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Markov dynamics on spaces of infinite configurations with marks

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Contents

Chapter 1

Introduction

1.1 Posing

The purpose of this thesis is to contribute to the development of the mathematical theory of evolving large population. In view of its numerous applications – in life and social sciences in particular – this theory has become popular within the last decades. Due to the intrinsic character of the processes that take place therein, as well as due to the large size, the evolution of such populations is naturally considered as stochastic. This feature predetermines the ways and the means of the mathematical modeling of such objects. In the present thesis, we work in the Markov approach in which the population states are probability measures defined on appropriate phase spaces, the evolution of which is obtained by solving (in one or another way) the corresponding Kolmogorov-Fokker-Planck equations [6]. The first key aspect of the thesis is that the age of the population members (time of their presence in the population) is explicitly taken into account. The second aspect is that the studied populations are infinite, which opens the possibility to clearly distinguish between the local and the global aspects of the theory, see [25]. The latter peculiarity of the theory leads, however, to essential technical complications as compared to the case of finite populations.

In view of the complexity of the evolution of large populations, its description is conducted in different spatio-temporal scales, cf. [2, 5]. At the macroscopic scale, the population is characterized by aggregated parameters, like density, mobility, etc. For such parameters, one derives – rather heuristically deduces – corresponding evolution equations intended to describe the population dynamics. The particular form of such equations corresponds to the model being considered. Below in this section, we outline some of them with the focus on the models containing the age parameter. The microscopic description is characterized by the use of the so-called individual based models in which each single population member is assigned its individual traits. In our case, such traits include also the member age. The most comprehensive description here amounts to constructing stochastic Markov processes. In certain sense, we work in the space between these two approaches.

Most of the material of this thesis was published in my articles [18, 19, 20]. In view of this, certain parts of the present text are direct quotations of the corresponding parts of these articles, supplied sometimes with suitable comments. For the sake of reader's convenience, some important formulas may be repeated. The majority of the proofs follow directly after the corresponding statements. The present thesis has the following structure. In the remaining part of this section we provide introductory material and then outline the aims of the thesis and its main results. The next chapter is dedicated to providing necessary technicalities including the Markovian terminology as well as our way of describing member's ages as marks. In Chapter 3 based on the results of [18], we describe the evolution of a model introduced herein in terms of correlation functions. This might be considered as an intermediate approach connecting the macro- and the microscopic theories. In Chapters 4 and 5 based on [19, 20], we provide the microscopic description of the evolution of another model of this kind introduced by us. This model is somewhat simpler than that studied in Chapter 3. In this case, we directly construct the Markov evolution of states. In Chapter 4, we consider the case where the population habitat is $X = \mathbb{R}^d$, which allows us to introduce a special class of the states – in fact, it is the same class as in Chapter 3, consisting of states possessing correlation functions – and then to show that the evolution leaves this class invariant. This can be considered as an additional information concerning the properties of the Markov evolution of this model. In Chapter 5, we study the case where X is just a locally compact Polish space. Here we prove the existence of a Markov process corresponding to this model. This is performed in the framework of the martingale approach [11, 13] combined with the theory of the Fokker-Planck equation [6] in the spirit of [26].

1.2 Age-structured population models

The use of population models traces back to 1798 when one of the most known model of this kind was proposed by Thomas Robert Malthus. The author considered a homogeneous population living isolated in an unchanging habitat with unlimited resources, in which the speeds of both procreation and mortality are proportional to the population size. This model predicts either an unlimited growth or an inevitable asymptotic extinction of the population. Due to these far from realistic assumptions and predictions it received rather restricted applications to real-world objects. Afterwards, a number of improvements and refinements were invented and implemented. Among such improvements we remark those which take into account the age structure of the populations being studied. Let us mention some motivations for this. In populations of living beings, individuals with different ages may have different survival capacities, mortality and reproduction rates. Many models of the infectious disease initiation and transmission include structuring the host population by the disease state dependent on the time since the disease initiation. Along with biological, biomedical and ecological applications, the age aspects are being taken into account also in many sociological and financial models, see [1, 4, 7, 21, 25, 29]. In particular, this kind of modeling is used to handling network security problems. W. Murray was the first who suggested to link modeling the ecology of the computer viruses to its counterpart dealing with the biological ones [29]. J. Kephart and S. White proposed to employ the SIS-kind models for studying the spread of computer viruses [21].

As mentioned above, the age structured populations can be studied at both microscopic or macroscopic scales. Below we outline some typical models of this kind representing each of these types.

1.2.1 Macroscopic phenomenology

The Lotka–McKendrick's model

The Lotka–McKendrick's model is a direct analogue of the Malthus model. The population dwells in an invariant habitat and its members differ in age, see [15]. The fertility $\beta(a)$ and mortality $\mu(a)$ rates are intrinsic parameters. In the simplest case, they do not depend on time. The evolution of the age-dependent population density $p(a,t)$ is described by the following equation

$$\begin{cases} \frac{\partial}{\partial t}p(a,t) + \frac{\partial}{\partial a}p(a,t) &= -\mu(a)p(a,t) \\ p(0,t) &= \int_0^{a_+} \beta(\varsigma)p(\varsigma,t)d\varsigma \\ p(a,0) &= p_0(a) \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial a}p(a,t)$ is to capture aging – a uniform drift with unit speed along the age axes $[0,a_+] \subset \mathbb{R}_+$. The basic and detailed analysis of this model can be found in [15]. This model laid the foundations for the forthcoming works in this field based in part on the theory of Volterra integral equations, for more details see [14].

Iannelli in [15] considered also some modifications of the Lotka–McKendrick's model, obtained by adding a time dependence of the vital rates $\beta(a,t)$ and $\mu(a,t)$. He studied the asymptotic behavior of these models, including strong and weak ergodicity, see Definition 3.1, page 37 and Definition 3.4 page 39 *ibid.* In Theorem 5.4.4 below, we will also consider ergodicity problem for the Markov process in one of our models.

The SIR model with aging

In this version of the SIR model (Susceptible, Infectious, Recovered) of infection spread an isolated population dwells in an invariant habitat, structured by age. Similarly as in (1.1) $p(a,t)$ is an age-density function at time t with age $a \in [0,a_+)$, with possible $a_+ = +\infty$. The population is divided into three groups: susceptible, infected and removed, which are described by their respective age-dependent densities $s(a,t)$, $i(a,t)$, $r(a,t)$ at time t, i.e.,

$$p(a,t) = s(a,t) + i(a,t) + r(a,t).$$

The evolution equation now reads

$$\begin{cases} \frac{\partial}{\partial t}s(a,t) + \frac{\partial}{\partial a}s(a,t) &= -(\mu(a) + \lambda(a,t))s(a,t), \\ \frac{\partial}{\partial t}i(a,t) + \frac{\partial}{\partial a}i(a,t) &= \lambda(a,t)s(a,t) - (\mu(a) + \sigma(a) + \delta(a))i(a,t), \\ \frac{\partial}{\partial t}r(a,t) + \frac{\partial}{\partial a}r(a,t) &= -\mu(a)r(a,t) + \sigma(a)i(a,t). \end{cases}$$

A detailed description of this model is provided in section 9.5 of [8], assuming that coefficients describe HIV infection and named it by analogy the SIA model from susceptible individuals, infective individuals, and AIDS cases. Then it was additionally assumed that there is a minimum age of interaction between the members of the population. In Section 9.5.3 of [8], this model was studied by C_0-semigroup methods. We will also use to these methods in the Theorem 5.3.8 below.

The COVID modeling

The very recent examples of the age-structured modeling are devoted to the worldwide spread of the COVID-19. Unfortunately, we have the opportunity to see this in the real life. During the COVID-19 pandemic, researchers were using and formulating mathematical models as a technique in gaining insight into the mode of spread of the virus, e.g. [1, 33]. The authors of [1] investigated the situation where vaccines and drugs could not be applied. Furthermore, they considered three different response strategies: total lock-down, partial lock-down, aiming at achieving herd immunity. They enriched the model SIR with economic components capturing the outbreak of the pandemic. This led them to the SIR model with wealth dynamics. The population was divided into two groups under and over the age of 65, assuming that members of the elderly group are more susceptible to infection. This model is very accurate and requires familiarization with many parameters, if the reader would like to delve into this topic, we recommend reading the entire article [1].

1.2.2 Microscopic individual-based theory: finite systems

The Jagers-Klebaner model

The authors of [17] proposed a modification of the known Bellman-Harris branching process model [3]. Therein, the particles are born and die at random. In the event of death of an individual, there arises a random number of independent offsprings. The birth parameters might depend on the population size and age structure. The generator G of the process is given by the formula:

$$\begin{aligned} GF((f,\mathcal{A})) &= F'((f,\mathcal{A}))(f',\mathcal{A}) + \sum_{j=1}^{z} b_{\mathcal{A}}(a_j)\{F(f(0)+(f,\mathcal{A})) - F((f,\mathcal{A}))\} \\ &+ \sum_{i=1}^{z} h_{\mathcal{A}}(a_j)\{\mathbb{E}_{\mathcal{A}}[F(Y(a_j)f(0)+(f,\mathcal{A})-f(a_j))] \\ &- F((f,\mathcal{A}))\}, \end{aligned} \tag{1.2}$$

where $F((f,\mu))$ is a test function on the space of measures μ, f is a function defined on $\mathbb{R}$, $\mathcal{A} = (a_1, a_2, \ldots, a_z)$ is the finite collection of individuals ages, $h_{\mathcal{A}}(a)$ is the hazard rate, $b_{\mathcal{A}}(a)$ is the intensity of birth, $Y(a)$ the number of offspring distributed depending on the age structure of the whole collection $\mathcal{A}$ and on the age of the particle which died, $\mathbb{E}_{\mathcal{A}}$ the expectation number of individuals depending on the initial collection. The states of the model are finite positive Borel measures on $\mathbb{R}_+$ equipped with the weak topology. In [17], the authors used some martingale's techniques to establish the asymptotic properties of this model.

The Méléard-Tran model

In [28], Méléard and Tran introduced an age-structured individual-based model, similar to (1.2). They considered a finite population in continuous time t where the individuals reproduce and die with rates depending on a quantitative trait $x \in \mathbb{R}$ and on their age $a \in \mathbb{R}_+$. The trait x is a special feature that characterizes individuals, it can mean body size, rate of food intake, etc. The model is described by the generator L having the form

$$\begin{aligned} LF_f(\mu) &= \langle \mu, \partial_a f(\cdot)\rangle F'_f(\mu) + \int_{X\times\mathbb{R}_+} \Big[\Big(d(x,a) \\ &+ \mu U(x,a)\Big)\Big(F_f\big(\mu - \delta_{(x,a)}\big) - F_f(\mu)\big) \\ &+ b(x,a)(1-p)\big(F_f(\mu+\delta_{(x,0)}) - F_f(\mu)\Big) \\ &+ b(x,a)p\int_{\mathbb{R}^d}\Big(F_f(\mu+\delta_{(x+h,0)}) - F_f(\mu)\Big)k(x,h)dh\Big]\mu(dx,da) \end{aligned}$$

where X is a closed subset of $\mathbb{R}^d$, μ is a finite measure on X, f is a continuous bounded real-valued function with bounded continuous derivatives with respect to a, $d(x,a)$ is the natural death rate, F is a cylindrical function. A large and finite population is considered and the population size is represented by the parameter n. Furthermore the birth and death rates are taken of order n. In this model, the trait can change depending on the occurrence of mutations with probability p. The individual compete with each other, which is captured in the rate $U(x,a)$.

1.2.3 Microscopic individual-based theory: infinite systems

The necessity of using individual-based models of infinite systems for modeling large real-world objects of this kind was realized in the first half of the XX-th century. It was then materialized in the concept of the 'infinite-volume limit' used in the statistical mechanics of physical particle system. In the present context, it was recognized and discussed in, e.g., [22, 25]. Briefly, the basic idea of this approach consist in getting the possibility to clearly distinguish between *local* and *global* aspects of the theory. As a finite system occupies a compact subset of the habitat (assuming the latter be locally compact), its description is always local rather than global. In dealing with infinite systems, one employs probability measures on an appropriate phase space to model system's states. Such models of states are quite abstract objects, not appropriate for a direct investigation. In view of this, it is convenient to restrict the theory to a class of states that possess *correlation functions* [24, 27], and then to describe the evolution of such states as the evolution of these correlation functions. The corresponding approach was realized in [5, 22, 25], see also the works quoted in these publications. Its additional advantage is that it is suitable for connecting micro-and macroscopic descriptions, as well as to find out intermediate versions of such theories. The only disadvantage is that the habitat should have some additional properties, e.g., it should be $\mathbb{R}^d$.

1.3 The aims and overview

In this work, we introduce and study two individual-based models, described by their Kolmogorov operators, similarly as in (1.2), of an infinite population the members of which are characterized by their age - time since appearance in the population. In Chapters 3 and 4, as the habitat we take $\mathbb{R}^d$ and use probability measures possessing correlation functions, see Definition 3.1.1. In this case, we describe these states with the help of such functions. In Chapter 3, based on [18], we introduce and study the model (3.1), where new members are born by the existing population members. This model turns to be quite hard to study. In view of this, we restrict ourselves to solving the evolution equation for the first two correlation functions in subsect. 3.3.1 and 3.3.2, which are usually studied in various macroscopic theories. However, in our approach the evolution equations are directly deduced from an individual-based model. These results are formulated in Proposition 3.2.1.

In Chapter 4, we introduce another model, see (4.1), in which particles arrive to and depart from X independently of each other. Their traits are (x, α), $x \in X$ and $\alpha \geq 0$ being time since appearance. In this chapter, we take $X = \mathbb{R}^d$ and use the same class of states as in Chapter 3. Due to the fact that the model is now simpler, we managed to prove in Theorem 4.2.2 that the corresponding Fokker-Planck equation has a global in time solution $t \mapsto \mu_t$ which describes the evolution of states of this model. Additionally I found a stationary state for this evolution. This chapter is based on [19].

In Chapter 5, based on [20], I consider the same model as in (4.1) with the habitat X being just a locally compact Polish space. In this case, one cannot use methods based on correlation functions. Therefore, the very construction of the theory needs to be modified, including introducing special metrics on the corresponding configuration spaces. The latter proved to be quite technical. The main result is contained in Theorem 5.4.4, where the existence of a unique Markov process describing the evolution of this model was proved by showing that the corresponding martingale problem is well-posed. Additionally, assuming that the departure rate is separated away from zero I proved that this evolution is temporarily ergodic.

To summarize: the main results of this thesis are

1. Proposition 3.2.1 and the solutions obtained in subsects. 3.3.1 and 3.3.2.

2. Theorem 4.2.2 and its proof.

3. Theorem 5.4.4 and its proof.

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Chapter 2

Preliminaries

2.1 The Markov evolution

2.1.1 Generalities

Throughout this work we use the following notations: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{R}^d$, $d \in \mathbb{N}$, is a standard Euclidean space. By a Polish space we mean a separable topological space the topology of which is consistent with a complete metric. Mostly, such a space is *locally compact*, which means that each of its points has a compact neighborhood. For a suitable $\Delta \subset E$, by $\mathbf{1}_\Delta$ we denote the indicator of Δ, i.e., the functions such that

$$\mathbf{1}_\Delta(x) := \begin{cases} 1 & \text{if } x \in \Delta\ , \\ 0 & \text{if } x \notin \Delta\ . \end{cases} \tag{2.1}$$

For a Polish space E, by $\mathcal{B}(E)$ we denote the corresponding Borel σ-field; $C_b(E)$ (resp. $B_b(E)$) stands for the set of all bounded and continuous (resp. bounded and measurable) functions $f : E \to \mathbb{R}$. By $C_b^+(E)$ and $B_b^+(E)$ we denote the set of positive elements of $C_b(E)$ and $B_b(E)$, respectively. Finally, $C_{cs}(E) \subset C_b(E)$ consists of all continuous compactly supported functions.

