Almost symplectic structures on the linear frame bundle from linear connection

Abstract. We describe all $\mathcal{M}_m$-natural operators $S: Q \rightsquigarrow \text{Symp} P^1$ transforming classical linear connections $\nabla$ on $m$-dimensional manifolds $M$ into almost symplectic structures $S(\nabla)$ on the linear frame bundle $P^1 M$ over $M$.

Let $V$ be a real vector space of even dimension. A bilinear form $\varpi: V \times V \to \mathbb{R}$ is called a symplectic form if it is antisymmetric and nondegenerate, i.e., it satisfies

$$\varpi(v, v) = 0 \text{ for all } v \in V \text{ and if } \varpi(v, u) = 0 \text{ for all } v \in V, \text{ then } u = 0.$$ 

A vector space $V$ is a symplectic vector space if it is equipped with a symplectic form, [1].

Let $\mathcal{M}_m$ denote the category of $m$-dimensional manifolds and their embeddings and $\mathcal{FM}$ denote the category of fibred manifolds and fibred maps between them.

For any $m$-dimensional manifold $M$ we have the linear frame bundle $P^1 M = \text{inv}J_0^1(\mathbb{R}^m, M)$ of the manifold $M$. This is a principal bundle with corresponding Lie group $GL(m) = G_m = \text{inv}J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0$, which acts on $P^1 M$ on the right via compositions of jets. Every map $\psi: M_1 \to M_2$ from the category $\mathcal{M}_m$ induces a map $P^1 \psi: P^1 M_1 \to P^1 M_2$ by $P^1 \psi(j_0^1 \varphi) = j_0^1(\psi \circ \varphi)$, where $\varphi: \mathbb{R}^m \to M_1$ is a map from the category $\mathcal{M}_m$. The correspondence $P^1: \mathcal{M}_m \to \mathcal{FM}$ is a bundle functor in the sense of [3].

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For any $2n$-dimensional manifold $N$ we have an almost symplectic structures bundle $\text{Symp}(N) = \bigcup_{y \in N} \text{Symp}(T_yN)$ over the manifold $N$, where $\text{Symp}(T_yN)$ denotes the set of symplectic forms $\varpi: T_yN \times T_yN \to \mathbb{R}$ on the tangent space $T_yN$. The bundle $\text{Symp}(N)$ is a subbundle (but not vector subbundle) of a vector bundle $T^*N \otimes T^*N$ of tensors of type $(0,2)$ over $N$. Sections $\Omega: N \to \text{Symp}(N)$ are called almost symplectic structures on the manifold $N$. Every embedding $\psi: N_1 \to N_2$ induces a fibred map $\text{Symp}(\psi): \text{Symp}(N_1) \to \text{Symp}(N_2)$ being restriction of $T^*\psi \otimes T^*\psi: T^*N_1 \otimes T^*N_1 \to T^*N_2 \otimes T^*N_2$ to $\text{Symp}(N)$. The correspondence $\text{Symp}: \mathcal{M}f_{2n} \to \mathcal{F}M$ is a bundle functor in the sense of [3].

Let $M$ be an $m$-dimensional manifold. We have the classical linear connection bundle $QM := (id_{T^*M} \otimes \pi^1)^{-1}(id_{T^*M}) \subset T^*M \otimes J^1TM$ of the manifold $M$, where $\pi^1: J^1TM \to TM$ is the projection of the first jet prolongation $J^1TM = \{j^1_1X: X \in \mathfrak{X}(M), x \in M\}$ of the tangent bundle $TM$ of the manifold $M$. Sections $\nabla: M \to QM$ correspond bijectively to classical linear connections on $M$. Every embedding $f: M_1 \to M_2$ induces a fibred map $Qf: QM_1 \to QM_2$ covering $f$. The correspondence $Q: \mathcal{M}f_{m} \to \mathcal{F}M$ is a bundle functor in the sense of [3].

Let $\{A^i_j\}$, $i,j = 1, \ldots, m$ be the standard basis in $\mathfrak{gl}(m) = \text{Lie}(\text{GL}(m))$.

For a principal fibre bundle $P^1M$ the action of group $\text{GL}(m)$ on $P^1M$ induces a homomorphism $\sigma$ of Lie algebra $\mathfrak{gl}(m)$ of group $\text{GL}(m)$ into Lie algebra $\mathfrak{X}(P^1M)$ of vector fields on $P^1M$. For every $A \in \mathfrak{gl}(m)$, a vector field $A^\ast = \sigma(A)$ is called the fundamental vector field corresponding to $A$. Since the action of group $\text{GL}(m)$ on $P^1M$ sends each fibre into itself, therefore $A^\ast$ is tangent to the fibre at each $u \in P^1M$, [2].

