doi: 10.2478/v10062-010-0008-8

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIV, NO. 1, 2010 SECTIO A 93–113

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Periodic solutions for second-order Hamiltonian systems with a p-Laplacian

Abstract. In this paper, by using the least action principle, Sobolev's inequality and Wirtinger's inequality, some existence theorems are obtained for periodic solutions of second-order Hamiltonian systems with a p-Laplacian under subconvex condition, sublinear growth condition and linear growth condition. Our results generalize and improve those in the literature.

1. Introduction. Consider the second-order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) + e(t), \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $e(t) \in L^1([0,T];\mathbb{R}^N)$ and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) *$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

2000 *Mathematics Subject Classification.* 34C25, 58E50.

Key words and phrases. Second-order Hamiltonian systems, p-Laplacian, periodic solution, Sobolev's inequality, Wirtinger's inequality, the least action principle.

The corresponding function φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t))dt + \int_0^T (e(t), u(t))dt$$

is continuously differentiable and weakly lower semicontinuous on H_T^1 (see [3]), where

$$H_T^1 = \big\{u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}, u(0) = u(T) \text{ and } \dot{u} \in L^2([0,T])\big\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left[\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right]^{1/2} \tag{1.2}$$

for each $u \in H_T^1$. Moreover, one has

$$(\varphi'(u), v) = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t)) + (e(t), v(t))]dt$$

for $u, v \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ (see [3]).

When $e(t) \equiv 0$, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods (see [1]–[15]). Many solvability conditions are given, such as the coercive condition (see [1]), the periodicity condition (see [11]), the convexity condition (see [2]), the subadditive condition (see [7]). Specially, when $e(t) \equiv 0$, in [13], Wu and Tang obtained the following theorem:

Theorem A. *Suppose $F = G(t,x) + H(t,x)$ with G and H satisfying assumption* (A) *and the following conditions:*

(A_1) *$G(t,\cdot)$ is (λ,μ)-subconvex for a.e. $t \in [0,T]$, where $\lambda, \mu > 1/2$ and $\mu < 2\lambda^2$;*

(A_2) *there exist $\alpha \in [0,1)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ such that*

$$|\nabla H(t,x)| \le f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

$$\frac{1}{|x|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] \to +\infty \quad \text{as} \quad |x| \to \infty. \tag{A_3}$$

Then problem (1.1) *has at least one solution which minimizes φ on H_T^1.*

Remark 1.1. A function $G : \mathbb{R}^N \to \mathbb{R}$ is called (λ,μ)-subconvex if

$$G(\lambda(x+y)) \le \mu(G(x) + G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in \mathbb{R}^N$ (see [13]).

Let

$$G(t,x)=h(t)|x|^{5/4},\quad H(t,x)=\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+(0.6T-t)|x|^{3/2}. \tag{1.3}$$

where $h(t)\in L^1([0,T];\mathbb{R}^+)$. Then $G(t,x)$ is $(2,2^{9/4})$-subconvex for a.e. $t\in[0,T]$. In fact, by Young's inequality,

$$\begin{aligned}
G(t,2(x+y)) &= h(t)|2(x+y)|^{5/4}\\
&= 2^{5/4}h(t)|x+y||x+y|^{1/4}\\
&\le 2^{5/4}h(t)(|x|+|y|)(|x|+|y|)^{1/4}\\
&\le 2^{5/4}h(t)(|x|+|y|)(|x|^{1/4}+|y|^{1/4})\\
&= 2^{5/4}h(t)(|x|^{5/4}+|y|^{5/4}+|x||y|^{1/4}+|y||x|^{1/4})\\
&\le 2^{5/4}h(t)\Bigg(|x|^{5/4}+|y|^{5/4}+\frac{4|x|^{5/4}}{5}\\
&\qquad\qquad +\frac{|y|^{5/4}}{5}+\frac{4|y|^{5/4}}{5}+\frac{|x|^{5/4}}{5}\Bigg)\\
&= 2^{9/4}h(t)(|x|^{5/4}+|y|^{5/4})\\
&= 2^{9/4}(G(t,x)+G(t,y)).
\end{aligned}$$

Obviously, $\lambda=2>1/2$ and $\mu=2^{9/4}<2^3=2\lambda^2$. Therefore, G satisfies (A_1). Moreover, it is easy to see that

$$\begin{aligned}
|\nabla H(t,x)| &\le \frac{7}{4}\left|\sin\left(\frac{2\pi t}{T}\right)\right||x|^{3/4}+\frac{3}{2}|0.6T-t||x|^{1/2}\\
&\le \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right|+\varepsilon\right)|x|^{3/4}+\frac{T^3}{\varepsilon^2}
\end{aligned}$$

for all $x\in\mathbb{R}^N$ and $t\in[0,T]$, where $\varepsilon>0$. The above inequality shows that (A_2) holds with $\alpha=3/4$ and

$$f(t)=\frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right|+\varepsilon\right),\qquad g(t)=\frac{T^3}{\varepsilon^2}.$$

However, $F(t,x)$ does not satisfy (A_3). In fact,

$$\begin{aligned}
&\liminf_{|x|\to\infty}|x|^{-2\alpha}\int_0^T F(t,x)dt\\
&=\liminf_{|x|\to\infty}|x|^{-3/2}\int_0^T\left[h(t)|x|^{5/4}+\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+(0.6T-t)|x|^{3/2}\right]dt\\
&=0.1T^2.
\end{aligned}$$

The above example shows that it is valuable to improve (A_3).

When $\alpha = 1$, $e(t) \equiv 0$, in [14], the authors obtained the following theorem:

Theorem B. *Suppose* $F = G(t, x) + H(t, x)$ *with* G *and* H *satisfying assumption* (A). *Assume that* (A_1) *and the following conditions hold:*
(B_2) *there exist* $f, g \in L^1([0, T]; \mathbb{R}^+)$ *with* $0 < \int_0^T f(t)dt < 12/T$ *such that*

$$|\nabla H(t, x)| \leq f(t)|x| + g(t)$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0, T]$;

$$(B_3) \qquad \frac{1}{|x|^2}\left[\frac{1}{\mu}\int_0^T G(t, \lambda x)dt + \int_0^T H(t, x)dt\right] \to +\infty \quad as \quad |x| \to \infty.$$

Then problem (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

Theorem B is not correct. In fact, by condition (A_1), in a similar way to [13], we get

$$G(t, x) \leq 2\mu(|x|^\beta + 1)a_0 b(t)$$

for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$, where $\beta < 2$, $a_0 = \max_{0 \leq s \leq 1} a(s)$. Then

$$\limsup_{|x|\to\infty} \frac{\int_0^T G(t, x)dt}{|x|^2} \leq 0. \tag{1.4}$$

By condition (B_2), we get

$$\begin{aligned} |H(t, x)| &\leq \int_0^1 |\nabla H(t, sx)||x|ds + |H(t, 0)| \\ &\leq \int_0^1 (f(t)|sx| + g(t))|x|ds + |H(t, 0)| \\ &= \frac{f(t)}{2}|x|^2 + g(t)|x| + |H(t, 0)|. \end{aligned}$$