Definition 2.1.1. *A family of functions, F, is said to separate the points of a Polish space E if for each distinct $x, y \in E$, one finds $f \in F$ such that $f(x) \neq f(y)$.*

By $\sigma\mathcal{F}$ we denote the smallest sub-field of $\mathcal{B}(E)$ such that each $f \in F$ is $\sigma\mathcal{F}$-measurable; by $\mathcal{P}(E)$ we mean the set of all probability Borel measures on $(E, \mathcal{B}(E))$. For $x \in E$, the Dirac measure δ_x with center at this x is defined by its values

$$\delta_x(\Delta) = \mathbf{1}_\Delta(x), \qquad \Delta \in \mathcal{B}(E), \tag{2.2}$$

see (2.1). For a given measure μ and a suitable function f, we write $\mu(f) = \int f d\mu$. Then $\delta_x(f) = f(x)$. For a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(E)$, by writing $\mu_n \Rightarrow \mu \in \mathcal{P}(E)$ we mean its weak convergence, i.e., $\mu_n(f) \to \mu(f)$ for all $f \in C_b(E)$.

Definition 2.1.2. *A family of functions $\mathcal{F}$ is said to be separating if $\mu_1(f) = \mu_2(f)$ holding for all $f \in \mathcal{F}$ implies $\mu_1 = \mu_2$ for any $\mu_1, \mu_2 \in \mathcal{P}(E)$.*

If $\mathcal{F}$ separates the points of E and its linear span is an algebra with respect to pointwise operations, then it is separating, see Ethier and Kurtz (1986), Theorem 4.5 on page 113, [13].

Proposition 2.1.3 (Ethier and Kurtz)**.** *Let V and $\mathcal{F}$ be a complete and separable metric spaces and a family of functions $F : V \to \mathbb{R}$, respectively. Assume that:*

(a) each $F \in \mathcal{F}$ is bounded and continuous;

(b) $F_1, F_2 \in \mathcal{F}$, their pointwise product is in $\mathcal{F}$;

(c) for each distinct $v_1, v_2 \in V$, there exists $F \in \mathcal{F}$ such that $F(v_1) \neq F(v_2)$;

(d) $\mathcal{F}$ contains $F \equiv 1$.

Then $\mathcal{F}$ is separating.

Definition 2.1.4. *A family of functions F is said to be convergence determining if $\mu_n(f) \to \mu(f)$ holding for all $f \in F$ implies $\mu_n \Rightarrow \mu$ as $n \to +\infty$.*

2.1.2 The Kolmogorov-Fokker-Planck formalism

Let E be a Polish space. Broadly speaking, a dynamical system on E is defined by a family of maps $\{T_t\}_{t\geq 0}$, $T_t : E \to E$, such that $T_t \circ T_s = T_{t+s}$ holding for all $t, s \geq 0$, and $T_0 = I$. Here $\circ$ and I denote composition and the identity map, respectively. The points $x \in E$ are then treated as states of the system; i.e., $x_t := Tx_0$ is its state at time t. In the stochastic (Markov) version, states are probability measures $\mu \in \mathcal{P}(E)$, and thus $\mu_t = T_t\mu_0$. Then the 'point' states $x \in E$ appear here as $\delta_x \in \mathcal{P}(E)$, see (2.2). That is, the evolution of the system is said to be deterministic if the family $\{T_t\}_{t\geq 0}$ preserves the set of point states, which means that $T_t\delta_x = \delta_{x_t}$ for all $t \geq 0$ and $x \in E$.

In the Kolmogorov-Fokker-Planck formalism, see [6, 22, 23], the construction of the evolution $\mu \mapsto \mu_t$ is performed by solving the Fokker-Planck equation

$$\mu_t(F) = \mu_0(F) + \int_0^t \mu_s(LF)ds, \qquad \mu_0 \in \mathcal{P}(\hat{\Gamma}), \tag{2.3}$$

where L is the so called Kolmogorov operator specific for the model being considered. It acts on the *test* functions, also called *observables*, usually taken from a suitable subset of $C_b(E)$. In fact, the choice of the domain of L predetermines the quality of the description of the stochastic evolution of a given model. Sometimes, one approaches to constructing the evolution in question indirectly by solving the Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \qquad F_t|_{t=0} = F_0, \tag{2.4}$$

which might be considered and 'dual' to (2.3). Then solutions of (2.3) are obtained from the solutions of (2.4) by setting $\mu_t(F) = \mu(F_t)$, where μ and F are taken as the initial conditions of (2.3) and (2.4), respectively. Finally, a Markov process, $\mathcal{X}$, with state space E is obtained by constructing a probability measure, P, on a space of paths with values in E such that the corresponding probability of the event $\mathcal{X}(t) \in \Delta$, $\Delta \in \mathcal{B}(E)$, is given by $\mu_t(\Delta)$, where μ_t is a solution of (2.3). This most comprehensive description will be obtained in Chapter 5 below by solving the so called martingale problem for L.

2.2 Measures on configuration spaces

2.2.1 The configuration spaces

In the description of the stochastic evolution of infinite populations they are viewed as random 'clouds' of point 'particles' placed in a suitable habitat. In our case, the latter is a locally compact Polish space X. In Chapters 3 and 4, we take $X = \mathbb{R}^d$, whereas in Chapter 5 it is just a general locally compact Polish space. Each mentioned cloud is supposed to be *locally finite*, which means that its intersection with a compact $\Lambda \subset X$ is supposed finite. The mathematical model of such a cloud is a configuration – a counting Radon measure on X. The set of all such measures Γ is then defined by the property that $\gamma(\Lambda) \in \mathbb{N}_0$ for each compact $\Lambda \subset X$. This can also be given the following geometric interpretation. Let δ be a complete metric of X consistent with its topology. Then for each $r > 0$, one finds $r' \in (0, r)$ such that the ball $B^\delta_{r'}(x) = \{y \in X : \delta(x, y) < r'\}$ has compact closure. Therefore, for a given $\gamma \in \Gamma$ and $x \in X$, there exists $r > 0$ such that $\gamma(B^\delta_r(x)) < \infty$. Then we define

$$p(\gamma) = \{x \in X : \inf_{r>0} \gamma(B^\delta_r(x)) := n_\gamma(x) \in \mathbb{N}\}.$$

Each γ is fully characterized by the pair $(p(\gamma), n_\gamma)$, in which $p(\gamma)$ is the *ground configuration* for γ. It is convenient to extend n_γ to the whole X by setting $n_\gamma(x) = 0$ whenever $x \notin p(\gamma)$. The mentioned geometric interpretation of γ is the collection of the points $x \in p(\gamma)$ such that at this x there is located $n_\gamma(x)$ indistinguishable 'particles' - population members. We refer the reader to [9, 10, 16, 27, 34] for more detail on this issue.

Our aim now is to define a suitable topology on Γ. It is the weak-hash (vague) topology. By definition, it is the weakest topology that makes continuous all the maps

$$\Gamma \ni \gamma \mapsto \int_X f(x)\gamma(dx) = \sum_{x \in p(\gamma)} n_\gamma(x) f(x) =: \sum_{x\in\gamma} f(x), \qquad f \in C_{cs}(X), \tag{2.5}$$

where the latter equality is the definition of $\sum_{x\in\gamma}$, see [27] for more detail. It is know, see [34], that Γ equipped with this topology is a Polish space. Next, we define

$$\breve{\Gamma} = \{\gamma \in \Gamma : n_\gamma(x) = 1 \text{ for all } x \in p(\gamma)\}. \tag{2.6}$$

That is, γ is in $\breve{\Gamma}$ if $\gamma = p(\gamma)$ as sets. Such configurations are called *simple*. The set $\breve{\Gamma}$ is a G_δ-subset of Γ, and hence is a Polish space in the subspace topology, see [34].

Along with infinite configurations, Γ contains also finite ones. Let Γ_0 stand for the set of all such configurations. It can be viewed as a subset of the set of all finite Borel measures on X, closed in the weak topology. Note that the subspace topology induced on Γ_0 be the weak-hash topology introduced in (2.5) coincides with the latter. At the same time, each $\gamma \in \Gamma_0$ can be interpreted as the element of X^m/Σ_m, where $m = |\gamma|$ is its cardinality and Σ_m is the corresponding symmetric group. We repeatedly exploit this interpretation below.

In the sequel, we will need the following extension of the last equality in (2.5). For $\gamma \in \Gamma$ and $x \in p(\gamma)$, by $\gamma \setminus x$ we understand the configuration γ' such that $n_{\gamma'}(y) = n_\gamma(y)$ for all $y \neq x$ and: (a) $p(\gamma') = p(\gamma)$ and $n_{\gamma'}(x) = n_\gamma - 1$ if $n_\gamma(x) > 1$; (b) $p(\gamma') = p(\gamma) \setminus x$ is $n_\gamma(x) = 1$. Similarly we define $\gamma' = \gamma \cup y$ by setting: (a) $p(\gamma') = p(\gamma) \cup \{y\}$, $n_{\gamma'}(y) = 1$ if $y \notin p(\gamma)$; (b) $p(\gamma') = p(\gamma)$ and $n_{\gamma'}(y) = n_\gamma(y) + 1$ if $y \in p(\gamma)$. Thereafter, we have

$$\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} f(x,y) = \int_X \int_X f(x,y)\gamma(dx)\gamma(dy) - \int_X f(x,x)\gamma(dx), \tag{2.7}$$

that can also be generalized to all $m \in \mathbb{N}$

$$\begin{aligned} &\sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}} f(x_1,\dots x_m) \\ &\quad = \sum_{G\in\mathbb{K}_m} (-1)^{l_G} \int_{X^{n_G}} f_G(y_1,\dots,y_{n_G})\gamma(dy_1)\cdots\gamma(dy_{n_G}) \\ &\quad = \sum_{G\in\mathbb{K}_m} (-1)^{l_G} \sum_{y_1\in\gamma}\cdots\sum_{y_{n_G}\in\gamma} f_G(y_1,\dots,y_{n_G}), \end{aligned} \tag{2.8}$$

where $\mathbb{K}_m$ is the collection of all graphs with vertices $\{1,2,\dots,m\}$, l_G and n_G are the number of edges and the connected components of G, respectively, whereas $f_G(y_1,\dots,y_{n_G})$ is obtained from $f(x_1,\dots,x_m)$ by setting the arguments $x_{l_1},\dots x_{l_{s_j}}$ of the latter equal y_j where $l_1,\dots l_{s_j}$ are the vertices of the j-th connected component of G. Note that the summations in the left-hand sides of (2.7) and (2.8) have the direct meaning if $\gamma \in \breve{\Gamma}$, see (2.6), where γ can be considered as just a locally finite subset of X.

Now for $\gamma \in \Gamma$ and $m \in \mathbb{N}$, by means of (2.8) we define the following counting measure on X^m equipped with the usual product topology. For a compact subset $\Delta \subset X^m$, we set

$$Q_\gamma^{(m)}(\Delta) = \sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}} \mathbf{1}_\Delta(x_1,\dots,x_m). \tag{2.9}$$

That is $Q_\gamma^{(m)}(\Delta)$ is the number of ordered strings $(x_1, \ldots, x_m)$ contained in Δ. It is know, see [27], that the map $\gamma \mapsto Q_\gamma^{(m)}(\Delta)$ is measurable for each compact Δ. However, it may be unbounded.

2.2.2 Measures on Γ

Let $\mathcal{B}(\Gamma)$ be the Borel σ-field of subsets of Γ relative to the weak-hash topology defined in (2.5). Recall that $\mathcal{P}(\Gamma)$ denotes the set of all probability measures on Γ.

Remark 2.2.1. As a G_δ-subset, $\breve{\Gamma}$ is in $\mathcal{B}(\Gamma)$. Hence, each $\mu \in \mathcal{P}(\Gamma)$ with the property $\mu(\breve{\Gamma}) = 1$ can be redefined as a probability measure on the Polish space $\breve{\Gamma}$.

As mentioned above the maps $\gamma \mapsto Q_\gamma^{(m)}(\Delta)$ need not be bounded hence μ-integrable for a given $\mu \in \mathcal{P}(\Gamma)$. Following [27] we will say that μ has *correlations* of order m if

$$\omega_\mu^{(m)}(\Delta) := \int_\Gamma Q_\gamma^{(m)}(\Delta)\mu(d\gamma) < \infty, \tag{2.10}$$

holding for all compact $\Delta \subset X^m$. Then the Radon measure $\omega_\mu^{(m)}$ on X^m is called the m-th order *correlation measure* corresponding to μ. It is clear that the existence of a given $\omega_\mu^{(m)}$ implies the existence of all $\omega_\mu^{(m')}$, $m' < m$. We say that μ has correlation functions of all orders if it has all corresponding $\omega_\mu^{(m)}$.

Let us now consider the case of $X = \mathbb{R}^d$. For $m \in \mathbb{N}$, let $\lambda^{(m)}$ be the Lebesgue measure on X^m, i.e.,

$$\lambda^{(m)}(dx_1, \ldots, dx_m) = dx_1 \cdots dx_m.$$

Assume that a given $\mu \in \mathcal{P}(\Gamma)$ has $\omega_\mu^{(m)}$ for this m, which is $\lambda^{(m)}$-absolutely continuous and the corresponding Radon-Nikodym derivative $k_\mu^{(m)}$ is $\lambda^{(m)}$-almost everywhere bounded. That is, $k_\mu^{(m)}$ is a symmetric element of $L^\infty(X^m) := L^\infty(X^m, \lambda^{(m)})$. Then we call $k_\mu^{(m)}$ the m-th order correlation function of μ. By the very definition, it follows that

$$k_\mu^{(m)}(x_1, \ldots, x_m) \geq 0.$$

The first two correlation functions have a special meaning characterizing the corresponding state μ. For a given compact $\Lambda \subset X$, let us consider $Q_\mu^{(1)}(\Lambda)$, which counts the points of γ contained in Λ, see (2.9) and (2.10). Then $\omega_\mu^{(1)}(\Lambda)$ is just the μ-expected value of the number of points contained in Λ. At the same time,

$$\omega_\mu^{(1)}(\Lambda) = \int_X \mathbf{1}_\Lambda(x) = \int_\Lambda k_\mu^{(1)}(x)dx,$$

which means that $k_\mu^{(1)}$ is just the particle density in state μ. In a similar way, by (2.7) one shows that

$$c_\mu(x, y) := k_\mu^{(2)}(x, y) - k_\mu^{(1)}(x)k_\mu^{(1)}(y) \tag{2.11}$$

is the truncated spatial correlation function in state μ. These two functions $k_\mu^{(1)}$ and c_μ are usually employed in describing population dynamics at macro-level by kinetic equations. Similarly as in Lemma 2.10 of [26], one proves the following statement, in which $\breve{\Gamma}$ stands for the set of simple configurations defined in (2.6).

