \leq \sum_{1 \leq m < n \leq N} \frac{1}{mn \ n^\gamma} = \sum_{m=1}^{N-1} \frac{1}{m} \sum_{n=m+1}^{N} \frac{1}{n^{1+\gamma}} \leq \frac{1}{\gamma} \sum_{m=1}^{N-1} \frac{1}{m^{1+\gamma}} < \infty. \tag{68}$$

In order to estimate $G_2$ in (67), observe that

$$G_2 \ll \sum_{1 \leq n \leq N} \sum_{1 \leq m \leq n/k_n \atop m \leq n/k_n - 1} \frac{k_n}{n(n - k_n)} \ll \sum_{1 \leq n \leq N} (n/k_n - 1) \frac{k_n}{n(n - k_n)} \tag{69}$$

$$\ll \sum_{n=1}^{N} \frac{1}{n} \ll \log N.$$ 

Thus, it remains to estimate component $G_3$. From its definition in (67), we get

$$G_3 \leq \sum_{1 \leq n \leq N} \frac{1}{n} \sum_{m=\lfloor n/k_n \rfloor}^{n-1} \frac{1}{m} \ll \sum_{1 \leq n \leq N} \frac{1}{n} \log \frac{n}{\lfloor n/k_n \rfloor} \ll \sum_{1 \leq n \leq N} \frac{1}{n} \log \frac{n}{n/k_n - 1}.$$ 

Therefore

$$G_3 \ll \sum_{1 \leq n \leq N} \frac{1}{n} \log \frac{n k_n}{n - k_n}. \tag{70}$$

It follows from (8) that, there exists constant $n_0$, such that $k_n \leq n/2$ for all $n > n_0$. Hence, for any $n > n_0$,

$$\log \frac{n k_n}{n - k_n} \leq \log \frac{n k_n}{n - n/2} = \log 2k_n,$$
and
\[(71) \quad \log \frac{n k_n}{n - k_n} \ll \log k_n.\]

Relations (70), (71) together with assumption (8) imply
\[(72) \quad G_3 \ll \sum_{n=1}^{N} \frac{1}{n} \log k_n \ll \sum_{n=1}^{N} \frac{1}{n} (\log n)^{1-\alpha} \ll (\log N)^{2-\alpha}\]

for some $\alpha > 0$. Due to (67)–(69) and (72)
\[(73) \quad \sum_2 \ll (\log N)^{2-\alpha}\]

for some $\alpha > 0$. It follows from (65), (66) and (73) that
\[\text{Var} \left( \sum_{n=1}^{N} \frac{1}{n} I \left( M_n(k_n) \leq u_n \right) \right) \ll (\log N)^{2-\alpha}\]

for some $\alpha > 0$. Thus, (64) holds for any $\varepsilon > 0$.

Consequently, by the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan [1], condition (63) is also satisfied. In turn, as (63) holds, then Lemma 3 and the regularity property of logarithmic means imply (1).

Thus, statement (i) of our assertion has been proved. $\square$

Proof of Theorem 1 (ii). Let $x$ be arbitrary real number. It is easy to check that, provided $\{a_n\}, \{b_n\}$ are such as in (2), then, under the assumptions of our theorem,
\[\lim_{n \to \infty} P \left( a_n \left( M_n(k_n) - b_n \right) \leq x \right) = \Phi(x)\]

(see also the remark on p. 416 in Stadtmueller [4]). This and Theorem 2.5.2 in Leadbetter et al. [3] imply that assumptions (11), (12) are satisfied with:
\[u_n := x/a_n + b_n, \quad \tau := x.\]

It is easily seen now that statement (ii) of Theorem 1 is a special case of its, the earlier proved, statement (i). $\square$

Remark. Suppose that $k_n = [\log n]^c$ for some $0 < c < 1$, and the number $\beta > 1$ satisfies the condition $c\beta < 1$. Let in addition, $X_1, X_2, \ldots$ be a stationary zero mean Gaussian sequence with unit variance and the covariance function $r(t) = e^{-\lambda t}$ for some $\lambda > 0$. Then, the assumptions (7)–(10) of Theorem 1 are fulfilled and the property in (3) holds with $\{a_n\}, \{b_n\}$ given by (2).

We now prove our second main result.

Proof of Theorem 2 (i). Let $k$ denote a fixed positive integer, $M_n^{(k)}$ stand for the $k$th largest maximum of $X_1, \ldots, X_n$. By applying assumption (13) on the covariance function $r(t) = \text{Cov}(X_1, X_{1+t})$ and assumption (14) on
the sequence \( \{u_n\} \), and by using similar methods to that applied in the proofs of Lemmas 1, 2, we can show that if \( m \leq n/k - 1 \), then:

\[
E \left| I \left( M_n^{(k)} \leq u_n \right) - I \left( M_{m,n}^{(k)} \leq u_n \right) \right| \ll \frac{1}{n^\gamma} + \frac{km}{n-k} \quad \text{for some } \gamma > 0,
\]

\[
\left| \text{Cov} \left( I \left( M_n^{(k)} \leq u_m \right), I \left( M_{m,n}^{(k)} \leq u_n \right) \right) \right| \ll \frac{1}{n^\gamma} \quad \text{for some } \gamma > 0.
\]