Let $\nabla$ be a classical linear connection on $m$-dimensional manifold $M$. For every $\xi \in \mathbb{R}^m$ we define the standard horizontal vector field $B(\xi)$ on $P^1M$ as follows. For each $u \in P^1M$, $u: \mathbb{R}^m \to T_{\pi(u)}M$, a vector field $B(\xi)_u$ is the unique horizontal vector at $u$ such that $T\pi((B(\xi))_u) = u(\xi)$, where $\pi: P^1M \to M$, [2].

The canonical form $\theta$ of bundle $P^1M$ is $\mathbb{R}^m$-valued 1-form on $P^1M$ defined by

$$\theta(X) = u^{-1}(T\pi(X)) \quad \text{for} \ X \in T_u(P^1M),$$

where $\pi: P^1M \to M$ and $u: \mathbb{R}^m \to T_{\pi(u)}(M)$, [2].

For a given connection $\nabla$ on $P^1M$ we define a 1-form $\omega$ on $P^1M$ with values in Lie algebra $\mathfrak{gl}(m)$ of group $\text{GL}(m)$ as follows. For each $X \in T_u(P^1M)$ we define $\omega(X)$ to be the unique $A \in \mathfrak{gl}(m)$ such that $(A^\ast)_u$ is equal to the vertical component of vector $X$. The form $\omega$ is called the connection form of the given connection $\nabla$, [2].

Let $B_1, \ldots, B_m$ be the standard horizontal vector fields corresponding to basic vectors $e_1, \ldots, e_m$ of space $\mathbb{R}^m$ and let $\{A^i_j\}$ be fundamental vector fields corresponding to basic vectors $\{A^i_j\}$ of Lie algebra $\mathfrak{gl}(m)$. It is easy
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to verify that \{B_l,A^j_i\} and \{\theta^i,\omega^j_i\} are dual to each other, i.e. they satisfy

\[
\begin{align*}
\theta^k(B_l) &= \delta^k_l, & \theta^k(A^j_i) &= 0, \\
\omega^k(B_l) &= 0, & \omega^k(A^j_i) &= \delta^k_i \delta^j_l,
\end{align*}
\]

where \theta^i are components of the canonical 1-form and \omega^j_i are components of the connection form.

**Proposition 1** ([2]). The \(m^2 + m\) vector fields \{\(B_k, A^j_i\); \(i, j, k = 1, \ldots, m\}\) define an absolute parallelism in the bundle \(P^1M\).

The following definition of a natural operator is particular case of an idea of natural operator which was considered in [3].

**Definition 1.** An \(\mathcal{M}f_m\)-natural operator \(S: Q \rightsquigarrow \text{Symp} P^1\) is a family of \(\mathcal{M}f_m\)-invariant regular operators \(S = (S_M)\)

\[
S_M: Q(M) \rightarrow \text{Symp}(P^1M)
\]

for any manifold \(M\) from the category \(\mathcal{M}f_m\), where \(Q(M)\) is the set of all linear connections on the manifold \(M\) (sections of \(Q(M) \rightarrow M\)) and \(\text{Symp}(P^1M)\) is the set of all almost symplectic structures on \(P^1M\) (sections of \(\text{Symp}(P^1M) \rightarrow P^1M\)). The invariance means that if \(\nabla_1 \in Q(M_1)\) and \(\nabla_2 \in Q(M_2)\) are \(\psi\)-related by \(\psi: M_1 \rightarrow M_2\), that is \(Q(\psi) \circ \nabla_1 = \nabla_2 \circ \psi\), then \(S(\nabla_1)\) and \(S(\nabla_2)\) are \(P^1\psi\)-related, that is \(\text{Symp}(P^1\psi) \circ S(\nabla_1) = S(\nabla_2) \circ P^1\psi\). The regularity means that smoothly parametrized families of classical linear connections are transformed by \(S\) on smoothly parametrized families of almost symplectic structures.

In the present note we will classify all natural operators \(S\) and obtained result will be modification of result in [4].

**Remark 1.** In [4] there were described geometric constructions on higher order frame bundles \(P^rM\). In the present paper we describe only case of linear frame bundle \(P^1M\). The generalization of this problem for \(P^rM\) is not possible, because dimension of \(P^rM\) for \(r > 1\) does not have to be even.