Proposition 2.2.2. *Let $X = \mathbb{R}^d$ and $\mu \in \mathcal{P}(\Gamma)$ have $k_\mu^{(2)}$. Then $\mu(\breve{\Gamma}) = 1$, i.e., the configurations in state μ are almost surely simple.*

The importance of this statement can be seen in the light of Remark 2.2.1. Let us outline the arguments yielding its proof. For $\epsilon \in (0,1)$, one defines

$$H(\gamma) = \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} h(x,y), \qquad h(x,y) = \frac{1}{|x-y|^{\epsilon d}}.$$

It is clear that $H(\gamma) = +\infty$ for each $\gamma \in \Gamma \setminus \breve{\Gamma}$. At the same time, by (2.9), (2.10) we have

$$\mu(H) = \int_\Gamma Q_\gamma^{(2)}(h)\mu(d\gamma) = \int_{X^2} k_\mu^{(2)}(x,y)h(x,y)dxdy < \infty.$$

The latter holds true as $k_\mu^{(2)} \in L^\infty(X^2)$ and h is absolutely integrable. It implies $\mu(\Gamma \setminus \breve{\Gamma}) = 0$. By similar arguments one can also extend Proposition 2.2.2 to the case where X is just a Polish space. That is, $\mu(\breve{\Gamma}) = 1$ if μ possesses the second order correlation measure verifying $\omega_\mu^{(2)}(D) = 0$ where $D = \{(x,y) \in X^2 : x = y\}$.

Finally, let us mention a probability measure on Γ which plays a special role in this work. Let ϱ be a positive Radon measure on X. The latter is just a general locally compact Polish space. The Poison measure π_ϱ with intensity measure ϱ is defined by its correlation measures

$$\omega_{\pi_\varrho}^{(m)} = \varrho^{\otimes m}, \tag{2.12}$$

which in particular means that for $\Delta = \Lambda_1 \times \Lambda_m$, all $\Lambda_i \subset X$ compact, one has

$$\omega_{\pi_\varrho}^{(m)}(\Delta) = \varrho(\Lambda_1)\cdots\varrho(\Lambda_m).$$

That is, in state π_ϱ the points choose their positions in X independently of each other. If $X = \mathbb{R}^d$ and $\varrho(dx) = \rho(x)dx$, $\rho \in L^\infty(X)$, then all $k_{\pi_\varrho}^{(m)}$ exist. By (2.11) and (2.12), it follows that

$$c_{\pi_\varrho}(x,y) = 0,$$

i.e., the spatial correlations are absent in this state.

2.2.3 Configuration spaces with marks

As mentioned above, our principal aim is to take into account also particles ages – the time of their presence in the population. To this end, we pass to the traits $\hat{x} = (x, \alpha)$, $x \in X$ and $\alpha \in \mathbb{R}_+ := [0, +\infty)$. It is then naturally to consider configurations $\hat{\gamma}$ as collections of such compound traits in which α appears as a mark. More on the general theory of marked configuration spaces can be found in [10]. According to this, a marked configuration $\hat{\gamma}$ is a counting Radon measure on $\hat{X} := X \times \mathbb{R}_+$ such that, for each compact $\Lambda \subset X$, the set $\lambda = \Lambda \times \mathbb{R}_+$ verifies $\hat{\gamma}(\lambda) \in \mathbb{N}_0$. That is, every compact Λ contains only a finite number of the elements of $\hat{\gamma}$ of different ages.

In Chapter 5, we deal with the general case where X is just a locally compact Polish space, without assuming a priori properties of the states. This means that the corresponding marked configurations may be multiple, and thus each $x \in X$ is characterized by a finite (possibly empty) configuration of ages $a = \{\alpha_1, \ldots, \alpha_n\}$. In Chapters 3 and 4, we deal with populations dwelling in $X = \mathbb{R}^d$ and with the states possessing all correlation functions. According to Proposition 2.2.2, such states are supported on simple configurations, which means that only those will be taken into account, see also Remark 2.2.1. In this case each age nonempty configuration a is a singleton. In view of this, we will denote there the age variable by a. In view of the mentioned differences in the properties of the marked configurations spaces, we make a more detailed presentation of them an the beginnings of the corresponding Chapters.

Chapter 3

The birth-and-death dynamics

In this chapter based on [18], we introduce a model and study the stochastic dynamics of an infinite population of point particles dwelling in $X = \mathbb{R}^d$, which amounts to the following. The particle having trait $\hat{x} = (x, a)$ at time $t = 0$: (a) can die at time t with probability $1 - e^{-m(\hat{x})t}$, where the death rate m is a suitable function; (b) can give birth to a new particle located in a compact Λ with probability $1 - e^{-\nu_{\hat{x}}(\Lambda)t}$, where ν is a positive measure kernel on $(\mathcal{B}(X), \hat{X})$. As mentioned above, we restrict our consideration to states possessing all correlation functions and hence satisfying the conditions of Proposition 2.2.2. In view of this and for the further simplicity, in this chapter we denote by Γ the space of all simple configurations defined in (2.6). Then the space of marked configurations $\hat{\Gamma}$ consists of the pairs $\hat{\gamma} = (\gamma, a)$, where $a : \gamma \to \mathbb{R}_+$ is the age map that assigns a_x to each $x \in \gamma$. Its topology will be made precise below.

The Kolmogorov operator corresponding to the model studied herein has the following form

$$
\begin{aligned}
(LF)(\hat{\gamma}) \;&=\; \sum_{x\in\gamma} \frac{\partial}{\partial a_x} F(\hat{\gamma}) + \sum_{x\in\gamma} m(\hat{x}) \left[F(\hat{\gamma}\setminus\hat{x}) - F(\hat{\gamma})\right] \\
&+\; \int_X \left[F(\hat{\gamma}\cup(x,0)) - F(\hat{\gamma})\right] \sum_{y\in\gamma} \nu_{\hat{y}}(dx).
\end{aligned} \tag{3.1}
$$

Here the first term on the right-hand side corresponds to aging, whereas the remaining ones describe the death and the birth of new particles as mentioned above. The model corresponding to (3.1) can also be used to describe the infection spread in an infinite population. In this interpretation, the population of healthy individuals is present as a vacuum, whereas $\hat{\gamma}$ corresponds to the population of infected individuals. The recovery depends only on the time since infection and is described by the function $m(\hat{x})$. The spread of infection from the infected population is captured by the kernel ν.

For this model, assuming that the evolution preserves the set of states possessing all correlation functions from the Kolmogorov equation (2.4) we derive the evolution equations for the correlation functions and solve them for $k^{(1)}$ and $k^{(2)}$, which yields the result usually obtained from phenomenological kinetic-like equations. However, in our case we obtained it directly from the individual-based model described in (3.1).

3.1 The technicalities

As mentioned above, in this chapter the configuration space $\hat{\Gamma}$ defined as the set of pairs (γ, a) where γ is a simple configuration on $X = \mathbb{R}^d$ and $a : \gamma \to \mathbb{R}_+$. The value of a at a given x is denoted by a_x. The space $\hat{X} = X \times \mathbb{R}_+$ is equipped with the usual product topology and the Lebesgue measure $d\hat{x} = dxda$.

The configuration space $\hat{\Gamma}$ is equipped with the following topology. Let $\mathcal{F}$ be the set of all bounded and continuous maps $f : X \times \mathbb{R}_+ \to \mathbb{R}$ such that each f vanishes for $x \in \Lambda^c := X \setminus \Lambda$ for an f-specific compact $\Lambda \subset X$. Then $\hat{\Gamma}$ is equipped with the weakest topology that makes continuous all the maps

$$\hat{\Gamma} \ni \hat{\gamma} \mapsto \sum_{x \in \gamma} f(x, a_x), \qquad f \in \mathcal{F}.$$

With this topology $\hat{\Gamma}$ is a Polish state.

Definition 3.1.1. *By $\mathcal{P}_*$ we denote the subset of $\mathcal{P}(\hat{\Gamma})$ consisting of those μ that have all correlation measures $\omega_\mu^{(m)}$, each of which is absolutely continuous with respect to the corresponding Lebesgue measure $\lambda^{(m)}(d\hat{x}_1 \cdots d\hat{x}_m) = d\hat{x}_1 \cdots d\hat{x}_m$ such that their Radon-Nikodym derivatives $k_\mu^{(m)}$ have the properties: for all $m \in \mathbb{N}$ the following holds*

(i) $k_\mu^{(m)}$ *is a symmetric element of* $L^\infty(\hat{X}^m) = L^\infty(\hat{X}^m, \lambda^{(m)})$*;*

(ii) $k_\mu^{(m)}((x_1, a_1) \ldots, (x_m, a_m)) \to 0$ *as* $\max_i a_i \to +\infty$*;*

(iii) for almost all $x_1, \ldots, x_m \in X$ *and* $a_1, \ldots a_m \in \mathbb{R}_+$*, the map*

$$(0, +\infty) \ni t \mapsto k_\mu^{(m)}((x_1, a_1) \ldots, (x_m, a_m + t)),$$

is continuously differentiable;

(iv) the following holds

$$\int_{\mathbb{R}_+^m} k_\mu^{(m)}((x_1, a_1) \ldots, (x_m, a_m)) da_1 \cdots da_m \in L^\infty(X^m);$$

(v) there exists a μ-dependent $\varepsilon > 0$ such that, for each $\theta \in L^1(\hat{X}) := L^1(\hat{X}, d\hat{x})$, $\|\theta\| = 1$, the series

$$1 + \sum_{m=1}^{\infty} \frac{\varepsilon^m}{m!} \int_{\hat{X}^m} k_\mu^{(m)}(\hat{x}_1, \ldots, \hat{x}_m)\theta(\hat{x}_1) \cdots \theta(\hat{x}_m) d\hat{x}_1 \cdots d\hat{x}_m$$

is absolutely convergent.

Now we introduce the following function

$$F^{\theta}(\hat{\gamma}) = \prod_{y\in\gamma}(1+\theta(\hat{y})), \qquad \hat{\gamma}\in\hat{\Gamma}. \tag{3.2}$$

Definition 3.1.2. *By Θ_0 we will denote the set of continuous functions $\theta : X \times \mathbb{R}_+ \to (-1, 0]$ having the following properties: (a) each $\theta(x, a)$ vanishes for $x \in \Lambda^c$ for some θ-specific compact $\Lambda \subset X$; (b) for each x, the map $a \mapsto \theta(x, a)$ is continuously differentiable.*

For $\theta \in \Theta_0$, is measurable and F^θ satisfies

$$0 < F^{\theta}(\hat{\gamma}) \leq 1.$$

Hence, it is μ-integrable for all $\mu \in \mathcal{P}(\hat{\Gamma})$. By employing Proposition 2.1.3, it can be proved that the set of functions $\{F^\theta : \theta \in \Theta_0\}$ is separating. According to item (v) of Definition 3.1.1, for $\mu \in \mathcal{P}_*$ this function can be μ-integrable also for $\theta \in L^1(\hat{X})$ for sufficiently small $\|\theta\|$. Indeed, by (2.8) and (2.10) it follows that

$$\begin{aligned}
\mu(F^{\theta}) &= 1 + \int_{\hat{\Gamma}} \left(\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\{\hat{x}_1,\ldots\hat{x}_m\}\subset\hat{\gamma}} \theta(\hat{x}_1)\cdots\theta(\hat{x}_m) \right) \mu(d\hat{\gamma}) \qquad (3.3)\\
&= 1 + \int_{\hat{\Gamma}} \left(\sum_{\eta\Subset\gamma} \prod_{x\in\eta} \theta(\hat{x}) \right) \mu(d\hat{\gamma})\\
&= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\hat{X}^m} k_{\mu}^{(m)}(\hat{x}_1,\ldots\hat{x}_m)\theta(\hat{x}_1)\cdots\theta(\hat{x}_m)d\hat{x}_1\cdots d\hat{x}_m\\
&=: \int_{\hat{\Gamma}_0} k_{\mu}(\hat{\eta}) \left(\prod_{x\in\eta} \theta(\hat{x}) \right) \lambda(d\hat{\eta}).
\end{aligned}$$

Here $\eta \Subset \gamma$ means that $\hat{\eta}$ is a finite and nonempty subset of $\hat{\gamma}$, $\hat{\Gamma}_0$ is the subset of $\hat{\Gamma}$ consisting of finite configurations, $k_\mu(\hat{\eta}) = k^{(m)}(\hat{x}_1, \ldots, \hat{x}_m)$ for $\hat{\eta} = \{\hat{x}_1, \ldots, \hat{x}_m\}$, and λ is the Lebesgue-Poisson measure on $\hat{\Gamma}_0$ defined by the formula

$$\int_{\hat{\Gamma}_0} G(\hat{\eta})\lambda(\hat{\eta}) = G(\varnothing) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\hat{X}^m} G^{(m)}(\hat{x})_1, \ldots, \hat{x}_m) d\hat{x}_1 \cdots \hat{x}_m,$$

where G and $G^{(m)}$ are defined analogously as k and $k^{(m)}$ above. In the sequel, we repeatedly use the following statement known as Minlos' lemma [30].

Lemma 3.1.3. *Let $n \in \mathbb{N}, n \geq 2$, then for all measurable functions $G : \hat{\Gamma}_0 \to \mathbb{R}, h : \hat{\Gamma}_0 \to \mathbb{R}$ it is true that:*

$$\int_{\hat{\Gamma}_0} \sum_{x\in\gamma} h(\hat{x})G(\hat{\gamma}\setminus\hat{x})\lambda(d\hat{\gamma}) = \int_{\hat{\Gamma}_0}\int_{\hat{X}} h(\hat{x})G(\hat{\gamma})d\hat{x}\lambda(d\hat{\gamma}). \tag{3.4}$$

We also use the following evident formula

$$\sum_{x\in\gamma} A(\hat{x}) \sum_{\eta\Subset\gamma\setminus x} B(\hat{\eta}) = \sum_{\eta\Subset\gamma} \sum_{x\in\eta} A(\hat{x}) B(\hat{\eta}\setminus\hat{x}), \tag{3.5}$$

holding for suitable functions h and G.

3.2 The model

To be able to guarantee that the evolution preserves the states possessing correlation functions with the properties as in Definition 3.1.1 we impose the following conditions on the kernel ν that appears in (3.1). Recall that by a measure kernel we mean a map $\mathcal{B}(X) \times \hat{X} \ni (\Lambda, \hat{y}) \mapsto \nu_{\hat{y}}(\Lambda)$, which is a finite positive Borel measure $\nu_{\hat{y}}$ for each $\hat{y} \in \hat{X}$ and a measurable function $\hat{y} \mapsto \nu_{\hat{y}}(\Lambda)$ for each $\Lambda \in \mathcal{B}(X)$. Namely, we assume that, for each $\hat{y}$, the measure $\nu_{\hat{y}}$ is such that

$$\nu_{\hat{y}}(dx) = b(\hat{y}|x)dx, \tag{3.6}$$

where the map $(\hat{y}, x) \mapsto b(\hat{y}|x) \in \mathbb{R}_+$ is measurable and satisfies

$$\forall a \in \mathbb{R}_+ \qquad b(y, a|x) \le \beta(x-y), \qquad \int_X \beta(x)dx =: \beta < \infty, \tag{3.7}$$

$$\forall x \in X \qquad \int_{\hat{X}} b(\hat{y}|x)d\hat{y} \le \bar{b} < \infty. \tag{3.8}$$

The condition in (3.7) is to control the procreation of the new members uniformly as to the age of the parental particles. The second condition controls the procreation from all age groups to given compact subset of $\Lambda \subset X$, i.e.,

$$\int_\Lambda \left(\int_{\hat{X}} b(\hat{y}|x)d\hat{y} \right) dx \le \bar{b}|\Lambda|,$$

where $|\Lambda|$ is the Euclidean volume of Λ.