Then

$$\limsup_{|x|\to\infty} \frac{\int_0^T H(t, x)dt}{|x|^2} \leq \frac{1}{2}\int_0^T f(t)dt. \tag{1.5}$$

Thus, it follows from (1.4) and (1.5) that

$$\limsup_{|x|\to\infty} \frac{1}{|x|^2}\left[\frac{1}{\mu}\int_0^T G(t, \lambda x)dt + \int_0^T H(t, x)dt\right] \leq \frac{1}{2}\int_0^T f(t)dt,$$

which contradicts condition (B_3) of Theorem B. Therefore, there are no functions satisfying Theorem B. Hence, it is necessary to improve Theorem B.

In our paper, by using the least action principle, we will further study the existence of solutions to problem (1.1) based on Theorem A and Theorem B.

In Section 2, we consider more general system

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' = \nabla F(t, u(t)) + e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.6}$$

where $p > 1$, $q > 1$, with p and q satisfying $1/p + 1/q = 1$, and T, $e(t)$ and $F(t,x)$ are the same as in problem (1.1). When $e(t) \equiv 0$, by using the minimax principle and the dual least action principle, it has been proved in [16] and [17], respectively, that system (1.6) has at least one solution. In this section, we improve two inequalities (which are often called Sobolev's inequality and Writinger's inequality) given in [3] (see Proposition 1.1 in [3]) and then by using them and the least action principle, we obtain some existence results for system (1.6).

In Section 3, we consider system (1.1), which is the special case $p = 2$ of (1.6). When $p = 2$, we have sharp estimates for Sobolev's inequality and Writinger's inequality (see Proposition 1.3 in [3]) so we can obtain better results than those following from the general Sobolev's inequality and Writinger's inequality. Even if $e(t) \equiv 0$, our Theorem 2.1 with $p = 2$ and $\alpha \in (0,1)$ and Theorem 3.1 with $\alpha \in (0,1)$ still improve Theorem A. Theorem 2.2 with $p = 2$ and Theorem 3.2 improve Theorem B. Our Theorems 2.3–2.5 and Theorems 3.3–3.5 cover the case when $\int_0^T e(t) = 0$ in Theorem 2.1 is deleted which leads to some new results. In Section 4, some examples will be given to illustrate our results.

2. Case $p > 1$. In this section, we consider system (1.6). Let

$$W_T^{1,p} = \{u : [0,T] \to \mathbb{R}^N \mid u(t) \text{ is absolutely continuous on } [0,T], \\ u(0) = u(T) \text{ and } \dot{u} \in L^p(0,T;\mathbb{R}^N)\}.$$

Then $W_T^{1,p}$ is a Banach space with the norm defined by

$$\|u\| = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{\frac{1}{p}}, \qquad u \in W_T^{1,p}.$$

It follows from [3] that $W_T^{1,p}$ is reflexive and uniformly convex.

The following two lemmas (that is Lemma 2.1 and Lemma 2.2) also have been proved in our another paper [18] which is in press.

Lemma 2.1 (see [18]). *Let $a > 0$, b, $c \geq 0$, $\varepsilon > 0$.*

(i) *If $\alpha \in (0,1]$, then*

$$(a + b + c)^\alpha \leq a^\alpha + b^\alpha + c^\alpha; \tag{2.1}$$

(ii) *If $\alpha \in (1,+\infty)$, then there exists $B(\varepsilon) > 1$ such that*

$$(a + b + c)^\alpha \leq (1+\varepsilon)a^\alpha + B(\varepsilon)b^\alpha + B(\varepsilon)c^\alpha. \tag{2.2}$$

Proof. It is easy to verify (i). In the sequel, we only prove (ii). Since

$$\lim_{x\to+\infty} \frac{x^{\alpha/(\alpha-1)} - 1}{\left[x^{1/(\alpha-1)} - 1\right]^\alpha} = 1,$$

there exists a constant $M = M(\varepsilon) > 1$ such that

$$\frac{M^{\alpha/(\alpha-1)} - 1}{\left[M^{1/(\alpha-1)} - 1\right]^{\alpha}} < 1 + \varepsilon.$$

Set

$$f(t) = (1+t)^{\alpha} - Mt^{\alpha}, \quad t \in [0,1].$$

Then

$$f(t) \le \frac{M^{\alpha/(\alpha-1)} - 1}{\left[M^{1/(\alpha-1)} - 1\right]^{\alpha}} < 1 + \varepsilon, \quad t \in [0,1].$$

It follows that

$$(1+t)^{\alpha} \le 1 + \varepsilon + Mt^{\alpha}, \quad t \in [0,1]. \tag{2.3}$$

If $a \le b + c$, then

$$(a+b+c)^{\alpha} \le 2^{\alpha}(b+c)^{\alpha} \le 2^{2\alpha-1}b^{\alpha} + 2^{2\alpha-1}c^{\alpha}.$$

This shows that (2.2) holds. If $a > b + c$, then by (2.3), we have

$$\begin{aligned}(a+b+c)^{\alpha} &\le a^{\alpha}\left(1 + \frac{b+c}{a}\right)^{\alpha} \le a^{\alpha}\left(1 + \varepsilon + M\frac{(b+c)^{\alpha}}{a^{\alpha}}\right) \\ &\le (1+\varepsilon)a^{\alpha} + 2^{\alpha-1}Mb^{\alpha} + 2^{\alpha-1}Mc^{\alpha}.\end{aligned}$$

This shows that (2.2) also holds. The proof is complete. □

Lemma 2.2 (see [18]). *Let* $u \in W_T^{1,p}$ *and* $\int_0^T u(t)dt = 0$. *Then*

$$\|u\|_{\infty} \le \left(\frac{T}{q+1}\right)^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p} \tag{2.4}$$

and

$$\int_0^T |u(s)|^p ds \le \frac{T^p \Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds, \tag{2.5}$$

where

$$\Theta(p,q) = \int_0^1 \left[s^{q+1} + (1-s)^{q+1}\right]^{p/q} ds.$$

Proof. Fix $t \in [0,T]$. For every $\tau \in [0,T]$, we have

$$u(t) = u(\tau) + \int_{\tau}^{t} \dot{u}(s)ds. \tag{2.6}$$

Set

$$\phi(s) = \begin{cases} s, & 0 \le s \le t, \\ T - s, & t \le s \le T. \end{cases}$$