This and the relation in (67), applied for \( k_n \equiv k \), yield

\[
\text{(74)} \quad \left| \text{Cov} \left( I \left( M_n^{(k)} \leq u_m \right), I \left( M_{m,n}^{(k)} \leq u_n \right) \right) \right| \ll \frac{1}{n^\gamma} + \frac{km}{n-k}
\]

for some \( \gamma > 0 \), provided \( m \leq n/k - 1 \). Suppose that \( \{\tilde{X}_i\} \) is an i.i.d. standard normal sequence and \( \tilde{M}_n^{(k)} \) denotes the \( k \)th largest maximum of \( \tilde{X}_1, \ldots, \tilde{X}_n \). It follows from Theorem 2.2.1 in Leadbetter et al. [3] that, under the assumptions of our theorem,

\[
\text{(75)} \quad \lim_{n \to \infty} P \left( \tilde{M}_n^{(k)} \leq u_n \right) = e^{-\tau k - 1 \sum_{s=0}^{k-1} \frac{\tau^s}{s!}},
\]

where \( \tau \) satisfies (14). By using similar methods to those applied in the estimation of \( D_1 \) in the proof of Lemma 1, it is easy to check that

\[
\text{(76)} \quad \left| P \left( M_n^{(k)} \leq u_n \right) - P \left( \tilde{M}_n^{(k)} \leq u_n \right) \right| \ll \frac{1}{n^\gamma}
\]

for some \( \gamma > 0 \). Relations (75) and (76) imply

\[
\text{(77)} \quad \lim_{n \to \infty} P \left( M_n^{(k)} \leq u_n \right) = e^{-\tau k - 1 \sum_{s=0}^{k-1} \frac{\tau^s}{s!},}
\]

provided \( \tau \) fulfils (14). Thus, in view of the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan [1], in order to prove (4), it is enough to show that

\[
\text{(78)} \quad \text{Var} \left( \sum_{n=k}^{N} \frac{1}{n} I \left( M_n^{(k)} \leq u_n \right) \right) \ll (\log N)^2 (\log \log N)^{-(1+\varepsilon)}
\]

for some \( \varepsilon > 0 \). We have

\[
\text{(79)} \quad \text{Var} \left( \sum_{n=k}^{N} \frac{1}{n} I \left( M_n^{(k)} \leq u_n \right) \right) \leq \sum_3 + \sum_4,
\]

where \( \sum_3, \sum_4 \) are defined as \( \sum_1, \sum_2 \) in (65), but for the case, when \( k_n \equiv k \). Obviously

\[
\text{(80)} \quad \sum_3 < \infty.
\]
In addition, by (74),
\[
\sum_{4} \ll \sum_{k \leq m < n \leq N, \atop m \leq n/k - 1} \frac{1}{mn} \frac{1}{n^\gamma} + \sum_{k \leq m < n \leq N, \atop m \leq n/k} \frac{km}{mn} - k + \sum_{k \leq m < n \leq N, \atop m > n/k - 1} \frac{1}{mn} =: H_1 + H_2 + H_3.
\]

By proceeding analogously as in the estimation of $G_1$, $G_2$ in the proof of Theorem 1 (i), we immediately get
\[
\tag{82} H_1 + H_2 \ll \log N.
\]

Furthermore, it is easily seen from the definition of $H_3$ in (81) that
\[
\tag{83} H_3 \ll \sum_{n=k}^{N} \frac{1}{n} \ll \log N.
\]

Thus, due to (81)–(83),
\[
\tag{84} \sum_{4} \ll \log N.
\]

By (79), (80) and (84), we obtain
\[
\text{Var} \left( \sum_{n=k}^{N} \frac{1}{n} I \left( M_n^{(k)} \leq u_n \right) \right) \ll \log N.
\]

Hence, the relation in (78) is satisfied. Consequently, by Lemma 3.1 in Csaki and Gonchigdanzan [1],
\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=k}^{N} \frac{1}{n} \left\{ I \left( M_n^{(k)} \leq u_n \right) - P \left( M_n^{(k)} \leq u_n \right) \right\} = 0 \text{ a.s.}
\]

This, (77) and the regularity property of logarithmic means yield (4), which is statement (i) of Theorem 2. \qed

**Proof of Theorem 2 (ii).** Let $x$ be arbitrary real number. Since (13) holds, it follows from Theorem 4.3.3 in Leadbetter et al. [3] that, provided \{a_n\}, \{b_n\} are such as in (5), we have
\[
\lim_{n \to \infty} n (1 - \Phi (x/a_n + b_n)) = e^{-x}.
\]

It is easily seen now that statement (ii) of Theorem 2 is a special case of its, the earlier proved, statement (i), with: $u_n := x/a_n + b_n$, $\tau := e^{-x}$. \qed
The almost sure central limit theorems...

References


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