Concerning the function $(x, a) \mapsto m(x, a)$ we assume that $x \mapsto m(x, a)$ is measurable for each fixed a and $a \mapsto m(x, a)$ continuous for each x. For $\hat{\eta} \in \hat{\Gamma}_0$, we then set

$$M(\hat{\eta}) = \sum_{x\in\eta} m(\hat{x}).$$

Our first aim is to pass from the Fokker-Planck equation (2.3) to the evolution equation for the correlation functions. Here one may follow the way elaborated in [26]. For $k : \hat{\Gamma}_0 \to \mathbb{R}$, we then define

$$(L^\Delta k)(\hat{\eta}) = -\sum_{x\in\eta} \frac{\partial}{\partial a_x} k(\hat{\eta}) - M(\hat{\eta})k(\hat{\eta}).$$

In order to use this operator on a regular basis, one would have to introduce an appropriate Sobolev-like Banach space of such functions keeping in mind that they satisfy the conditions mentioned in Definition 3.1.1. In this chapter, however, we restrict ourselves to deriving only equations for $k^{(1)}$ and $k^{(2)}$, which are usually 'deduced' by employing heuristic arguments. Keeping this in mind, we prove the following statement.

Proposition 3.2.1. *Assume that $\mu \in \mathcal{P}_*$ is such that its correlation function k_μ satisfies the renewal condition, cf. [15],*

$$k_\mu(\hat{\eta} \setminus \hat{x} \cup (x,0)) = \int_{\widehat{X}} b(\hat{y}|x) k_\mu(\hat{\eta} \setminus \hat{x} \cup \hat{y}) d\hat{y} + k_\mu(\hat{\eta} \setminus \hat{x}) \sum_{y \in \eta \setminus x} b(\hat{y}|x). \tag{3.9}$$

Assume further that both integrals

$$\begin{aligned} I_\mu(\theta) &:= \mu(LF^\theta) \qquad (3.10) \\ J_\mu(\theta) &:= \int_{\hat{\Gamma}_0} (L^\Delta k_\mu)(\hat{\eta}) \left(\prod_{x \in \eta} \theta(\hat{x}) \right) \lambda(d\hat{\eta}), \end{aligned}$$

absolutely converge for all $\theta \in \Theta$. Then $I_\mu(\theta) = J_\mu(\theta)$.

Proof. Define

$$\begin{aligned} (\tilde{L}^\Delta k_\mu)(\hat{\eta}) &= (L^\Delta k_\mu)(\hat{\eta}) + \sum_{x \in \eta} \delta(a_x) \qquad (3.11) \\ &\times \left[-k_\mu(\hat{\eta}) + \int_{\widehat{X}} b(\hat{y}|x) k_\mu(\hat{\eta} \setminus \hat{x} \cup \hat{y}) d\hat{y} + k_\mu(\hat{\eta} \setminus \hat{x}) \sum_{y \in \eta \setminus x} b(\hat{y}|x) \right], \end{aligned}$$

where $\delta(a_x)$ is Dirac's δ-function, which means that the right-hand side of (3.11) is a distribution. Note however, that $\tilde{L}^\Delta k = L^\Delta k$ if k satisfies (3.9). The Kolmogorov operator (3.1) with ν as in (3.6) acts on F^θ (3.2) as follows

$$\begin{aligned} (LF)^\theta(\hat{\gamma}) &= \sum_{x \in \gamma} \frac{\partial \theta(\hat{x})}{\partial a_x} \prod_{y \in \gamma \setminus x} (1 + \theta(\hat{y})) + \sum_{x \in \gamma} m(\hat{x})(1 - (1 + \theta(\hat{x}))) \prod_{y \in \gamma \setminus x} (1 + \theta(\hat{y})) \\ &+ \int_{\hat{X}} \sum_{x \in \gamma} b(\hat{x}|\hat{y}) \delta(a_y)(1 + \theta(\hat{y}) - 1) \prod_{z \in \gamma} (1 + \theta(\hat{z})) d\hat{y}. \end{aligned}$$

Now we split LF^θ into three parts. The first one is

$$\begin{aligned}
\int_{\hat\Gamma} L_1F^\theta(\hat\gamma)\mu(d\hat\gamma) &= \int_{\hat\Gamma}\sum_{x\in\gamma}\frac{\partial\theta(\hat x)}{\partial a_x}\prod_{y\in\gamma\setminus x}(1+\theta(\hat y))\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma}\sum_{x\in\gamma}\frac{\partial\theta(\hat x)}{\partial a_x}\sum_{\eta\subset\gamma\setminus x}\prod_{y\in\eta}\theta(\hat y)\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma}\sum_{\eta\subset\gamma}\sum_{x\in\eta}\frac{\partial\theta(\hat x)}{\partial a_x}\prod_{y\in\eta\setminus x}\theta(\hat y)\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma_0}k_\mu(\hat\eta)\sum_{x\in\gamma}\frac{\partial\theta(\hat x)}{\partial a_x}\prod_{y\in\eta\setminus x}\theta(\hat y)\lambda(d\hat\eta)\\
&= \int_{\hat\Gamma_0}\Big(\int_{\hat X}k_\mu(\hat\eta\cup\hat x)\frac{\partial\theta(\hat x)}{\partial a_x}d\hat x\Big)\prod_{y\in\eta}\theta(\hat y)\lambda(d\hat\eta)\\
&= -\int_{\hat\Gamma_0}\int_{\hat X}k_\mu(\hat\eta\cup\hat x)\delta(a_x)+\frac{\partial}{\partial a_x}k_\mu(\hat\eta\cup\hat x)\theta(\hat x)d\hat x\prod_{y\in\eta}\theta(\hat y)\lambda(d\hat\eta)\\
&= -\int_{\hat\Gamma_0}\Big(\sum_{x\in\eta}\delta(a_x)k_\mu(\hat\eta)+\frac{\partial}{\partial a_x}k_\mu(\hat\eta)\theta(\hat x)d\hat x\Big)\prod_{y\in\eta}\theta(\hat y)\lambda(d\hat\eta).
\end{aligned}$$

This is obtained using (3.5) with $A(x)=\frac{\partial\theta(\hat x)}{\partial a_x}$, $B(\hat\eta)=\prod_{y\in\eta}\theta(\hat y)$ and Lemma 3.1.3. The second part is

$$\begin{aligned}
\int_{\hat\Gamma} L_2F^\theta(\hat\gamma)\mu(d\hat\gamma) &= \int_{\hat\Gamma}\sum_{x\in\gamma}(-\theta(\hat x))m(\hat x)\prod_{y\in\gamma\setminus x}(1+\theta(\hat y))\mu(d\hat\gamma)\\
&= -\int_{\hat\Gamma}\sum_{x\in\gamma}\theta(\hat x)m(\hat x)\sum_{\eta\subset\gamma\setminus x}\prod_{y\in\eta}\theta(\hat y)\mu(d\hat\gamma)\\
&= -\int_{\hat\Gamma}\sum_{\eta\subset\gamma}\sum_{x\in\eta}m(\hat x)\prod_{y\in\eta}\theta(\hat y)\mu(d\hat\gamma)\\
&= -\int_{\hat\Gamma_0}\sum_{x\in\eta}m(\hat x)k_\mu(\hat\eta)\prod_{y\in\eta}\theta(\hat y)\lambda(d\hat\eta).
\end{aligned}$$

Finally, the third part

$$\begin{aligned}
\int_{\hat\Gamma} L_3 F^\theta(\hat\gamma)\mu(d\hat\gamma) &= \int_{\hat\Gamma}\int_{\hat X}\sum_{x\in\gamma} b(\hat x|\hat y)\delta(a_y)\theta(\hat y)\prod_{z\in\gamma}(1+\theta(\hat z))d\hat y\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma}\int_{\hat X}\sum_{x\in\gamma} b(\hat x|\hat y)\delta(a_y)\theta(\hat y)(1+\theta(\hat x))\prod_{z\in\gamma\setminus x}(1+\theta(\hat z))d\hat y\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma}\int_{\hat X}\sum_{x\in\gamma} b(\hat x|\hat y)\delta(a_y)\theta(\hat y)(1+\theta(\hat x))\sum_{\eta\subset\gamma\setminus x}\prod_{z\in\eta}\theta(\hat z)d\hat y\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma}\int_{\hat X}\sum_{\eta\subset\gamma}\sum_{x\in\eta} b(\hat x|\hat y)\delta(a_y)(1+\theta(\hat x))\prod_{z\in\eta\setminus x\cup y}\theta(\hat z)d\hat y\mu(d\hat\gamma)\\
&= \int_{\hat\Gamma_0}\int_{\hat X} k_\mu(\hat\eta)\sum_{x\in\eta} b(\hat x|\hat y)\delta(a_y)\prod_{z\in\eta\setminus x\cup y}\theta(\hat z)d\hat y\lambda(d\hat\eta)\\
&+ \int_{\hat\Gamma_0}\int_{\hat X} k_\mu(\hat\eta)\sum_{x\in\eta} b(\hat x|\hat y)\delta(a_y)\prod_{z\in\eta\cup y}\theta(\hat z)d\hat y\lambda(d\hat\eta)\\
&= \int_{\hat\Gamma_0}\int_{\hat X}\sum_{y\in\eta} k_\mu(\hat\eta\cup\hat x\setminus\hat y)b(\hat x|\hat y)\delta(a_y)\prod_{z\in\eta}\theta(\hat z)d\hat x\lambda(d\hat\eta)\\
&+ \int_{\hat\Gamma_0}\int_{\hat X}\sum_{y\in\eta} k_\mu(\hat\eta\setminus\hat y)b(\hat x|\hat y)\delta(a_y)\prod_{z\in\eta}\theta(\hat z)d\hat y\lambda(d\hat\eta).
\end{aligned}$$

Putting everything together and taking into account (3.9) we get

$$\begin{aligned}
\int_{\hat\Gamma}(L_1+L_2+L_3)F^\theta(\hat\gamma)\mu(d\hat\gamma) &= \int_{\hat\Gamma_0}\tilde L^\Delta k_\mu(\hat\eta)\left(\prod_{x\in\eta}\theta(\hat x)\right)\lambda(d\hat\eta)\\
&= \int_{\hat\Gamma_0} L^\Delta k_\mu(\hat\eta)\left(\prod_{x\in\eta}\theta(\hat x)\right)\lambda(d\hat\eta),
\end{aligned}$$

which yields the proof, see (3.10). □

3.3 The result

As in [26], by Proposition 3.2.1 one may obtain solutions of the Fokker-Planck equation (2.3) by solving the Cauchy problem

$$\frac{\partial}{\partial t}k_t(\hat\eta) = (L^\Delta k_t)(\hat\eta), \qquad k_t|_{t=0} \;=\; k_{\mu_0}, \tag{3.12}$$

where k_{μ_0} is the correlation function of the initial condition of (2.3). As mentioned above, a regular theory of this equation should be based on the use of appropriate Banach spaces. After developing such a theory, one should also prove that k_t is indeed the correlation function for a certain state, which is usually a hard technical problem, see [25]. Here we obtain from (3.12) the equations for $k_t^{(1)}$ and $k_t^{(2)}$, which can be viewed as the usual PDE of transport type.

For $\tau \geq 0$, set

$$\psi_\tau(\hat{\eta}) = k_{t-\tau}(\hat{\eta}_\tau), \tag{3.13}$$

where

$$\hat{\eta}_\tau = \{(x, a_x - \tau) : (x, a_x) \in \hat{\eta}\}, \qquad \tau \leq \min\{a_x, t\}.$$

To get a classical solution of (3.12) we have to assume that k_t is continuously differentiable with respect to each a_x and therefore it is continuously differentiable with respect to t. By taking the derivative in (3.13) by (3.12) we get

$$\frac{d}{d\tau}\psi_\tau(\hat{\eta}) = M(\hat{\eta}_\tau)\psi_\tau(\hat{\eta})$$

which can be solved

$$\psi_\tau(\hat{\eta}) = \psi_0(\hat{\eta}) \exp\left(\int_0^\tau M(\hat{\eta}_\vartheta) d\vartheta\right).$$

By (3.13) this yields, where $a_{\hat{\eta}} = \min_{x \in \eta} a_x$:

$$k_t(\hat{\eta}) = \begin{cases} k_0(\hat{\eta}_t) \exp\left(-\int_0^t M(\hat{\eta}_\vartheta) d\vartheta\right), & t \leqslant a_{\hat{\eta}} \\ k_{t-a_{\hat{\eta}}}(\hat{\eta}_{a_{\hat{\eta}}}) \exp\left(-\int_0^{a_{\hat{\eta}}} M(\hat{\eta}_\vartheta) d\vartheta\right), & t > a_{\hat{\eta}}, \end{cases} \tag{3.14}$$

which is a functional equation. Below we study this equation in the two particular cases mentioned above.

3.3.1 The evolution of densities

By setting $\hat{\eta} = \{\hat{x}\}$ in (3.14), we get the following formula describing the evolution of the first correlation function

$$k_t^{(1)}(x, a) = \begin{cases} k_0^{(1)}(x, a-t) \exp\left(-\int_0^t m(x, a-\theta) d\theta\right), & t \leqslant a \\ k_{t-a}^{(1)}(x, 0) \exp\left(-\int_0^a m(x, a-\theta) d\theta\right), & t > a, \end{cases} \tag{3.15}$$

subject to the following renewal condition obtained from (3.9)

$$k_t^{(1)}(x, 0) = \int_{\hat{X}} b(\hat{y}|x) k_t^{(1)}(\hat{y}) d\hat{y}.$$

By (3.15) we have

$$
\begin{aligned}
k_t^{(1)}(x,0) &= \int_{\mathbb{R}^d}\left(\int_0^\infty b(y,a_y|x)k_t^{(1)}(y,a_y)da_y\right)dy \\
&= \int_{\mathbb{R}^d}\int_0^t b(y,a_y|x)k_{t-a_y}^{(1)}(y,0)e^{-\int_0^{a_y} m(y,a_y-\theta)d\theta}da_y dy \qquad (3.16)\\
&+ \int_{\mathbb{R}^d}\int_t^{+\infty} b(y,a_y|x)k_0^{(1)}(y,a_y-t)e^{-\int_0^t m(y,a_y-\theta)d\theta}da_y dy.
\end{aligned}
$$

By (3.7), the second term in (3.16) is bounded uniformly with respect to t and x by

$$
\beta\cdot \operatorname*{ess\,sup}_{y\in\mathbb{R}^d}\left|\int_0^\infty k_0^{(1)}(y,a)da\right|,
$$

see also item (iv) of Definition 3.1.1. Set

$$
k_t^{(1)}(x,0) := u_t(x).
$$

Now (3.16) can be written in the following form:

$$
u = Au + v, \tag{3.17}
$$

where

$$
\begin{aligned}
(Au)_t(x) &= \int_{\mathbb{R}^d}\int_0^t b(y,a_y|x)u_{t-a_y}(y)e^{-\int_0^{a_y} m(y,a_y-\theta)d\theta}da_y dy,\\
v_t(x) &= \int_{\mathbb{R}^d}\int_t^{+\infty} b(y,a_y|x)k_0^{(1)}(y,a_y-t)e^{-\int_0^t m(y,a_y-\theta)d\theta}da_y dy.
\end{aligned}
$$