Integrating (2.6) over $[0,T]$ and using the Hölder inequality, we obtain

$$
\begin{aligned}
T|u(t)| &= \left|\int_0^T u(\tau)d\tau + \int_0^T \int_\tau^t \dot{u}(s)dsd\tau\right| \\
&\le \int_0^t \int_\tau^t |\dot{u}(s)|dsd\tau + \int_t^T \int_t^\tau |\dot{u}(s)|dsd\tau \\
&= \int_0^t s|\dot{u}(s)|ds + \int_t^T (T-s)|\dot{u}(s)|ds \\
&= \int_0^T \phi(s)|\dot{u}(s)|ds \\
&\le \left(\int_0^T [\phi(s)]^q ds\right)^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p} \\
&= \frac{1}{(q+1)^{1/q}} \left[t^{q+1} + (T-t)^{q+1}\right]^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p}.
\end{aligned} \tag{2.7}
$$

Since $t^{q+1} + (T-t)^{q+1} \le T^{q+1}$ for $t \in [0,T]$, it follows from (2.7) that (2.4) holds. On the other hand, from (2.7), we have

$$
\begin{aligned}
T^p \int_0^T |u(t)|^p dt &\le \frac{1}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds\right) \int_0^T \left[t^{q+1} + (T-t)^{q+1}\right]^{p/q} dt \\
&\le \frac{T^{1+p(q+1)/q}}{(q+1)^{p/q}} \left(\int_0^T |\dot{u}(s)|^p ds\right) \int_0^1 \left[s^{q+1} + (1-s)^{q+1}\right]^{p/q} ds \\
&= \frac{T^{2p}\Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(s)|^p ds.
\end{aligned}
$$

It follows that (2.5) holds. The proof is complete. □

Remark 2.1. Clearly, our Lemma 2.1 improves Proposition 1.1 in [3]. In fact, according to the proof of Proposition 1.1 in [3], it is easy to show that if $\int_0^T u(t)dt = 0$, then

$$
\|u\|_\infty \le T^{1/q} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p} \tag{2.8}
$$

and

$$
\int_0^T |u(s)|^p ds \le T^p \int_0^T |\dot{u}(s)|^p ds. \tag{2.9}
$$

Obviously, our result is better.

Lemma 2.3 (see [16]). *In Sobolev's space $W_T^{1,p}$, for $u \in W_T^{1,p}$, $\|u\| \to \infty$ if and only if*

$$
\left(|\bar{u}|^p + \int_0^T |\dot{u}(t)|^p dt\right)^{1/p} \to \infty.
$$

Let

$$\tilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} \;\middle|\; \int_0^T u(t)dt = 0 \right\}.$$

It is easy to show that $\tilde{W}_T^{1,p}$ is a subset of $W_T^{1,p}$ and $W_T^{1,p} = \mathbb{R}^N \oplus \tilde{W}_T^{1,p}$. For $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. By Lemma 2.2, we have

$$\int_0^T |\tilde{u}(t)|^p dt \le \frac{T^p \Theta(p,q)}{(q+1)^{p/q}} \int_0^T |\dot{u}(t)|^p dt \quad \text{for every} \quad u \in W_T^{1,p} \tag{2.10}$$

and

$$\|\tilde{u}\|_\infty^p \le \left(\frac{T}{q+1}\right)^{p/q} \int_0^T |\dot{u}(t)|^p dt \quad \text{for every} \quad u \in W_T^{1,p}. \tag{2.11}$$

Let $\varphi_p : W_T^{1,p} \to \mathbb{R}$ be defined by

$$\varphi_p(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t,u(t))dt + \int_0^T (e(t),u(t))dt. \tag{2.12}$$

Then φ_p is continuously differentiable and weakly lower semicontinuous in $W_T^{1,p}$ (see [3]). Moreover,

$$\begin{aligned} \langle \varphi_p'(u), v\rangle = & \int_0^T \left(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)\right) dt \\ & + \int_0^T (\nabla F(t,u(t)), v(t))dt + \int_0^T (e(t),v(t))dt \end{aligned} \tag{2.13}$$

for $u, v \in W_T^{1,p}$. It is well known that the solutions of problem (1.6) correspond to the critical points of φ_p (see [3]).

Next, for the sake of convenience, we denote

$$M_1 = \int_0^T f(t)dt, \quad M_2 = \int_0^T g(t)dt, \quad M_3 = \int_0^T |e(t)|dt.$$

Theorem 2.1. *Suppose $F = G(t,x) + H(t,x)$ with G and H satisfying assumption* (A) *and $e \in L^1([0,T];\mathbb{R})$ satisfies $\int_0^T e(t)dt = 0$. Assume the following conditions hold:*

(I_1) $G(t,\cdot)$ is (λ,μ)-subconvex for a.e. $t \in [0,T]$, where $\lambda, \mu > 1/2$ and $\mu < 2^{p-1}\lambda^p$;

(I_2) there exist $\alpha \in (0, p-1)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ such that

$$|\nabla H(t,x)| \le f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(I_3) $$\liminf_{|x|\to\infty} \frac{1}{|x|^{q\alpha}}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{T}{q(q+1)}\left(\int_0^T f(t)dt\right)^q.$$

Then (1.6) *has at least one solution which minimizes φ_p on $W_T^{1,p}$.*

Proof. By (I_3), we can choose constants $\varepsilon > 0$, $a_1 > [T/(q+1)]^{1/q}$ such that

$$\liminf_{|x|\to\infty} \frac{1}{|x|^{q\alpha}} \left[\frac{1}{\mu} \int_0^T G(t, \lambda x)dt + \int_0^T H(t,x)dt \right] > \frac{[(1+\varepsilon)a_1 M_1]^q}{q}. \tag{2.14}$$