For given connection \(\nabla \in Q(M)\) with respect to the global basis of vector fields \(\{B_k, A^j_i\\}\) on \(P^1M\) we have a canonical (in \(\nabla\)) fibred diffeomorphism

\[
K_\nabla: P^1M \times \text{Symp}(\mathbb{R}^{m^2 + m}) \rightarrow \text{Symp}(P^1M)
\]

covering \(id_{P^1M}\) defined by the condition that the matrix of map \(K_\nabla(u(x), \varpi)\) in the basis \(\{B_k(\nabla)(u(x)), A^j_i(\nabla)(u(x))\}\) is the same as the one of the symplectic form \(\varpi\) in the canonical basis of space \(\mathbb{R}^{m^2 + m}\).
Let \( Z^s = J^s_0(Q(R^m)), s = 0, 1, \ldots, \infty \) be the set of \( s \)-jets \( j^s_0 \nabla \) of all classical linear connections \( \nabla \) on \( R^m \) satisfying
\[
\sum_{j,k=1}^m \nabla^j_{jk}(x)x^jx^k = 0 \quad \text{for} \quad i = 1, \ldots, m,
\]
it means that the usual coordinates \( x^1, \ldots, x^m \) on \( R^m \) are \( \nabla \)-normal with center \( 0 \in R^m \).

**Example 1.** General construction: Let \( \mu: Z^\infty \rightarrow \text{Symp}(R^{m^2+m}) \) be a map satisfying the following local finite determination property.

For any \( \rho \in Z^\infty \) we can find an open neighborhood \( U \subset Z^\infty \) of jet \( \rho \), a natural number \( s \) and a smooth map \( f: \pi_s(U) \rightarrow \text{Symp}(R^{m^2+m}) \) such that \( \mu = f \circ \pi_s \) on \( U \), where \( \pi_s: Z^\infty \rightarrow Z^s \) is the jet projection. (A simple example of such \( \mu \) is \( \mu = f \circ \pi_s \) for smooth \( f: Z^s \rightarrow \text{Symp}(R^{m^2+m}) \) and for finite number \( s \).

Given a classical linear connection \( \nabla \) on an \( m \)-dimensional manifold \( M \) we define an almost symplectic structure \( S^{(\mu)}(\nabla) \) on \( P^1M \) as follows. Let \( u(x) \in (P^1M)_x, x \in M \). Choose a \( \nabla \)-normal coordinate system \( \psi \) on \( M \) with center \( x \) such that \( P^1\psi(u(x)) = I^0 = j^1_0(id_{R^m}) \). Such a coordinate system \( \psi \) exists. Then \( \text{germ}_x(\psi) \) is uniquely determined. We put
\[
S^{(\mu)}(\nabla)_{u(x)} = \text{Symp}(P^1(\psi^{-1}))(K_{\psi, \nabla}(I^0, \mu(j^\infty_0(\psi, \nabla)))).
\]
Since \( \text{germ}_x(\psi) \) is uniquely determined, then above definition is correct. The family \( S^{(\mu)}: Q \rightsquigarrow \text{Symp} P^1 \) is an \( \mathcal{M}_{f_{m}} \)-natural operator.

**Theorem 1.** Any \( \mathcal{M}_{f_{m}} \)-natural operator \( S: Q \rightsquigarrow \text{Symp} P^1 \) is of the form \( S^{\prec \mu} \) for some uniquely determined (by \( S \)) function \( \mu: Z^\infty \rightarrow \text{Symp}(R^{m^2+m}) \) satisfying local finite determination property.

**Proof.** Let \( S: Q \rightsquigarrow \text{Symp} P^1 \) be an \( \mathcal{M}_{f_{m}} \)-natural operator. Define \( \mu: Z^\infty \rightarrow \text{Symp}(R^{m^2+m}) \) by
\[
(I^0, \mu(j^\infty_0(\nabla))) = K_{\nabla}^{-1}(S(\nabla)(I^0)).
\]
Then by non-linear Peetre theorem, [3], \( \mu \) satisfies local finite determination property. Then by definitions of \( \mu \) and \( S^{\prec \mu} \) we have that \( S(\nabla)(I^0) = S^{\prec \mu}(\nabla)(I^0) \) for any classical linear connection \( \nabla \) on \( R^m \) such that the identity map \( id_{R^m} \) is a \( \nabla \)-normal coordinate system with center \( 0 \in R^m \). Then by the invariance of \( S \) and \( S^{\prec \mu} \) with respect to normal coordinates we deduce that \( S = S^{\prec \mu} \).

**Remark 2.** Symplectic geometry methods are key ingredients in the study of dynamical systems, mathematical physics, analytical mechanics, differential geometry; [1], [5].
References


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