Both u and v are positive elements of the Banach space $\mathcal{U}_\alpha$ of functions $\mathbb{R}_+\times\mathbb{R}^d\mapsto\mathbb{R}$, which are $C^1(\mathbb{R}_+)$ for almost all fixed x and $L^\infty(\mathbb{R}^d)$ for all a, equipped with the norm

$$
\|u\|_\alpha = \sup_{t\geq 0} e^{-\alpha t}\|u_t\|_{L^\infty(\mathbb{R}^d)}, \qquad \alpha > 0. \tag{3.18}
$$

By (3.7), the operator norm of A satisfies

$$
\|A\|_\alpha \leq \beta \operatorname*{ess\,sup}_{y\in\mathbb{R}^d}\int_0^\infty e^{-\alpha a-\int_0^a m(y,\theta)d\theta}da =: \beta q(\alpha).
$$

If the mortality rate $m(y,a)$ is separated away from zero, i.e., satisfies $m(y,\alpha)\geq m_*$, then

$$
q(\alpha)\leq 1/(\alpha+m_*). \tag{3.19}
$$

For $\beta q(\alpha)<1$, we can solve (3.17) in the form

$$
u = (I-A)^{-1}v,
$$

where v is determined by the initial condition of (3.12). By item (i) of Definition 3.1.1 and (3.8), it follows that

$$\|v\|_\alpha \le \|v\|_0 \le \bar{b}\|k_0^{(1)}\|_{L^\infty(\hat{X})}. \tag{3.20}$$

Now (3.16) in the space $\mathcal{U}_\alpha$ takes the form:

$$k_t^{(1)}(x,0) = u_t(x) = (Bv)_t(x) := \sum_{n=0}^{\infty} (A^n v)_t(x). \tag{3.21}$$

Finally, the evolution of the particle density is described by the formula

$$k_t{}^{(1)}(x,a) = \begin{cases} k_0{}^{(1)}(x,a-t)e^{-\int_0^t m(x,a-\theta)d\theta}, & t \leqslant a \\ \sum_{n=0}^{\infty} (A^n v)_{t-a}(x) e^{-\int_0^a m(x,a-\theta)d\theta}, & t > a. \end{cases} \tag{3.22}$$

The solution (3.22) may increase in time. By (3.18), (3.20) and (3.21), for $t > a$ it satisfies

$$k_t(x,a) \le \frac{\|v\|_\alpha}{1-\beta q(\alpha)} e^{\alpha t} \le \frac{\bar{b}}{1-\beta q(\alpha)} e^{\alpha t}\|k_0^{(1)}\|_{L^\infty(\hat{X})} e^{\alpha t}.$$

For $m_* > \beta$, by (3.19) we have $\beta q(0) < 1$, which means that $k_t^{(1)}(x,a)$ remains bounded in this case.

3.3.2 The evolution of the second correlation function

In this case, we take $\hat{\eta} = \{(x,a_x),(y,a_y)\}$ with $a_x \le a_y$. By (3.14) we then get

$$k_t^{(2)}(y,a_y,x,a_x) = \begin{cases} k_0^{(2)}(\hat{\eta}_t) e^{-\int_0^t (m(y,a_y-\theta)+m(x,a_x-\theta))d\theta}, & t \leqslant a_x \\ k_{t-a_x}^{(2)}(y,a_y-a_x,x,0) e^{-\int_0^{a_x}(m(y,a_y-\theta)+m(x,a_x-\theta)d\theta}, & t > a_x. \end{cases} \tag{3.23}$$

To find $k_{t-a_x}^{(2)}(y,a_y-a_x,x,0)$, we use (3.9) which yields

$$k_t^{(2)}(y,a,x,0) = \int_{\mathbb{R}^d}\int_0^{+\infty} b(z,a_z|x) k_t^{(2)}(y,a;z,a_z)\, da_z dz \tag{3.24}$$

$$+ \quad k_t^{(1)}(y,a) b(y,a|x).$$

For $t \leq a$, by (3.22) for $k_t^{(1)}(y,a)$ and (3.23) for $k_t^{(2)}(y,a;z,a_z)$ we bring (3.24) to the following

$$\begin{aligned} k_t^{(2)}(y,a,x,0) &= b(y,a|x)k_0^{(1)}(y,a-t)e^{-\int_0^t m(y,a-\theta)d\theta} \\ &+ \int_{\mathbb{R}^d}\int_0^t b(z,a_z|x)k_{t-a_z}^{(2)}(y,a-a_z;z,0) \\ &\times \exp\Big(-\int_0^{a_z}[m(z,a_z)+m(y,a_y-\theta)]d\theta\Big)da_zdz \\ &+ \int_{\mathbb{R}^d}\int_t^{+\infty} b(z,a_z|x)k_0^{(2)}(y,a-t;z,a_z-t) \\ &\times \exp\Big(-\int_0^t [m(y,a-\theta)+m(z,a_z-\theta)]\,d\theta\Big)da_zdz. \end{aligned} \tag{3.25}$$

On the other hand, for $t > a$, we rewrite (3.24) as follows:

$$\begin{aligned} k_t^{(2)}(y,a,x,0) &= b(y,a|x)\,(Bv)_{t-a}(y)e^{-\int_0^a m(y,a-\theta)d\theta} \\ &+ \int_{\mathbb{R}^d}\int_0^a b(z,a_z|x)k_{t-a_z}^{(2)}(y,a-a_z;z,0) \\ &\times \exp\left(-\int_0^{a_z}[m(z,a_z-\theta))+m(y,a-\theta)]d\theta\right)da_zdz \\ &+ \int_{\mathbb{R}^d}\int_a^t b(z,a_z|x)k_{t-a}^{(2)}(y,0;z,a_z-a) \\ &\times \exp\left(-\int_0^a [m(y,a-\theta)+m(z,a_z-\theta)]d\theta\right)da_zdz \\ &+ \int_{\mathbb{R}^d}\int_t^{+\infty} b(z,a_z|x)k_{t-a}^{(2)}(y,0;z,a_z-a) \\ &\times \exp\left(-\int_0^a [m(y,a-\theta)+m(z,a_z-\theta)]d\theta\right)da_zdz. \end{aligned} \tag{3.26}$$

Now we would like to solve (3.25) and (3.26) as a single equations in the space of continuously differentiable functions $\mathbb{R}_+ \ni t \mapsto w_t \in C^1(\mathbb{R}_+)\otimes L^\infty((\mathbb{R}^d)^2)\otimes L^1(\mathbb{R}_+)$. Let:

$$w_t(y,a,x) := k_t{}^{(2)}(y,a,x,0),$$

then we have:

$$w_t(y,a,x) = (A_2w)_t(y,a,x) + f_t(y,a,x). \tag{3.27}$$

For $t \leq a$ define:

$$\begin{aligned}
(A_2 w)_t(y,a,x) &= \int_{\mathbb{R}^d}\int_0^t b(z,a_z|x) w_{t-a_z}(z,y,a-a_z) \\
&\times \exp\left(-\int_0^a (m(y,a-\theta)) + m(z,a_z-\theta))d\theta\right) da_z dz,
\end{aligned}$$

$$\begin{aligned}
f_t(y,a,x) &= \int_{\mathbb{R}^d}\int_t^{+\infty} b(z,a_z|x) k_0^{(2)}(y,a-t,z,a_z-t) \\
&\times \exp\left(-\int_0^t m(y,a-\vartheta) + m(z,a_z-\vartheta)d\vartheta\right) da dy \\
&+ k_t{}^{(1)}(y,a) b(x,y,a),
\end{aligned} \tag{3.28}$$

and for $t > a$:

$$\begin{aligned}
(A_2 w)_t(y,a,x) &= \int_{\mathbb{R}^d}\int_0^a b(z,a_z|x) w_{t-a_z}(z,y,a-a_z) \\
&\times \exp\left(-\int_0^{a_z} (m(y,a-\theta)) + m(z,a_z-\theta))d\theta\right) da_z dz \\
&+ \int_{\mathbb{R}^d}\int_a^t b(z,a_z|x) w_{t-a}(y,z,a_z-a) \\
&\times \exp\left(-\int_0^a (m(y,a-\theta)) + m(z,a_z-\theta))d\theta\right) da_z dz \\
&+ \int_{\mathbb{R}^d}\int_t^\infty b(z,a_z|x) w_{t-a}(y,z,a_z-a) \\
&\times \exp\left(-\int_0^a (m(y,a-\theta)) + m(z,a_z-\theta))d\theta\right) da_z dz,
\end{aligned}$$

$$f_t(y,a,x) = k_t{}^{(1)}(y,a) b(y,a,x). \tag{3.29}$$

Let $\mathcal{W}_\alpha$ be a space with the norm:

$$\|w\|_\alpha = \sup_{t\geq 0} e^{-\alpha t} \operatorname*{ess\,sup}_{(x,y)\in(\mathbb{R}^d)^2} \int_0^\infty |w_t(y,a,x)| da.$$

It is clear that:

$$\int_0^\infty (A_2 w)_t(y,a,x) da = \int_0^t (A_2 w)_t(y,a,x) da + \int_t^\infty (A_2 w)_t(y,a,x) da.$$

Therefore

$$\begin{aligned}\int_0^\infty (A_2 w)_t(y,a,x)da &= \int_0^t \int_{\mathbb{R}^d} \int_0^a b(z,a_z|x) w_{t-a_z}(z,y,a-a_z) \\ &\times \exp\left(-\int_0^{a_z} m(y,a-\theta)) - m(z,a_z-\theta))d\theta\right) da_z dz da \\ &+ \int_0^t \int_{\mathbb{R}^d} \int_a^t b(z,a_z|x) w_{t-a}(y,z,a_z-a) \\ &\times \exp\left(-\int_0^{a} m(y,a-\theta)) - m(z,a_z-\theta))d\theta\right) da_z dz da \\ &+ \int_0^t \int_{\mathbb{R}^d} \int_t^\infty b(z,a_z|x) w_{t-a}(y,z,a_z-a) \\ &\times \exp\left(-\int_0^{a} m(y,a-\theta)) - m(z,a_z-\theta))d\theta\right) da_z dz da \\ &+ \int_t^\infty \int_{\mathbb{R}^d} \int_0^t b(z,a_z|x) w_{t-a_z}(z,y,a-a_z) \\ &\times \exp\left(-\int_0^{a} m(y,a-\theta)) - m(z,a_z-\theta))d\theta\right) da_z dz da.\end{aligned}$$

To estimate the previous integral we use (3.8) and the fact that $e^{-a} < 1$, where $a > 0$. Let $\alpha = \frac{\beta}{q}$, for some fixed $q < 1$ therefore:

$$\|A_2 w\|_\alpha \leq q\|w\|_\alpha.$$

We can write the solution for (3.27) as:

$$k_t(x,0,y,a) = w_t(x,y,a) = \sum_{n=0}^\infty (A_2{}^n f)_t(x,y,a).$$

The solution for $k^{(2)}(x,a_x,y,a_y)$ takes the form:

$$k^{(2)}(x,a_x,y,a_y) = \begin{cases} k_0^{(2)}(x,a_x-t;y,a_y-t) \\ \times \exp\Big(-\int_0^t (m(x,a_x-\theta)+m(y,a_y-\theta))d\theta\Big), & \text{for } t \leq a_x; \\ \Big(\sum_{n=0}^\infty (A_2{}^n f)_{t-a_x}(y,a_y-a_x,x) + k_{t-a_x}^{(1)}(y,a)b(y,a_y|x)\Big) \\ \times \exp\Big(-\int_0^{a_x} (m(x,a_x-\theta)+m(y,a_y-\theta))d\theta\Big), & \text{for } a_x < t. \end{cases}$$

For $a_x < t \leq a_y$ with the appropriate formula for f_t (3.28):

$$\begin{aligned}(A_2 f)_t(y,a,x) &= \int_{\mathbb{R}^d}\int_0^t b(z,a_z|x) f_{t-a_z}(z,y,a-a_z) \\ &\times \exp\left(-\int_0^a [m(y,a-\theta)) + m(z,a_z-\theta)]d\theta\right) da_z dz.\end{aligned}$$

For $a_x \leq a_y < t$, f_t like in (3.29):

$$\begin{aligned}(A_2 f)_t(y,a,x) &= \int_{\mathbb{R}^d}\int_0^a b(z,a_z|x) f_{t-a_z}(z,y,a-a_z) \\ &\times \exp\left(-\int_0^{a_z} [m(y,a-\theta)) + m(z,a_z-\theta)]d\theta\right) da_z dz \\ &+ \int_{\mathbb{R}^d}\int_a^t b(z,a_z|x) f_{t-a}(y,z,a_z-a) \\ &\times \exp\left(-\int_0^a [m(y,a-\theta)) + m(z,a_z-\theta))d\theta]\right) da_z dz \\ &+ \int_{\mathbb{R}^d}\int_t^\infty b(z,a_z|x) f_{t-a}(y,z,a_z-a) \\ &\times \exp\left(-\int_0^a [m(y,a-\theta)) + m(z,a_z-\theta)]d\theta\right) da_z dz.\end{aligned}$$

Chapter 4

The independent appearance model: evolution in a spacial class of states

In this and the next chapters we introduce and study the following modification of the model defined in (3.1). The corresponding Kolmogorov operator is taken in the form

$$(LF)(\hat{\gamma}) = \sum_{x\in\gamma} \frac{\partial}{\partial a_x} F(\hat{\gamma}) + \sum_{x\in\gamma} m(\hat{x}) \left[F(\hat{\gamma}\setminus\hat{x}) - F(\hat{\gamma}) \right] + \int_X \left[F(\hat{\gamma}\cup(y,0)) - F(\hat{\gamma}) \right] \chi(dy), \tag{4.1}$$

where the first two terms have the same meaning as the corresponding terms of L given in (3.1), whereas the third one described an independent appearance of new particles in the habitat governed by the measure χ. In this chapter, we take $X = \mathbb{R}^d$ and

$$\chi(dy) = b(y)dy, \tag{4.2}$$

with an appropriate positive density b and the Lebesgue measure dy. In the next chapter, we consider the case of a general locally compact Polish space X and hence a general arriving measure χ. Our present choice of X and χ (4.2) will allow us to solve the Fokker-Planck equation (2.3) and thus to construct the evolution of states $\mu_0 \mapsto \mu_t$ which preserves a special class of them. We begin describing this class by introducing 'tempered configurations'.

4.1 Tempered configurations and measures

In this section, we select a subset of $\hat{\Gamma}_* \subset \hat{\Gamma}$ such that the measures in question have the property $\mu(\hat{\Gamma}_*) = 1$, and thus forget of the remaining configurations.

4.1.1 Tempered configurations

Let $\psi : X \to \mathbb{R}_+$ be:

(a) continuous, bounded and strictly positive;

(b) integrable, i.e., satisfy $\int_X \psi(x)dx < \infty$.