Let $\beta = \log_{2\lambda}(2\mu)$. Then $0 < \beta < p$. In a similar way as in [13], by the (λ, μ)-subconvexity of $G(t, \cdot)$, we can prove that

$$G(t,x) \leq (2\mu|x|^\beta + 1)a_0 b(t) \tag{2.15}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, where $0 < \beta < p$ and $a_0 = \max_{0\leq s\leq 1} a(s)$. It follows from (I_1), (2.15) and (2.11) that

$$\begin{aligned} \int_0^T G(t, u(t))dt &\geq \frac{1}{\mu} \int_0^T G(t, \lambda \bar{u})dt - \int_0^T G(t, -\tilde{u}(t)) \\ &\geq \frac{1}{\mu} \int_0^T G(t, \lambda \bar{u})dt - (2\mu\|\tilde{u}\|_\infty^\beta + 1)a_0 \int_0^T b(t)dt \\ &\geq \frac{1}{\mu} \int_0^T G(t, \lambda \bar{u})dt - C_1\|\dot{u}\|_{L^p}^\beta - C_2 \end{aligned} \tag{2.16}$$

for some positive constants C_1 and C_2. By (I_2), Lemma 2.1 and (2.11), we get

$$\begin{aligned} &\left| \int_0^T [H(t,u(t)) - H(t,\bar{u})]dt \right| \\ &= \left| \int_0^T \int_0^1 (\nabla H(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t))dsdt \right| \\ &\leq \int_0^T \int_0^1 f(t)|\bar{u} + s\tilde{u}(t)|^\alpha \cdot |\tilde{u}(t)|dsdt + \int_0^T \int_0^1 g(t)|\tilde{u}(t)|dsdt \\ &\leq M_1(1+\varepsilon)|\bar{u}|^\alpha \|\tilde{u}\|_\infty + M_1 B(\varepsilon)\|\tilde{u}\|_\infty^{\alpha+1} + M_2\|\tilde{u}\|_\infty \\ &\leq \frac{1}{pa_1^p}\|\tilde{u}\|_\infty^p + \frac{[(1+\varepsilon)a_1M_1]^q}{q}|\bar{u}|^{q\alpha} + M_1B(\varepsilon)\|\tilde{u}\|_\infty^{\alpha+1} + M_2\|\tilde{u}\|_\infty \\ &\leq \frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q} \|\dot{u}\|_{L^p}^p + \frac{[(1+\varepsilon)a_1M_1]^q}{q}|\bar{u}|^{q\alpha} \\ &\quad + \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1B(\varepsilon)\|\dot{u}\|_{L^p}^{\alpha+1} + \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p}. \end{aligned} \tag{2.17}$$

It follows from (2.12), (2.16), (2.17) and $\int_0^T e(t)dt = 0$ that

$$\begin{aligned}
\varphi_p(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T G(t,u(t))dt + \int_0^T [H(t,u(t)) - H(t,\bar{u})]dt \\
&\quad + \int_0^T H(t,\bar{u})dt + \int_0^T (e(t), \bar{u} + \tilde{u}(t))dt \\
&\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p + \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt - C_1\|\dot{u}\|_{L^p}^\beta - C_2 \\
&\quad - \frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot{u}\|_{L^p}^p - \frac{[(1+\varepsilon)a_1M_1]^q}{q}|\bar{u}|^{q\alpha} \\
&\quad - \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1B(\varepsilon)\|\dot{u}\|_{L^p}^{\alpha+1} - \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p} \\
&\quad + \int_0^T H(t,\bar{u})dt - \left(\frac{T}{q+1}\right)^{1/q} M_3\|\dot{u}\|_{L^p} \\
&= \left(\frac{1}{p} - \frac{1}{pa_1^p}\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}\|_{L^p}^p - C_1\|\dot{u}\|_{L^p}^\beta - C_2 \\
&\quad - \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1B(\varepsilon)\|\dot{u}\|_{L^p}^{\alpha+1} - \left(\frac{T}{q+1}\right)^{1/q} (M_2+M_3)\|\dot{u}\|_{L^p} \\
&\quad + |\bar{u}|^{q\alpha}\left\{\frac{1}{|\bar{u}|^{q\alpha}}\left[\frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt + \int_0^T H(t,\bar{u})dt\right] \right. \\
&\qquad \left. - \frac{[(1+\varepsilon)a_1M_1]^q}{q}\right\}.
\end{aligned}$$

By Lemma 2.3, $\|u\| \to \infty$ if and only if $\left(|\bar{u}|^p + \|\dot{u}\|_{L^p}^p\right)^{1/p} \to \infty$. Hence, the above inequality, $a_1 > [T/(q+1)]^{1/q}$ and (2.14) imply that

$$\varphi_p(u) \to +\infty, \quad \text{as} \quad \|u\| \to \infty.$$

By Theorem 1.1 in [3], the proof of Theorem 2.1 is complete. □

Remark 2.2. Clearly, when $p = 2$ and $\alpha \in (0,1)$, our Theorem 2.1 improve Theorem A. We choose $p = 4$, $\lambda = 1$, $\mu = 3/2$. There exist functions satisfying our Theorem 2.1 but not satisfying Theorem A. For example, let

$$G(t,x) = h(t) + 1 + \sin|x|^2, \quad H(t,x) = (0.5T - t)|x|^{7/2} + 2T^3|x|^{10/3},$$

where $h \in L^1([0,T],\mathbb{R}^+)$ satisfies $h(t) \geq 1$ for a.e. $t \in [0,T]$.

Theorem 2.2. *Suppose $F = G(t,x) + H(t,x)$ with G and H satisfying assumption* (A). *Assume (I_1) and the following conditions hold:*

(I_4) *there exist $f, g \in L^1([0,T];\mathbb{R}^+)$ with $\int_0^T f(t)dt < \left(\frac{q+1}{T}\right)^{p/q}\frac{1}{D_{p-1}}$ such* *that*

$$|\nabla H(t,x)| \le f(t)|x|^{p-1} + g(t)$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$, *where*

$$D_{p-1} = \begin{cases} 1, & p \in (1,2], \\ 2^{p-2}, & p \in (2,+\infty); \end{cases}$$

(I_5)

$$\liminf_{|x|\to\infty} \frac{1}{|x|^p}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{T D_{p-1}^q \left(\int_0^T f(t)dt\right)^q}{q\left[(q+1)^{p/q} - D_{p-1}T^{p/q}\int_0^T f(t)dt\right]^{q/p}}.$$

Then (1.6) *has at least one solution which minimizes* φ_p *on* $W_T^{1,p}$.

Proof. By (I_5), we can choose an $a_2 > T^{1/q}/[(q+1)^{p/q} - M_1 D_{p-1}T^{p/q}]^{1/p}$ such that

$$\liminf_{|x|\to\infty} \frac{1}{|x|^p}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{[D_{p-1}a_2M_1]^q}{q}. \tag{2.18}$$

By (I_1), we can get (2.16). By (I_4) and (2.11), we get

$$\begin{aligned}
&\left|\int_0^T [H(t,u(t)) - H(t,\bar u)]dt\right| \\
&= \left|\int_0^T\int_0^1 (\nabla H(t,\bar u + s\tilde u(t)), \tilde u(t))dsdt\right| \\
&\le \int_0^T\int_0^1 f(t)|\bar u + s\tilde u(t)|^{p-1}\cdot|\tilde u(t)|dsdt + \int_0^T\int_0^1 g(t)|\tilde u(t)|dsdt \\
&\le D_{p-1}M_1|\bar u|^{p-1}\|\tilde u\|_\infty + \frac{M_1D_{p-1}}{p}\|\tilde u\|_\infty^p + M_2\|\tilde u\|_\infty \qquad (2.19)\\
&\le \frac{1}{pa_2^p}\|\tilde u\|_\infty^p + \frac{[D_{p-1}a_2M_1]^q}{q}|\bar u|^p + \frac{M_1D_{p-1}}{p}\|\tilde u\|_\infty^p + M_2\|\tilde u\|_\infty \\
&\le \frac{1}{pa_2^p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot u\|_{L^p}^p + \frac{[D_{p-1}a_2M_1]^q}{q}|\bar u|^p \\
&\quad + \frac{M_1D_{p-1}}{p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot u\|_{L^p}^p + \left(\frac{T}{q+1}\right)^{1/q}M_2\|\dot u\|_{L^p}.
\end{aligned}$$

It follows from (2.16) and (2.19) that

$$\begin{aligned}
\varphi_p(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T G(t,u(t))dt + \int_0^T [H(t,u(t)) - H(t,\bar{u})]dt \\
&\quad + \int_0^T H(t,\bar{u})dt + \int_0^T (e(t), \bar{u} + \tilde{u}(t))dt \\
&\geq \frac{1}{p}\|\dot{u}\|_{L^p}^p + \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt - C_1\|\dot{u}\|_{L^p}^\beta - C_2 \\
&\quad - \frac{1}{pa_2^p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot{u}\|_{L^p}^p - \frac{[D_{p-1}a_2M_1]^q}{q}|\bar{u}|^p \\
&\quad - \frac{M_1D_{p-1}}{p}\left(\frac{T}{q+1}\right)^{p/q}\|\dot{u}\|_{L^p}^p - \left(\frac{T}{q+1}\right)^{1/q} M_2\|\dot{u}\|_{L^p} \\
&\quad + \int_0^T H(t,\bar{u})dt - \left(\frac{T}{q+1}\right)^{1/q} M_3\|\dot{u}\|_{L^p} - M_3|\bar{u}| \\
&= \left[\frac{1}{p} - \frac{1}{pa_2^p}\left(\frac{T}{q+1}\right)^{p/q} - \frac{M_1D_{p-1}}{p}\left(\frac{T}{q+1}\right)^{p/q}\right]\|\dot{u}\|_{L^p}^p \\
&\quad - C_1\|\dot{u}\|_{L^p}^\beta - C_2 - \left(\frac{T}{q+1}\right)^{1/q}(M_2+M_3)\|\dot{u}\|_{L^p} \\
&\quad + |\bar{u}|^p\left\{\frac{1}{|\bar{u}|^p}\left[\frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt + \int_0^T H(t,\bar{u})dt\right] - \frac{[D_{p-1}a_2M_1]^q}{q}\right\} \\
&\quad - M_3|\bar{u}|.
\end{aligned}$$

As $\|u\| \to \infty$ if and only if $\left(|\bar{u}|^p + \|\dot{u}\|_{L^p}^p\right)^{1/p} \to \infty$, the above inequality, $a_2 > T^{1/q}/[(q+1)^{p/q} - M_1D_{p-1}T^{p/q}]^{1/p}$ and (2.18) imply that

$$\varphi_p(u) \to +\infty, \quad \text{as} \quad \|u\| \to \infty.$$

By Theorem 1.1 in [3], the proof of Theorem 2.2 is complete. □

Remark 2.3. We choose $p = 4$, $\lambda = 1$, $\mu = 3/2$. There exist functions satisfying our Theorem 2.2. For example, let

$$G(t,x) = h(t) + 1 + \sin|x|^2, \quad H(t,x) = T^4|x|^4 + (k(t),x).$$

where $k \in L^1([0,T],\mathbb{R}^N)$ and $h \in L^1([0,T],\mathbb{R}^+)$ satisfies $h(t) \geq 1$ for a.e. $t \in [0,T]$.

Next, we consider the case when $\int_0^T e(t)dt = 0$ in Theorem 2.1 is deleted. We will consider three cases: $\alpha \in (1/q, p-1)$, $\alpha = 1/q$ and $\alpha \in (0, 1/q)$.

Theorem 2.3. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (I_1), (I_3) *and the following condition hold:*

(I_2') *there exist* $\alpha \in (1/q, p-1)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t);$$

Then (1.6) *has at least one solution which minimizes* φ_p *on* $W_T^{1,p}$.

Proof. By (I_3), we can choose an $a_3 > [T/(q+1)]^{1/q}$ such that

$$\liminf_{|x|\to\infty} \frac{1}{|x|^{q\alpha}} \left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{[(1+\varepsilon)a_3M_1]^q}{q}. \tag{2.20}$$

By (I_1), we can get (2.16). By (I_2'), we can get (2.17) with $\alpha \in (1/q, p-1)$. It follows from (2.16) and (2.17) with $\alpha \in (1/q, p-1)$ that

$$\begin{aligned}
\varphi_p(u) &= \frac{1}{p}\|\dot u\|_{L^p}^p + \int_0^T G(t,u(t))dt + \int_0^T [H(t,u(t)) - H(t,\bar u)]dt \\
&\quad + \int_0^T H(t,\bar u)dt + \int_0^T (e(t), \bar u + \tilde u(t))dt \\
&\ge \left(\frac{1}{p} - \frac{1}{pa_3^p}\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot u\|_{L^p}^p - C_1\|\dot u\|_{L^p}^\beta \\
&\quad - C_2 - \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1B(\varepsilon)\|\dot u\|_{L^p}^{\alpha+1} \\
&\quad - \left(\frac{T}{q+1}\right)^{1/q}(M_2+M_3)\|\dot u\|_{L^p} - M_3|\bar u| \\
&\quad + |\bar u|^{q\alpha}\left\{\frac{1}{|\bar u|^{q\alpha}}\left[\frac{1}{\mu}\int_0^T G(t,\lambda\bar u)dt + \int_0^T H(t,\bar u)dt\right] - \frac{[(1+\varepsilon)a_3M_1]^q}{q}\right\}.
\end{aligned}$$

As $\|u\| \to \infty$ if and only if $\left(|\bar u|^p + \|\dot u\|_{L^p}^p\right)^{1/p} \to \infty$, the above inequality, $a_3 > [T/(q+1)]^{1/q}$, $\alpha \in (1/q, p-1)$ and (2.20) imply that

$$\varphi_p(u) \to +\infty, \quad \text{as} \quad \|u\| \to \infty.$$

By Theorem 1.1 in [3], the proof of Theorem 2.3 is complete. □

Remark 2.4. Theorem 2.3 shows that in Theorem 2.1, $\int_0^T e(t)dt = 0$ can be deleted when $\alpha \in (1/q, p-1)$.