One can take $\psi(x) = e^{-|x|}$ as an example of such a function. Then we set

$$\Psi(\hat{\gamma}) = \sum_{\hat{x}\in\hat{\gamma}} \psi(x), \qquad \hat{\gamma} \in \hat{\Gamma}, \tag{4.3}$$

which can take infinite values for some $\hat{\gamma}$. Thereby, the set of tempered configurations is defined as

$$\hat{\Gamma}_* = \{\hat{\gamma} \in \hat{\Gamma} : \Psi(\hat{\gamma}) < \infty\},$$

where $\hat{\Gamma}$ is the set of all (possibly multiple) configurations. Similarly as in [26], subsect. 2.3, we equip this set with the following metric

$$\upsilon_*(\hat{\gamma}, \hat{\gamma}') = \sup_g \Big| \sum_{\hat{x}\in\hat{\gamma}} g(\hat{x})\psi(x) - \sum_{\hat{x}\in\hat{\gamma}'} g(\hat{x})\psi(x) \Big|, \tag{4.4}$$

where the supremum is taken over the subset of the set of bounded Lipschitz-continuous functions $C^{BL}(\hat{X})$ consisting of those $g : \hat{X} \to \mathbb{R}$ for which

$$\sup_{\hat{x}\in\hat{X}} |g(\hat{x})| + \sup_{\hat{x}\neq\hat{y}\in\hat{X}} \frac{|g(x,\alpha) - g(y,\alpha')|}{|x-y| + |\alpha - \alpha'|} \leq 1.$$

It is possible to prove, [26], Proposition 2.7, that the metric space $(\hat{X}, \upsilon_*)$ is complete and separable. Let $\mathcal{B}(\hat{\Gamma}_*)$ be the corresponding Borel σ-field of such subsets of $\hat{\Gamma}_*$. By Kuratowski's theorem (Theorem 3.9 in [31]), one then proves that $\hat{\Gamma}_* \in \mathcal{B}(\hat{\Gamma})$ and $\mathcal{B}(\hat{\Gamma}_*)$ coincides with the Borel $\sigma-$field related to the topology on $\hat{\Gamma}_*$ induced by the vague topology of $\hat{\Gamma}$. This allows one to redefine each $\mu \in P(\hat{\Gamma})$ with the property $\mu(\hat{\Gamma}_*) = 1$ as a measure on $(\hat{\Gamma}_*, B(\hat{\Gamma}_*))$, see [26], Corollary 2.8 for further details.

Let $\mathcal{P}_* \subset \mathcal{P}(\hat{\Gamma})$ be as in Definition 3.1.1. By Proposition 2.2.2, we know that each $\mu \in \mathcal{P}_*$ is supported on the set of simple configurations, which allows us to assume – in this chapter also – that $\hat{\Gamma}$ and $\hat{\Gamma}_*$ consist of single configurations only. It turns out that the set of measures has another useful property.

Proposition 4.1.1. *For each $\mu \in \mathcal{P}_*$, it follows that $\mu(\hat{\Gamma}_*) = 1$.*

Proof. Similarly as in Proposition 2.2.2, the proof consist in showing that $\mu(\Psi) < \infty$, which readily follows by (2.9) and (2.10)

$$\mu(\Psi) = \int_{\hat{X}} k_\mu^{(1)}(\hat{x})\psi(x)d\hat{x} \leq \int_X \psi(x)dx \left(\sup_{x\in X} \int_{\mathbb{R}_+} k_\mu^{(1)}(x,\alpha)d\alpha \right) < \infty, \tag{4.5}$$

see item (iv) of Definition 3.1.1. □

As mentioned above, each $\mu \in \mathcal{P}_*$ can be redefined as a probability measure on $(\hat{\Gamma}_*, \mathcal{B}(\hat{\Gamma}_*))$ with single configurations, which we assume from now on. Recall that the functions F^θ are defined in (3.2).

Definition 4.1.2. *For a given measurable $q : \hat{X} \to (0, 1)$ and $\mu \in \mathcal{P}(\hat{\Gamma})$, the measure μ^q defined by the relation $\mu^q(F^\theta) = \mu(F^{\theta_q})$ is called an independent $q-$thinning of μ. Here $\theta_g(\hat{x}) = \theta(\hat{x})q(\hat{x})$ and $\theta \in \Theta_0$, see Definition 3.1.2.*

For $\mu \in \mathcal{P}_*$, its q-thinning amounts to multiplying $k_\mu(\hat{\eta})$ by $\prod_{x \in \eta} q(\hat{x})$. An important subclass of $\mathcal{P}_*$ constitute Poisson measures π_ϱ with intensity measures such that

$$\varrho(d\hat{x}) = \rho(\hat{x})d\hat{x}, \tag{4.6}$$

with ρ satisfying the conditions of Definition 3.1.1 in the part related to $k_\mu^{(1)}$. For such measures, one has, cf. (2.12),

$$k_{\pi_\varrho}(\hat{\eta}) = \prod_{x \in \eta} \rho(\hat{x}). \tag{4.7}$$

Then by (3.3) it follows that

$$\pi_\varrho(F^\theta) = \exp\Big(\int_{\hat{X}} \rho(\hat{x})\theta(\hat{x})d\hat{x}\Big). \tag{4.8}$$

Note that each $\mu \in \mathcal{P}(\hat{\Gamma})$ can have a correlation function understood as a distribution. To see this, let us first define

$$\delta(\xi; \hat{\eta}) = \begin{cases} \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \delta(\hat{x}_j - \hat{y}_{\sigma(j)}), & \text{if } \ |\hat{\eta}| = |\hat{\xi}| = n; \\ 0 & \text{otherwise.} \end{cases}$$

In the first line, $\hat{\xi} = \{\hat{x}_1, \ldots, \hat{x}_n\}$, $\quad \hat{\eta} = \{\hat{y}_1, \ldots, \hat{y}_n\}$, $\quad \Sigma_n$ is the symmetric group, and $\delta(\hat{x} - \hat{y})$ is the usual Dirac $\delta-$function on $\mathbb{R}^d \times \mathbb{R}_+$. The correlation function $k_{\hat{\gamma}}$ of the $\delta-$measure $\delta_{\hat{\gamma}} \in \mathcal{P}(\hat{\Gamma})$ is then

$$k_{\hat{\gamma}}(\hat{\xi}) = \sum_{\eta \subset p(\hat{\xi})} \delta(\hat{\xi}; \hat{\eta}). \tag{4.9}$$

By (4.9) we then have

$$\delta_{\hat{\gamma}}(F^\theta) = \int_{\hat{\Gamma}_0} k_{\hat{\gamma}}(\hat{\xi}) \prod_{\hat{x} \in \hat{\xi}} \theta(\hat{x}) \lambda(d\hat{\xi}) = \prod_{\hat{x} \in \hat{\gamma}} (1 + \theta(\hat{x})).$$

By means of $k_{\hat{\gamma}}$ one can define the correlation function for any μ by the formula

$$k_\mu(\hat{\xi}) = \int_{\hat{\Gamma}} k_{\hat{\gamma}}(\hat{\xi})\mu(d\hat{\gamma}). \tag{4.10}$$

4.1.2 Convolution of measures

For $\mu_1, \mu_2 \in \mathcal{P}(\hat{\Gamma})$, their *convolution* is defined as

$$(\mu_1 \star \mu_2)(F) = \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} F(\hat{\gamma}_1 \cup \hat{\gamma}_2)\mu_1(d\hat{\gamma}_1)\mu_2(d\hat{\gamma}_2). \tag{4.11}$$

As mentioned above, the class F^θ, see (3.2), we $\theta \in \Theta$ is separating hence measure-defining. Thus, it is enough to define $\mu_1 \star \mu_2$ for such F^θ. By (3.2) it readily follows that

$$(\mu_1 \star \mu_2)(F^\theta) = \mu_1(F^\theta)\mu_2(F^\theta). \tag{4.12}$$

Then for $\mu_1, \mu_2 \in \mathcal{P}_*$, one readily gets that

$$(\mu_1 \star \mu_2)(F^\theta) = \int_{\hat{\Gamma}_0} \Big(\sum_{\xi \subset p(\hat{\eta})} k_{\mu_1}(\hat{\eta} \setminus \hat{\xi}) k_{\mu_2}(\hat{\xi}) \Big) \prod_{x \in \eta} \theta(\hat{x}) \lambda(d\hat{\eta}). \tag{4.13}$$

Proposition 4.1.3. *It follows that $\mu_1 \star \mu_2 \in \mathcal{P}_*$, whenever $\mu_1, \mu_2 \in \mathcal{P}_*$.*

Proof. We begin by slightly formalizing Definition 3.1.1. According to its item (iv), it follows that each

$$\varkappa_\mu^{(n)}(x_1, \ldots, x_n) := \int_{(\mathbb{R}_+)^n} k_\mu^{(n)}((x_1, a_1), \ldots, (x_n, a_n)) da_1 \cdots da_n, \qquad n \in \mathbb{N},$$

is essentially bounded. Let $\varkappa_\mu^{(n)}$ be this bound. Then item (v) of Definition 3.1.1 is equivalent to the following

$$\varkappa_\mu^{(n)} \le n! \varepsilon^{-n}, \qquad n \in \mathbb{N},$$

holding for some $\varepsilon > 0$. At the same time, by (4.13) it follows that

$$k_{\mu_1 \star \mu_2}(\hat{\eta}) = \sum_{\xi \subset p(\hat{\eta})} k_{\mu_1}(\hat{\eta} \setminus \hat{\xi}) k_{\mu_2}(\hat{\xi}),$$

which then yields

$$\varkappa_{\mu_1 \star \mu_2}^{(n)} \le \sum_{m=0}^{n} \binom{n}{m} \varkappa_{\mu_1}^{(m)} \varkappa_{\mu_2}^{n-m} \le (n+1)! \varepsilon^{-n} \le n! \le n! \Big(\frac{\varepsilon}{2}\Big)^{-n},$$

where $\varepsilon = \min\{\varepsilon_{\mu_1}; \varepsilon_{\mu_2}\}$. This yields the validity of item (v) of the mentioned definition. □

4.2 The result

In this section, we formulate and prove a statement describing the evolution of our model introduced in (4.1), (4.2). It is determined by the model parameters m and b which are supposed nonnegative, measurable and bounded. Additionally, we assume that the map $\mathbb{R}_+ \ni \alpha \mapsto m(x, \alpha)$ is continuous for each fixed x.

4.2.1 The statement

Along with the class of functions Θ_0 introduced in Definition 3.1.2, we will use the following class.

Definition 4.2.1. *By Θ we mean the collection of all $\theta : \hat{X} \to \mathbb{R}$ that have the following form*

$$\theta(x,\alpha) = \vartheta(x)e^{-\tau\psi(x)\phi(\alpha)} + e^{-\tau\psi(x)\phi(\alpha)} - 1, \tag{4.14}$$

where $\vartheta : X \to (-1, 0]$ is a continuous functions with compact support, ψ is as in (4.3), $\tau \geq 0$ and $\phi(\alpha) = \frac{\alpha}{1+\alpha}$.

Then we define $\mathcal{F} = \{F^\theta : \theta \in \Theta\}$ with F^θ having the form

$$F^\theta(\hat{\gamma}) = \exp\Big(\sum_{\hat{x}\in\hat{\gamma}} \log(1+\vartheta(x)) - \tau \sum_{\hat{x}\in\hat{\gamma}} \psi(x)\varphi(\alpha)\Big).$$

Our aim is to solve the corresponding Fokker-Planck equation (2.3) for F^θ with $\theta \in \Theta$. Note that $0 < F^\theta(\hat{\gamma}) \leq 1$ for each $\hat{\gamma} \in \hat{\Gamma}_*$ and $\mu(F^\theta) \leq 1$ for all $\mu \in \mathcal{P}_*$. It is possible to show, cf. [12], Theorem 18, that each F^θ is υ_*-continuous (see (4.4)). The pointwise product of F^θ and $F^{\theta'}$ is $F^{\theta''}$ with θ'' corresponding to $\vartheta''(x) = \vartheta(x) * \vartheta'(x)$ defined in (5.23) below and $\tau'' = \tau + \tau'$. Assume that $\hat{\gamma}_1 \neq \hat{\gamma}_2$, both are in $\hat{\Gamma}_*$. Then one finds $\hat{x}$ which belongs to exactly one of these configurations, say $\hat{\gamma}_1$. If there is no $\hat{y} \in \hat{\gamma}_2$ with $p(\hat{y}) = p(\hat{x})$, one takes $\tau = 0$ and ϑ such that $\vartheta(p(\hat{x})) \neq 0$ and $\vartheta(p(\hat{y})) = 0$ for all $\hat{y} \in \hat{\gamma}_2$. Otherwise, one takes $\tau > 0$ and $\vartheta(p(\hat{x})) = \vartheta(p(\hat{y})) \neq 0$ and $\vartheta(p(\hat{z})) = 0$ for all $\hat{z} \in \hat{\gamma}_1 \cup \hat{\gamma}_2$ such that $\vartheta(p(\hat{z})) \neq \vartheta(p(\hat{x}))$. In both cases, the corresponding F^θ separates $\hat{\gamma}_1$ and $\hat{\gamma}_2$, see property (c) of Proposition 2.1.3. Clearly, $F^\theta \equiv 1$ for $\tau = 0$ and $\vartheta \equiv 0$. Then by Proposition 2.1.3 $F = \{F^\theta : \theta \in \Theta\}$ is separating.

Let us prove now that LF^θ is $\mu-$integrable for each $\mu \in \mathcal{P}_*$. By (4.1) we have

$$\begin{aligned}(LF^\theta)(\hat{\gamma}) &= \sum_{\hat{x}\in\hat{\gamma}} \Big(\frac{\partial}{\partial\alpha}\theta(x,\alpha) - m(x,\alpha)\theta(x,\alpha)\Big) F^\theta(\hat{\gamma}\setminus\hat{x}) \\ &+ F^\theta(\hat{\gamma}) \int_X b(x)\theta(x,0)dx =: H_1(\hat{\gamma}) + H_2(\hat{\gamma}).\end{aligned} \tag{4.15}$$

Since b is bounded, H_2 is also bounded. Since ϑ is continuous and compactly supported, it is ψ-bounded. Hence, by (4.14) one concludes that, for all $\hat{x} \in \hat{X}$, the following holds

$$|\theta(\hat{x})| \leq d_\theta \psi(x), \quad \Big|\frac{\partial}{\partial\alpha}\theta(\hat{x})\Big| \leq \tau\psi(x), \tag{4.16}$$

where d_θ depends only on the choice of ϑ and τ. By (4.16) we then have

$$|H_1(\hat{\gamma})| \leq D_\theta \Psi(\hat{\gamma}), \tag{4.17}$$

holding with an appropriate D_θ. By (4.5) this yields the property in question. Now for $\theta \in \Theta$ and m as in (4.1), we set

$$\theta_t(x,a) = \theta(x, a+t) \exp\Big(- \int_\alpha^{\alpha+t} m(x,\alpha) d\alpha \Big), \qquad t \geq 0, \tag{4.18}$$

and then define a map $\mathcal{P}_* \ni \mu \to \mu^t \in \mathcal{P}_*, t \geq 0$ by the following relation

$$\mu^t(F^\theta) = \mu(F^{\theta_t}), \qquad \theta \in \Theta. \tag{4.19}$$

Since the family $\{F^\theta : \theta \in \Theta\}$ is separating, each μ^t is uniquely determined by (4.18), (4.19). Note that the correlation function of μ^t can be expressed through that of μ as follows

$$k_{\mu^t}(\hat{\eta}) = \mathcal{J}_t(\hat{\eta}) k_{\mu(\hat{\eta}^t)} \exp\Big(- \sum_{\hat{x} \in \hat{\eta}} \int_\alpha^{\alpha+t} m(x,\alpha) d\alpha \Big), \tag{4.20}$$

where $\hat{\eta}^t = \{(x, \alpha - t) : x \in p(\hat{\eta}\}$,

$$\mathcal{J}_t(\hat{\xi}) = \prod_{x \in p(\hat{\xi})} J_t(x,\alpha), \quad J_t(x,\alpha) := 1 - \mathbf{1}_{[0,t)}(\alpha),$$

By Definition 3.1.1, the map $\mu \mapsto \mu^t$ preserves $\mathcal{P}_*$ and is a combination of a thinning and an age shift. Now we are at a position to formulate our result.