Theorem 2.4. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (I_1) *and the following conditions hold:*

(I_2'') *there exist* $f, g \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$|\nabla H(t,x)| \le f(t)|x|^{1/q} + g(t);$$

(I_3'')

$$\begin{aligned}
&\liminf_{|x|\to\infty} \frac{1}{|x|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] \\
&\qquad > \frac{T}{q(q+1)}\left(\int_0^T f(t)dt\right)^q + \int_0^T |e(t)|dt.
\end{aligned}$$

Then (1.6) *has at least one solution which minimizes* φ_p *on* $W_T^{1,p}$.

Proof. By (I_3''), we can choose constants $\varepsilon > 0$ and $a_4 > [T/(q+1)]^{1/q}$ such that

$$\liminf_{|x|\to\infty} \frac{1}{|x|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{[(1+\varepsilon)a_4 M_1]^q}{q} + \int_0^T |e(t)|dt. \tag{2.21}$$

By (I_1), we can get (2.16). By (I_2''), we can get (2.17) with $\alpha = 1/q$. It follows from $p > (q+1)/q$, (2.16) and (2.17) with $\alpha = 1/q$ that

$$\begin{aligned}\varphi_p(u) \geq &\left(\frac{1}{p} - \frac{1}{pa_4^p}\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}\|_{L^p}^p - C_1\|\dot{u}\|_{L^p}^\beta - C_2 \\ &- \left(\frac{T}{q+1}\right)^{(q+1)/q^2} M_1 B(\varepsilon)\|\dot{u}\|_{L^p}^{(q+1)/q} - \left(\frac{T}{q+1}\right)^{1/q}(M_2+M_3)\|\dot{u}\|_{L^p} \\ &+ |\bar{u}|\left\{\frac{1}{|\bar{u}|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt + \int_0^T H(t,\bar{u})dt\right] - \frac{[(1+\varepsilon)a_4M_1]^q}{q} - M_3\right\}.\end{aligned}$$

As $\|u\| \to \infty$ if and only if $\left(|\bar{u}|^p + \|\dot{u}\|_{L^p}^p\right)^{1/p} \to \infty$, the above inequality, $a_4 > [T/(q+1)]^{1/q}$, and (2.21) imply that

$$\varphi_p(u) \to +\infty, \quad \text{as} \quad \|u\| \to \infty.$$

By Theorem 1.1 in [3], the proof of Theorem 2.4 is complete. □

Remark 2.5. We choose $p = 4$, $\lambda = 1$, $\mu = 3/2$. There exist functions satisfying our Theorem 2.4. For example, let

$$G(t,x) = h(t) + 1 + \sin|x|^2, \quad H(t,x) = (0.5T - t)|x|^{7/4} + \frac{|x|^3}{1+|x|^2}\cdot l(t),$$

where $l \in L^1([0,T],\mathbb{R}^+)$, $h \in L^1([0,T],\mathbb{R}^+)$ satisfies $h(t) \geq 1$ for a.e. $t \in [0,T]$, and let $e(t) \in L^1([0,T];\mathbb{R}^N)$ satisfy

$$\int_0^T |e(t)|dt < \int_0^T l(t)dt - \frac{9}{28}\cdot\left(\frac{7}{16}\right)^{4/3} T^{11/3}.$$

Theorem 2.5. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (I_1) *and the following conditions hold:*

(I_2''') *there exist* $\alpha \in (0, 1/q)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$|\nabla H(t,x)| \leq f(t)|x|^\alpha + g(t);$$

$$\liminf_{|x|\to\infty} \frac{1}{|x|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \int_0^T |e(t)|dt. \tag{I_3'''}$$

Then (1.6) *has at least one solution which minimizes* φ_p *on* $W_T^{1,p}$.

Proof. Choose an $a_5 > [T/(q+1)]^{1/q}$. By (I_1), we can get (2.16). By (I_2'''), we can get (2.17) with $\alpha \in (0, 1/q)$. It follows from (2.16) and (2.17) with $\alpha \in (0, 1/q)$ that

$$\begin{aligned}\varphi_p(u) \geq & \left(\frac{1}{p} - \frac{1}{pa_5^p}\left(\frac{T}{q+1}\right)^{p/q}\right)\|\dot{u}\|_{L^p}^p - C_1\|\dot{u}\|_{L^p}^{\beta} \\ & - C_2 - \left(\frac{T}{q+1}\right)^{(\alpha+1)/q} M_1 B(\varepsilon)\|\dot{u}\|_{L^p}^{\alpha+1} \\ & - \left(\frac{T}{q+1}\right)^{1/q}(M_2+M_3)\|\dot{u}\|_{L^p} - \frac{[(1+\varepsilon)a_5 M_1]^q}{q}|\bar{u}|^{q\alpha} \\ & + |\bar{u}|\left\{\frac{1}{|\bar{u}|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})dt + \int_0^T H(t,\bar{u})dt\right] - M_3\right\}.\end{aligned}$$

As $\|u\| \to \infty$ if and only if $\left(|\bar{u}|^p + \|\dot{u}\|_{L^p}^p\right)^{1/p} \to \infty$, the above inequality, $a_5 > [T/(q+1)]^{1/q}$, $\alpha \in (0, 1/q)$ and (I_3''') imply that

$$\varphi_p(u) \to +\infty, \quad \text{as} \quad \|u\| \to \infty.$$

By Theorem 1.1 in [3], the proof of Theorem 2.5 is complete. □

Remark 2.6. We choose $p = 4$, $\lambda = 1$, $\mu = 3/2$. There exist functions satisfying our Theorem 2.5. For example, let

$$G(t,x) = h(t) + 1 + \sin|x|^2, \quad H(t,x) = (0.5T - t)|x|^{5/4} + \frac{|x|^3}{1+|x|^2}\cdot l(t),$$

where $l \in L^1([0,T], \mathbb{R}^+)$, $h \in L^1([0,T], \mathbb{R}^+)$ satisfies $h(t) \geq 1$ for a.e. $t \in [0,T]$ and let $e(t) \in L^1([0,T]; \mathbb{R}^N)$ satisfy

$$\int_0^T |e(t)|dt < \int_0^T l(t)dt.$$

3. Case $p = 2$. For $u \in H_T^1 = W_T^{1,2}$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u} = u(t) - \bar{u}$. Then we have the following estimates sharper than (2.11) and (2.10) with $p = 2$.

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \quad \text{(Sobolev's inequality)} \tag{3.1}$$

$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \quad \text{(Wirtinger's inequality)} \tag{3.2}$$

(see Proposition 1.3 in [3]).

Consequently, for the special case $p = 2$, we can obtain better results. The proofs are similar to those in Section 2. We only need to replace (2.11) with (3.1) in the proof. Hence, we just give the results.