Theorem 4.2.2. *For each $\mu_0 \in \mathcal{P}_*$, the Fokker-Planck equation (2.3) has a solution of the following form*

$$\mu_t = \mu_0^t \star \pi_{\varrho_t}, \tag{4.21}$$

where μ_0^t is obtained from μ_0 according to (4.19) and π_{ϱ_t} is the Poisson measure, see (4.6), (4.7) and (4.8), with the intensity measure

$$\varrho_t(dx, d\alpha) = \hat{b}(x,\alpha) \mathbf{1}_{[0,t)}(\alpha) dx d\alpha := b(x) \exp\Big(- \int_0^\alpha m(x,\varsigma) d\varsigma \Big) \mathbf{1}_{[0,t)}(\alpha) dx d\alpha, \tag{4.22}$$

where $\mathbf{1}_{[0,t)}(\alpha)$ is defined in (2.1). If $m(\hat{x}) \geq m_*$ for some $m_* > 0$, the evolution given in (4.21) has a stationary state π_ϱ with $\varrho(dx, d\alpha) = \hat{b}(x,\alpha) dx d\alpha$, see (4.22).

4.2.2 Comments

Let us make some comments to this statement. According to (4.12), (4.8) and (4.21) it follows that

$$\mu_t(F^\theta) = \exp\Big(\int_X \int_{[0,t)} \hat{b}(x,\alpha) \theta(x,\alpha) dx d\alpha \Big) \mu_0(F^{\theta_t}). \tag{4.23}$$

Hence, the solution satisfies the initial condition $\mu_t|_{t=0} = \mu_0$, see (4.18). If $\mu_0(\varnothing) = 1$, i.e., the initial state is an empty habitat, by (4.23) it follows that $\mu_t = \pi_{\varrho_t}$ with ϱ_t given in (4.22). Let us show that this μ_t satisfies (2.3). For π_ϱ, by the following easy to prove formula, cf. (3.4),

$$\int_{\hat{\Gamma}_0} \sum_{\xi \subset \eta} G(\xi, \eta \setminus \xi) \lambda(d\xi) = \int_{\hat{\Gamma}_0} \int_{\hat{\Gamma}_0} G(\xi, \eta) \lambda(d\hat{\xi}) \lambda(d\hat{\eta}),$$

we have that

$$\begin{aligned} \pi_\varrho(H_1) &= \int_{\hat{\Gamma}_0} \Big(\prod_{x\in\eta} \hat{\varrho}(\hat{x}) \Big) \sum_{x\in\eta} \Big[\frac{\partial}{\partial\alpha} - m(x,\alpha) \Big] \theta(\hat{x}) \prod_{y \in \eta \setminus x} \theta(\hat{y}) \lambda(d\hat{\eta}) \qquad (4.24) \\ &= \int_{\hat{\Gamma}_0} \Big(\prod_{x\in\eta} \rho(\hat{x}) \Big) \int_{\hat{X}} \rho(\hat{x}) \Big[\frac{\partial}{\partial\alpha} - m(x,\alpha) \Big] \theta(\hat{x}) \prod_{y\in\eta} \theta(\hat{y}) \lambda(d\hat{\eta}) \\ &= -\Big(\int_X \rho(x,0)\theta(x,0) dx \Big) \pi_\rho(F^\theta) \\ &- \Big(\int_{\hat{X}} \theta(\hat{x}) \Big[\frac{\partial}{\partial\alpha} + m(x,\alpha) \Big] \rho(\hat{x}) dx \Big) \pi_\varrho(F^\theta). \end{aligned}$$

And also

$$\pi_\varrho(H_2) = \Big(\int_X b(x)\theta(x,0) dx \Big) \pi_\varrho(F^\theta). \qquad (4.25)$$

In the sense of distributions, we have that

$$\frac{\partial}{\partial\alpha} \mathbf{1}_{[0,t)}(\alpha) = -\frac{\partial}{\partial t} \mathbf{1}_{[0,t)}(\alpha).$$

Then for $\rho_t(x,\alpha) = \hat{b}(x,\alpha)\mathbf{1}_{[0,t)}(\alpha)$, see (4.22), one obtains

$$\Big[\frac{\partial}{\partial\alpha} + m(x,\alpha) \Big] \rho_t(x,\alpha) = -\frac{\partial}{\partial t} \rho_t(x,\alpha). \qquad (4.26)$$

By (4.24), (4.25) and the latter equality it follows that

$$\begin{aligned} \pi_{\varrho_t}(LF^\theta) &= \exp\Big(\int_{\hat{X}} \rho_t(\hat{x})\theta(\hat{x}) d\hat{x} \Big) \frac{\partial}{\partial t} \int_{\hat{X}} \rho_t(\hat{x})\theta(\hat{x}) d\hat{x} \qquad (4.27) \\ &= \frac{\partial}{\partial t} \exp\Big(\int_{\hat{X}} \rho_t(\hat{x})\theta(\hat{x}) d\hat{x} \Big), \end{aligned}$$

by which one readily concludes that $\mu_t = \pi_{\varrho_t}$ satisfies (2.3).

4.2.3 The proof

The proof of Theorem 4.2.2 is divided into the following steps

(i) proving that for each $\theta \in \Theta$, the map $t \mapsto \mu_t(F^\theta)$ has a continuous derivative at each $t > 0$ (by Lebesgue's dominated convergence theorem);

(ii) proving that this derivative satisfies (4.28), see below;

(iii) showing that π_ϱ is a stationary state;

(iv) proving the weak convergence of probability measures on $\hat{\Gamma}_*$

(v) showing that the family $(\mu_t)_{t\geq 0}$ is tight (by Prohorov's theorem).

The proof of the first part will be done by showing that:

(a) for each $\theta \in \Theta$, the map $t \mapsto \mu_t(F^\theta)$ has a continuous derivative at each $t > 0$;

(b) this derivative satisfies, cf. (4.27),

$$\frac{d}{dt}\mu_t(F^\theta) = \mu_t(LF^\theta). \tag{4.28}$$

By (4.11), (4.12), (4.21), (4.22) and (4.23) we have

$$\mu_t(F^\theta) = \mu_0(F^{\theta_t})\pi_{\varrho_t}(F^\theta) =: \mu_0(F^{\theta_t})Q_\theta(t). \tag{4.29}$$

In view of (4.27), the continuous differentiability in question will thus follow by the same property of $t \mapsto \mu_0(F^{\theta_t})$. By (4.18) we have

$$\begin{aligned} \frac{\partial}{\partial t}F^{\theta_t}(\hat{\gamma}) &= \sum_{x\in p(\hat{\gamma})}\Big(\frac{\partial}{\partial \alpha}\theta_t(\hat{x})\Big)F^{\theta_t}(\hat{\gamma}\setminus\hat{x}) - \sum_{x\in p(\hat{\gamma})} m(x,\alpha)\theta_t(\hat{x})F^{\theta_t}(\hat{\gamma}\setminus\hat{x}) \\ &=: \sum_{x\in p(\hat{\gamma})}\sigma_t(\hat{x})F^{\theta_t}(\hat{\gamma}\ \hat{x}) =: S_t(\hat{\gamma}). \end{aligned} \tag{4.30}$$

Similarly as in (4.17) we then conclude that $\left|\frac{\partial}{\partial t}F^{\theta_t}(\hat{\gamma})\right| \leq D'_\theta\Psi(\hat{\gamma})$, with a certain $D'_\theta > 0$. By Lebesgue's dominated convergence theorem this yields

$$\begin{aligned} \frac{d}{dt}\mu_0(F^{\theta_t}) &= \mu_0\Big(\frac{\partial}{\partial t}F^{\theta_t}\Big) = \mu_0(S_t) \qquad (4.31) \\ &= \int_{\hat{\Gamma}_0} k_{\mu_0}(\hat{\eta})\sum_{x\in p(\hat{\eta})}\Big(\sigma_t(\hat{x})\prod_{y\in p(\hat{\eta}\setminus\hat{x})}\theta_t(\hat{y})\Big)\lambda(d\hat{\eta}), \end{aligned}$$

as well as the continuity of the map $t \mapsto \mu_0(\frac{\partial}{\partial t}F^{\theta_t})$. Here k_{μ_0} is the correlation function of μ_0 understood in the sense of (4.10). Now let us turn to proving (4.28). By (4.29) and (4.31) we have

$$\mathrm{LHS}(4.28) = \mu_0(S_t)Q_\theta(t) + \mu_t(F^\theta)\int_X b(x)\theta(x,t)\exp\Big(-\int_0^t m(x,\varsigma)d\varsigma\Big)dx. \quad (4.32)$$

At the same time, by (4.15) it follows that

$$H_1(\hat\gamma_1 \cup \hat\gamma_2) = H_1(\hat\gamma_1)F^\theta(\hat\gamma_2) + H_1(\hat\gamma_2)F^\theta(\hat\gamma_1),$$

which by (4.11) and (4.21) yields

$$\mathrm{RHS}(3.17) = \mu_0^t(H_1)\pi_{\varrho_t}(F^\theta) + \mu_0^t(F^\theta)\pi_{\varrho_t}(H_1) + \mu_t(F^\theta)\int_X b(x)\vartheta(x)dx, \quad (4.33)$$

Note that

$$\pi_{\varrho_t}(F^\theta) = \exp\Big(\int_{\hat X}\rho_t(\hat x)\theta(\hat x)d\hat x\Big) = Q_\theta(t), \quad (4.34)$$

see (4.8), (4.22) and (4.29). By (4.15) we have that

$$H_1(\hat\gamma) = \sum_{x\in\gamma} h_\theta(\hat x)F^\theta(\hat\gamma\setminus\hat x),$$

$$h_\theta(x,\alpha) := \frac{\partial}{\partial\alpha}\theta(x,\alpha) - m(x,\alpha)\theta(x,\alpha).$$

By (4.20) one then gets

$$\begin{aligned}
\mu_0^t(H_1) &= \int_{\hat\Gamma_0} k_{\mu_0^t}(\hat\eta)\Big(\sum_{x\in\eta} h_\theta(\hat x)\prod_{y\in\eta\setminus x}\theta(\hat y)\Big)\lambda(d\hat\eta) \quad (4.35)\\
&= \int_{\hat\Gamma_0}\Big(\int_{\hat X} k_{\mu_0^t}(\hat\eta\cup\hat x)h_\theta(\hat x)d\hat x\Big)\prod_{y\in\eta}\theta(\hat y)\lambda(d\hat\eta)\\
&= \int_{\hat\Gamma_0}\Big(\int_X\int_t^{+\infty} k_{\mu_0}(\hat\eta^t\cup(x,\alpha-t))\\
&\times \exp\Big(-\int_{\alpha-t}^{\alpha} m(x,\varsigma)d\varsigma\Big)h_\theta(x,\alpha)dxd\alpha\Big)\\
&\times \mathcal{J}_t(\hat\eta)\prod_{y\in\eta}\theta(y,\alpha_y)\exp\Big(-\int_{\alpha_y-t}^{\alpha_y} m(y,\varsigma)d\varsigma\Big)h_\theta(y,\alpha)dxd\alpha\Big)\\
&= \int_{\hat\Gamma_0} K_t(\hat\eta)\prod_{y\in\eta}\theta_t(\hat y)\lambda(d\hat\eta).
\end{aligned}$$

Here $\hat{\eta}^t$ and $\mathcal{J}_t$ are as in (4.20) and θ_t is defined in (4.18), whereas

$$\begin{aligned} K_t(\hat{\eta}) &:= \int_X \int_t^{+\infty} k_{\mu_0}(\hat{\eta}^t \cup (x, \alpha - t)) h_\theta(x,\alpha) \exp\Big(- \int_{\alpha-t}^{\alpha} m(x,\varsigma)d\varsigma\Big) dx d\alpha \\ &= \int_X \int_t^{+\infty} k_{\mu_0}(\hat{\eta}^t \cup (x, \alpha)) h_\theta(x,\alpha+t) \exp\Big(- \int_{\alpha}^{\alpha+t} m(x,\varsigma)d\varsigma\Big) dx d\alpha \end{aligned}$$

By (4.18) and (4.30) we have

$$h_\theta(x,\alpha+t) \exp\Big(- \int_{\alpha}^{\alpha+t} m(x,\varsigma)d\varsigma\Big) = \sigma_t(x,\alpha).$$

We use this in the latter expression and then in (4.35); thus, we arrive at the following

$$\mu_0^t(H_1) = \int_{\hat{\Gamma}_0} k_{\mu_0}(\hat{\eta}) \Big(\prod_{x\in\eta} \sigma_t(\hat{x}) \prod_{y \in \eta\setminus x} \theta_t(\hat{y}) \Big) \lambda(d\hat{\eta}) = \mu_0(S_t), \tag{4.36}$$

see (4.31). Now similarly as in (4.24) we obtain

$$\begin{aligned} \pi_{\varrho_t}(H_1) &= \Big(\int_{\hat{X}} h_\theta(\hat{x}) \varrho_t(d\hat{x}) \Big) \pi_{\varrho_t}(F^\theta) \qquad (4.37) \\ &= \Big(\int_{\hat{X}} \int_0^t b(x) \exp\Big(- \int_0^\alpha m(x,\varsigma)d\varsigma \Big) \\ &\times \Big[\frac{\partial}{\partial \alpha} - m(x,\alpha) \Big] \theta(x,\alpha) dx d\alpha \Big) \pi_{\varrho_t}(F^\theta) \\ &= \Big(\int_{\hat{X}} b(x) \Big[\exp\Big(- \int_0^t m(x,\varsigma)d\varsigma \Big) \theta(x,t) - \theta(x,0) \Big] dx \Big) \pi_{\varrho_t}(F^\theta). \end{aligned}$$

Finally, we use (4.36) and (4.37) in (4.33), take into account (4.34) and (4.32), and conclude that (4.28) holds true.

To prove that π_ϱ is a stationary solution of (2.3) we again use (4.15) and (4.24). For

$$\hat{b}(x,\alpha) = b(x) \exp\Big(- \int_0^\alpha m(x,\vartheta)d\vartheta \Big),$$

we have, cf. (4.26),

$$\Big[\frac{\partial}{\partial \alpha} + m(x,\alpha) \Big] \hat{b}(x,\alpha) = 0,$$

which by (4.24) yields $\pi_\varrho(LF^\theta) = 0$, and hence the property in question.

Chapter 5

The independent appearance model: a Markov process

In this chapter we study the model defined in (4.1) with X being just a locally compact Polish space. In this case, we do not have the property as in Proposition 2.2.2. That is why, we will deal with spaces of multiple configurations and then understand the corresponding sums as in (2.8). Here we also will not use tempered configurations and thus special classes of measures. The presentation of the material is close to our paper [20] and the notations used here are mostly independent of the previous part of the thesis.

Since the configurations here are multiple, we take this into account and instead of (4.1) we write the following formula for the corresponding Kolmogorov operator

$$\begin{aligned}(LF)(\hat{\gamma}) &= \sum_{(x,\alpha)\in\hat{\gamma}} \frac{\partial}{\partial\alpha} F(\hat{\gamma}) + \sum_{(x,\alpha)\in\hat{\gamma}} m(x,\alpha)\left[F(\hat{\gamma}\setminus(x,\alpha)) - F(\hat{\gamma})\right] \\ &+ \int_X \left[F(\hat{\gamma}\cup(x,0)) - F(\hat{\gamma})\right]\chi(dx).\end{aligned} \tag{5.1}$$

The model parameters are subject to the following assumptions:

(i) The departure rate $X \times \mathbb{R}_+ \ni (x,\alpha) \to m(x,\alpha) \in \mathbb{R}_+$ is continuous and bounded, i.e., such that $m(x,\alpha) \le m_*$ for some $m_* > 0$ and all (x,α). Moreover, there exits $\kappa : [0,1] \to \mathbb{R}_+$ such that $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$ and the following holds

$$\forall_{x\in X} |m(x,\alpha) - m(x,\alpha')| \le \kappa(|\alpha-\alpha'|), \qquad |\alpha-\alpha'| \in (0,1). \tag{5.2}$$

(ii) The arriving measure χ is just a positive Radon measure.