Theorem 3.1. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A) *and* $e \in L^1([0,T];\mathbb{R})$ *satisfies* $\int_0^T e(t)dt = 0$. *Assume* (A_1), (A_2) *with* $\alpha \in (0,1)$ *and the following condition hold:*

$$(A_3') \quad \liminf_{|x|\to\infty} \frac{1}{|x|^{2\alpha}} \left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{T}{24}\left(\int_0^T f(t)dt\right)^2.$$

Then (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

Theorem 3.2. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (A_1), (B_2) *and the following condition hold:*

$$(B_3') \quad \liminf_{|x|\to\infty} \frac{1}{|x|^{2}} \left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{T\left(\int_0^T f(t)dt\right)^2}{2\left(12 - T\int_0^T f(t)dt\right)}.$$

Then (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

Theorem 3.3. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (A_1), (A_3') *and the following condition hold:*

(A_2') *there exist* $\alpha \in (1/2,1)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t);$$

Then (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

Theorem 3.4. *Suppose* $F = G(t,x) + H(t,x)$ *with* G *and* H *satisfying assumption* (A). *Assume* (A_1) *and the following conditions hold:*

(A_2'') *there exist* $f, g \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$|\nabla H(t,x)| \le f(t)|x|^{1/2} + g(t);$$

$$(A_3'') \quad \liminf_{|x|\to\infty} \frac{1}{|x|} \left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \frac{T}{24}\left(\int_0^T f(t)dt\right)^2 + \int_0^T |e(t)|dt.$$

Then (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

Remark 3.1. There exist functions satisfying our Theorem 3.4. For example, let

$$G(t,x) = \frac{|x|^{6/5}}{1+|x|^{6/5}} \cdot h(t), \quad H(t,x) = (0.5T - t)|x|^{3/2} + \frac{|x|^3}{1+|x|^2} \cdot l(t).$$

where $h, l \in L^1([0,T],\mathbb{R}^+)$, and let $e(t) \in L^1([0,T];\mathbb{R}^N)$ satisfy

$$\int_0^T |e(t)|dt < \int_0^T l(t)dt - \frac{9T^5}{1536}.$$

Theorem 3.5. *Suppose $F = G(t,x) + H(t,x)$ with G and H satisfying assumption* (A). *Assume (A_1) and the following conditions hold:*

(A_2''') *there exist $\alpha \in (0,1/2)$, $f, g \in L^1([0,T];\mathbb{R}^+)$ such that*

$$|\nabla H(t,x)| \le f(t)|x|^\alpha + g(t);$$

$$(A_3''') \qquad \liminf_{|x|\to\infty} \frac{1}{|x|}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] > \int_0^T |e(t)|dt.$$

Then (1.1) *has at least one solution which minimizes φ on H_T^1.*

Remark 3.2. There exist functions satisfying our Theorem 3.5. For example, let

$$G(t,x) = (h(t),x), \quad H(t,x) = (0.5T - t)|x|^{5/4} + \frac{|x|^3}{1+|x|^2}\cdot l(t).$$

where $h \in L^1([0,T],\mathbb{R}^N)$ with $\int_0^T h(t)dt = 0$ and $l \in L^1([0,T],\mathbb{R}^+)$ and let $e(t) \in L^1([0,T];\mathbb{R}^N)$ satisfy

$$\int_0^T |e(t)|dt < \int_0^T l(t)dt.$$

4. Examples. In this section, we verify three examples. The others can be verified by using the similar way.

Example 4.1. Let G and H be as in Remark 2.2. Then $G(t,x)$ is $(1,3/2)$-convex for a.e. $t \in [0,T]$ and satisfies (I_1). In fact, since $h(t) \ge 1$ for a.e. $t \in [0,T]$, it is easy to get

$$\begin{aligned} G(t,x+y) &= h(t) + 1 + \sin|x+y|^2 \\ &\le \frac{3}{2}(2h(t) + 2 + \sin|x|^2 + \sin|y|^2) \\ &= \frac{3}{2}(G(t,x) + G(t,y)). \end{aligned}$$

Obviously, $\lambda = 1 > 1/2, \mu = 3/2 > 1/2$ and $\mu = 3/2 < 2^3 = 2^{p-1}\lambda^p$.

Next we show that H satisfies (I_2). By Young's inequality, it is easy to obtain

$$\begin{aligned} |\nabla H(t,x)| &\le \frac{7}{2}|0.5T - t||x|^{5/2} + \frac{20}{3}T^3|x|^{7/3} \\ &\le \frac{7}{2}(|0.5T - t| + \varepsilon)|x|^{5/2} + A_1(\varepsilon) \end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $A_1(\varepsilon) > 1$. Let

$$f(t) = \frac{7}{2}(|0.5T - t| + \varepsilon), \qquad g(t) = A_1(\varepsilon).$$

Then H satisfies condition (I_2) with $\alpha = 5/2$.

Note that

$$\begin{aligned}
&|x|^{-q\alpha}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] \\
&= |x|^{-10/3}\left[\frac{2}{3}\int_0^T G(t,x) + \int_0^T H(t,x)dt\right] \\
&= |x|^{-10/3}\int_0^T \left[\frac{2}{3}h(t) + \frac{2}{3} + \frac{2}{3}\sin|x|^2 + (0.5T - t)|x|^{7/2} + 2T^3|x|^{10/3}\right] dt.
\end{aligned}$$

Then if $T > 0.00244$, we can choose $\varepsilon > 0$ sufficient small such that

$$\begin{aligned}
&\liminf_{|x|\to\infty} |x|^{-10/3}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] \\
&= 2T^4 \\
&> \frac{9T}{28}\left(\frac{7T^2}{8} + \frac{7T\varepsilon}{2}\right)^{4/3} \\
&= \frac{T}{q(q+1)}\left(\int_0^T f(t)dt\right)^q.
\end{aligned}$$

This shows that (I_3) holds. By Theorem 2.1, problem (1.6) has at least one solution.

Example 4.2. Let $G(t,x)$ and $H(t,x)$ be as in (1.3). Then arguing as in the introduction, we get

$$\begin{aligned}
&\liminf_{|x|\to\infty} |x|^{-2\alpha}\int_0^T F(t,x)dt \\
&= \liminf_{|x|\to\infty} |x|^{-3/2}\int_0^T \left[h(t)|x|^{5/4} + \sin\left(\frac{2\pi t}{T}\right)|x|^{7/4} + (0.6T - t)|x|^{3/2}\right] dt \\
&= 0.1T^2.
\end{aligned}$$

Moreover,

$$\frac{T}{24}\left(\int_0^T f(t)dt\right)^2 = \frac{T}{24}\left[\frac{7}{4}\int_0^T \left(\left|\sin\left(\frac{2\pi t}{T}\right)\right| + \varepsilon\right) dt\right]^2 = \frac{49T^3}{384}\left(\frac{2}{\pi} + \varepsilon\right)^2.$$

If $T < 48\pi^2/245$, we choose $0 < \varepsilon < \sqrt{192/245T} - 2/\pi$, then

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = 0.1T^2$$

$$> \frac{49T^3}{384}\left(\frac{2}{\pi}+\varepsilon\right)^2 = \frac{T}{24}\left(\int_0^T f(t)dt\right)^2.$$

This shows that (A_3') holds. By Theorem 3.3, problem (1.1) has at least one solution. If $\int_0^T e(t)dt = 0$, we can also use Theorem 3.1 to obtain the conclusion.

Example 4.3. Let

$$G(t,x) = h(t)|x|^{5/4}, \quad H(t,x) = (0.6T - t)|x|^2 - t|x|^{3/2} + (k(t), x).$$

where $h \in L^1([0,T], \mathbb{R}^+)$ and $k(t) \in L^1([0,T]; \mathbb{R}^N)$. Then arguing as in the introduction, we show that $G(t,x)$ is $(2, 2^{9/4})$-subconvex and satisfies (A_1). It is easy to see that

$$|\nabla H(t,x)| \le 2|0.6T - t||x| + \frac{3t}{2}|x|^{1/2} + |k(t)|$$

$$\le 2\left(|0.6T - t| + \varepsilon\right)|x| + \frac{T^2}{2\varepsilon} + |k(t)|$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\varepsilon > 0$. Let

$$f(t) = 2\left(|0.6T - t| + \varepsilon\right), \qquad g(t) = \frac{T^2}{2\varepsilon} + |p(t)|.$$

Note that

$$|x|^{-2}\left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right]$$

$$= |x|^{-2}\left[2^{-9/4}\int_0^T G(t,2x)dt + \int_0^T H(t,x)dt\right]$$

$$= |x|^{-2}\int_0^T \left[\frac{h(t)}{2}|x|^{5/4} + (0.6T - t)|x|^2 - t|x|^{3/2} + (k(t), x)\right] dt$$

$$= \frac{|x|^{-3/4}}{2}\int_0^T h(t)dt + 0.1T^2 - 0.5T^2|x|^{-1/2} + \left(\int_0^T k(t)dt, |x|^{-2}x\right).$$

On the other hand, we have

$$\int_0^T f(t)dt = 2\int_0^T \left(|0.6T - t| + \varepsilon\right)dt = 0.52T^2 + 2\varepsilon T,$$

and

$$\frac{T\left(\int_0^T f(t)dt\right)^2}{2\left(12 - T\int_0^T f(t)dt\right)} = \frac{T^3(0.52T + 2\varepsilon)^2}{2[12 - T^2(0.52T + 2\varepsilon)]}.$$

If $T^3 < 6$, we choose $\varepsilon > 0$ sufficient small such that

$$\int_0^T f(t)dt = 0.52T^2 + 2\varepsilon T < \frac{12}{T}$$

and

$$\liminf_{|x|\to+\infty} |x|^{-2} \left[\frac{1}{\mu}\int_0^T G(t,\lambda x)dt + \int_0^T H(t,x)dt\right] = 0.1T^2$$

$$> \frac{T(0.52T^2 + 2\varepsilon T)^2}{2[12 - T(0.52T^2 + 2\varepsilon T)]} = \frac{T\left(\int_0^T f(t)dt\right)^2}{2\left(12 - T\int_0^T f(t)dt\right)}.$$

This shows that (B_2) and (B_3') hold. By Theorem 3.2, problem (1.1) has at least one solution.

Acknowledgment. This work is supported by the Graduate degree thesis Innovation Foundation of Central South University (No: 3960-71131100014) and the Outstanding Doctor degree thesis Implantation Foundation of Central South University (No: 2008yb032) and partially supported by the NNSF (No: 10771215) of China.

References

[1] Berger, M.S., Schechter, M., *On the solvability of semilinear gradient operator equations*, Adv. Math. **25** (1977), 97–132.
[2] Mawhin, J., *Semi-coercive monotone variational problems*, Acad. Roy. Belg. Bull. Cl. Sci. **73** (1987), 118–130.
[3] Mawhin, J., Willem, M., *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York, 1989.
[4] Mawhin, J., Willem, M., *Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 431–453.
[5] Rabinowitz, P. H., *On subharmonic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **33** (1980), 609–633.
[6] Rabinowitz, P. H., *Minimax methods in critical point theory with applications to differential equations*, in: CBMS Regional Conf. Ser. in Math., Vol. 65, American Mathematical Society, Providence, RI, 1986.
[7] Tang, C. L., *Periodic solutions of nonautonomous second order systems with γ-quasisubadditive potential*, J. Math. Anal. Appl. **189** (1995), 671–675.
[8] Tang, C. L., *Periodic solutions of nonautonomous second order systems*, J. Math. Anal. Appl. **202** (1996), 465–469.
[9] Tang, C. L., *Periodic solutions of nonautonomous second order systems with sublinear nonlinearity*, Proc. Amer. Math. Soc. **126** (1998), 3263–3270.
[10] Tang, C. L., Wu, X. P., *Periodic solutions for second order systems with not uniformly coercive potential*, J. Math. Anal. Appl. **259** (2001), 386–397.
[11] Willem, M., *Oscillations forcees de systèmes hamiltoniens*, Publ. Math. Fac. Sci. Besancon, Anal. Non Lineaire Annee 1980-1981, Expose No. 4, 16 p. (1981) (French).
[12] Wu, X., *Saddle point characterization and multiplicity of periodic solutions of non-autonomous second order systems*, Nonlinear Anal. TMA **58** (2004), 899–907.

[13] Wu, X. P., Tang, C. L., *Periodic solutions of a class of nonautonomous second order systems*, J. Math. Anal. Appl. **236** (1999), 227–235.

[14] Zhao F., Wu, X., *Periodic solutions for a class of non-autonomous second order systems*, J. Math. Anal. Appl. **296** (2004), 422–434.

[15] Zhao F., Wu, X., *Existence and multiplicity of periodic solution for non-autonomous second-order systems with linear nonlinearity*, Nonlinear Anal. **60** (2005), 325–335.

[16] Xu, B., Tang, C. L., *Some existence results on periodic solutions of ordinary $p-$Lapalcian systems*, J. Math. Anal. Appl. **333** (2007), 1228–1236.

[17] Tian, Y., Ge, W., *Periodic solutions of non-autonoumous second-order systems with a p-Lapalcian*, Nonlinear Anal. TMA **66** (2007), 192–203.

[18] Zhang, X., Tang, X., *Periodic solutions for an ordinary p-Laplacian system*, Taiwanese J. Math. (in press).

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Received August 1, 2009