The result of the present chapter can be outlined as follows. We introduce a Banach space $\mathcal{C}$ of bounded continuous functions $F : \hat{\Gamma} \to \mathbb{R}$, in which we define L as a closed and densely defined linear operator that satisfies the conditions of the Hille-Yosida theorem, and hence is the generator of a C_0−semigroup $\{S(t)\}_{t\geq 0}$. Then the solution of (2.4) is obtained in the form $F_t = S(t)F_0$. For a class of functions $\mathcal{F}_\Theta$, F_t corresponding to $F_0 \in \mathcal{F}_\Theta$ is obtained in an explicit way. This allows us to explicitly construct the corresponding Markov transition function $p_t^{\hat{\gamma}}$ and obtain finite-dimensional laws of a Markov process X with values in $\hat{\Gamma}$, which describes the stochastic evolution of our model. Possible objects of this kind are specified as stochastic processes that solve the martingale problem for L. Then we show that this problem is well-posed, i.e., uniqueness holds. The main ingredient of the proof here is showing that the corresponding Fokker-Planck equation for L has a unique solution, which we do by employing the resolvent of L. Assuming that $m(x, \alpha) \geq m_0 > 0$, we also show that the process X has a unique stationary state, explicitly constructed in the paper, such that the laws of $\mathcal{X}(t)$ weakly converge to this state as $t \to +\infty$.

5.1 The space of marks and the metric

The space $\hat{X}$ is equipped with the product topology assuming that the topology of $\mathbb{R}_+$ be defined by the metric which we introduce now.
For $\alpha \geq 0$, we set

$$\omega(\alpha) = \min\left\{\alpha; \frac{1}{\alpha}\right\},$$

and then

$$\begin{aligned} r(\alpha, 0) &= \omega(\alpha), \\ r(\alpha, \alpha_0) &= \min\{|\alpha - \alpha_0|; \omega(\alpha) + \omega(\alpha_0)|\}, \end{aligned} \tag{5.3}$$

where $|\beta|$ is the usual absolute value of $\beta \in \mathbb{R}$.

Proposition 5.1.1. *The above introduced r is a metric such that $(\mathbb{R}_+, r)$ be a compact metric space.*

Proof. To prove the first part we just have to check the validity of the triangle inequality

$$r(\alpha_1, \alpha_2) \leq r(\alpha_1, \alpha_3) + r(\alpha_2, \alpha_3). \tag{5.4}$$

This technical exercise is made in Appendix. To prove the compactness, we have to show that:

(a) r is complete;

(b) the space $(\mathbb{R}_+, r)$ is totally bounded.

Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an r-Cauchy sequence.
Here one may have the following possibilities:

(i) there exists $\alpha < \infty$ such that $\alpha_n \leq \bar{\alpha}$, such that $n \in \mathbb{N}$;

(ii) the considered sequence contains a subsequence that diverges in the usual sense.

In case (i), $\{\alpha_n\}_{n\in\mathbb{N}}$ contains a subsequence, say $\{\alpha_{nk}\}_{n\in\mathbb{N}}$, such that $|\alpha_{nk}-\alpha_*| \to 0$ as $k \to +\infty$ for some $\alpha_* \leq \bar{\alpha}$. At the same time, for $\varepsilon < \frac{2}{\bar{\alpha}}$, $r(\alpha_n, \alpha_m) < \varepsilon$ implies $|\alpha_n - \alpha_m| < \varepsilon$, see (5.3), which means that $r(\alpha_n, \alpha_*) \to 0$ as $n \to +\infty$.
In case (ii), the divergent subsequence converges in r to zero, which implies that the whole sequence converges to zero in r. Hence, the latter metric is complete. To prove (b), we set $\tilde{B}_\varepsilon(\alpha) = \{\alpha' \in \mathbb{R}+ : r(\alpha, \alpha') < \varepsilon\}$. Fix $\varepsilon \in (0, 1)$ and take the least $k \in \mathbb{N}$ such that $k+1 > \frac{1}{\varepsilon^2}$. Then $\mathbb{R}_+ = \bigcup_{j=0}^{k} \tilde{B}_\varepsilon(j\varepsilon)$, which yields the property in question. □

Let us now compare r with the absolute-value metrics of $\mathbb{R}_+$. By $C(\mathbb{R}_+, \mathcal{T}_r)$ we will mean the set of all bounded r-continuous functions, whereas $C_b(\mathbb{R}_+, \mathcal{T}_{|\cdot|})$ is going to stand for the set of all bounded $|\cdot|$-continuous functions.

Proposition 5.1.2. *$\mathcal{T}_r$ is coarser than $\mathcal{T}_{|\cdot|}$, and hence the embedding $(\mathbb{R}_+, \mathcal{T}_{|\cdot|}) \hookrightarrow (\mathbb{R}_+, \mathcal{T}_r)$ is continuous, whereas both latter topological spaces are Borel isomorphic. Moreover,*

$$C(\mathbb{R}_+, \mathcal{T}_r) = \{u \in C_b(\mathbb{R}_+, \mathcal{T}_{|\cdot|}) : \lim_{\alpha\to+\infty} u(\alpha) = u(0)\}. \tag{5.5}$$

Proof. The validity of the first statement and (5.5) readily follows by the fact that each $|\cdot|$-convergent sequence is also r-convergent, and each r-convergent sequence either converges in $|\cdot|$ to the same limit $\alpha \neq 0$, or has two $|\cdot|$-accumulation points: 0 and ∞. Since the mentioned embedding is continuous and injective, it is also Borel-measurable. By Kuratowski's theorem, see Parthasarathy (1967), page 21, [31], its inverse is also measurable and thus is the isomorphism in question. This, in particular, means that the corresponding Borel σ-fields coincide. □

Definition 5.1.3. *For a suitable $u \in C(\mathbb{R}_+, \mathcal{T}_r)$, we introduce the map $\alpha \to u'(\alpha)$, meaning the usual derivative for $\alpha > 0$ and the right-hand side one if $\alpha = 0$. A given u is said to be continuously differentiable on $\mathbb{R}_+$ if $u' \in C(\mathbb{R}_+, \mathcal{T}_r)$.*

Let us consider the following functions

$$u_n(\alpha) = \frac{\alpha^2}{1 + n\alpha^3}, \quad \alpha \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{5.6}$$

It is clear that:

(a) each u_n is continuously differentiable, see Definition 5.1.3;

(b) u_n is decreasing for $\alpha > \sqrt[3]{\dfrac{2}{n}}$. Moreover, $u_n(\alpha) \leq \dfrac{\sqrt[3]{4}}{3\sqrt[3]{n^2}}$ and

$$u_n'(\alpha) = \frac{2\alpha - n\alpha^4}{(1+n\alpha^3)^2}, \quad |u_n'(\alpha)| \leq \frac{c}{\sqrt[3]{n}}, \tag{5.7}$$

the latter holding for some $c > 0$ and all $\alpha \geq 0$.

Now let $\{\sigma_k\}_{k\in\mathbb{N}} =: \Sigma \subset [0, +\infty)$ be countable and such that:

(i) $\sigma_1 = 0$;

(ii) $\forall_{k\in\mathbb{N}} \quad \sigma_k < \sigma_{k+1}; \quad \sigma_n \to \bar{\sigma} < \infty$ as $n \to +\infty$.

Next, for $k, n \in \mathbb{N}$, we set

$$w_{n,k}(\alpha) = e^{-\sigma_k u_n(\alpha)}. \tag{5.8}$$

Then $w_{k,n}$ is continuously differentiable and the following holds

$$|w_{n,k}'(\alpha)| = \frac{\bar{\alpha}c}{\sqrt[3]{n}} w_{n,k}(\alpha), \tag{5.9}$$

where $\mathcal{C}$ is the same as in (5.7).
Next, let a be a finite collection of points $\alpha_l \in \mathbb{R}_+$. That is, $a = \{\alpha_l\}_{1\leq l\leq m}, \alpha_l \leq \alpha_{l+1}$ for all l. For $\alpha \in a$, by $n_a(\alpha) \in \mathbb{N}$ we will denote the multiplicity of α in a, i.e., the number of elements of a coinciding with this α. We extend it to all $\alpha \geq 0$ by setting $n_a(\alpha) = 0$ whenever α is not in a. Two such a and a' are equal if they consist of exactly the same elements, with the same multiplicities.

Proposition 5.1.4. *Let a and a' be as just described. Then they are equal if*

$$\sum_{\alpha\in a} w_{k,n}(\alpha) = \sum_{\alpha\in a'} w_{n,k}(\alpha), \tag{5.10}$$

holding for all $k, n \in \mathbb{N}$.

Proof. For a as above and $\zeta \in \mathbb{C}$, consider

$$f_{n,a}(\zeta) = \sum_{\alpha\in a} e^{-\zeta u_n(\alpha)}, n \in \mathbb{N}.$$

Each such f is an exponential type entire function. By (5.8) and (5.10) we have that $(f_{n,a} - f_{n,a'})|_\Sigma = 0$, holding for all $n \in \mathbb{N}$. Since Σ has a limiting point, this implies $f_{n,a}(\zeta) = f_{n,a'}(\zeta)$ for all $\zeta \in \mathbb{R}$ and $n \in \mathbb{N}$. Obviously, $\lim_{\zeta\to+\infty} f_{n,a}(\zeta) = n_a(0)$, where $n_a(0) \geq 0$ is the multiplicity of $\alpha = 0$ in a. Then the just mentioned equality yields $n_a(0) = n_{a'}(0)$ and also

$$\sum_{\alpha\in a\setminus\{0\}} e^{-\zeta u_n(\alpha)} = \sum_{\alpha\in a'\setminus\{0\}} e^{-\zeta u_n(\alpha)}. \tag{5.11}$$

Let α_* and α'_* be the least positive elements of a and a', respectively. Take $n > \frac{2}{\alpha_-^3}$, $\alpha_- := \min\{\alpha_*; \alpha'_*\}$. Then, for such n and all $\alpha > \alpha_-$, one has $u_n(\alpha_-) > u_n(\alpha)$. Now we multiply both sides of (5.11) by $e^{\zeta u_n(\alpha_-)}$ and pass to the limit $\zeta \to -\infty$. This yields that $\alpha_* = \alpha'_*$ and $n_a(\alpha_*) = n_{a'}(\alpha'_*)$. Thereafter, we subtract the coinciding terms from both sides of (5.11) and proceed to comparing the remaining least elements of a and a'. This eventually yields the equality to be proved. □

Let $\mathcal{A}$ be the set of all $a = \{\alpha_l\}_{1\leq l\leq m}$, $m \in \mathbb{N}_0$, $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \alpha_m$. Define

$$\rho(a, a') = \sum_{k,n\in\mathbb{N}} \frac{2^{-k-n}\rho_{k,n}(a, a')}{1 + \rho_{k,n}(a, a')}, \tag{5.12}$$

$$\rho_{k,n}(a, a') := \Big| \sum_{\alpha\in a} w_{k,n}(\alpha) - \sum_{\alpha\in a'} w_{k,n}(\alpha) \Big|.$$

By Proposition 5.1.4 it follows that ρ is a metric on $\mathcal{A}$. Each $a \in \mathcal{A}$ can be considered as a finite counting measure defined on the compact space $(\mathbb{R}_+, r)$, for which $a(\Delta) = \sum_{\alpha\in a} \mathbf{1}_\Delta(a) = |a \cup \Delta|$, holding for all Borel subsets Δ. The weak topology of A is defined as the coarsest topology that makes continuous all the maps $a \to \sum_{\alpha\in a} w(a), w \in C(\mathbb{R}_+, \mathcal{T}_r)$. In the weak topology, A is a closed subset of the space of all finite positive measures on $C(\mathbb{R}_+, \mathcal{T}_r)$.

Proposition 5.1.5. $(\mathcal{A}, \rho)$ *is a complete metric space. The corresponding metric topology coincides with the weak topology that turns A into a locally compact Polish space.*

Proof. As each $w_{k,n}$ is in $C(\mathbb{R}_+, \mathcal{T}_r)$, the weak convergence of a sequence $\{a_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$ to a certain $a \in \mathcal{A}$ yields $\rho(a, a_m) \to 0, m \to +\infty$. Assume now that $\{a_m\}_{m\in\mathbb{N}}$ is a ρ-Cauchy sequence. By taking $\sigma = 0$ we then get from the latter that, for some $m_* \in \mathbb{N}$, the cardinalities of all $a_m, m > m_*$, coincide. By Prohorov's theorem this yields that $\{a_m\}_{m\in\mathbb{N}}$ contains a subsequence that weakly converges to some a. Hence, the whole sequence converges in ρ to this a. Then the metric is complete and the corresponding metric topology is exactly the weak topology of $\mathcal{A}$. The separability and local compactness follow by the fact that $C(\mathbb{R}_+, \mathcal{T}_r)$ is compact. □

Let Γ be the set of all locally finite simple configurations on X. That is, each $\gamma \in \Gamma$ is a subset of X such that each compact $\Lambda \subset X$ contains a finite number of the elements of γ. Let now $\breve{\gamma}$ be the pair $(\gamma, n), \gamma \in \Gamma$ and $n : \gamma \to \mathbb{N}$. The value of n at a given $x \in \gamma$ can be considered as the multiplicity of $x \in \breve{\gamma}$. That is, $\breve{\gamma}$ is a configuration with multiple locations, for which γ is the ground configuration. Sometimes, we will write $n_{\breve{\gamma}}(x)$ to explicitly indicate that we mean the multiplicity of x in the mentioned $\breve{\gamma}$. By $\breve{\Gamma}$ we denote the set of all such multiple configurations. For $\breve{\gamma} = (\gamma, n)$, we write $\gamma = p(\breve{\gamma})$. The weak-hash, vague, topology of $\breve{\Gamma}$ is defined as the coarsest topology that makes continuous all the maps

$$\breve{\gamma} \to \sum_{x\in p(\breve{\gamma})} n(x)g(x), \qquad g \in C_{cs}(X).$$

It is well-known, see [34], Lemma 1.2, that with this topology $\breve{\Gamma}$ is a Polish space, whereas $\hat{\Gamma}$ is a G_δ subset of $\breve{\Gamma}$, by which it is also Polish. Following Lenard [27] we will also consider $\breve{\gamma}$ as configurations of point particles, in which distinct particles may have the same location. Such particles can be enumerated, which allows one to write

$$\sum_{x\in p(\breve{\gamma})} n(x)g(x) = \sum_{x\in\breve{\gamma}} g(x), \tag{5.13}$$

where in the second sum we mean a certain enumeration of this sort. In the same sense, we will write

$$\sum_{x\in\breve{\gamma}} \sum_{x\in\breve{\gamma}\setminus x_1} \cdots \sum_{x\in\breve{\gamma}\setminus\{x_1,x_2,\ldots,x_{m-1}\}} g(x_1,\ldots,x_m), \quad m\in\mathbb{N},$$

where in expressions like $\breve{\gamma}\setminus x$ we treat x the singleton $\{x\}$, whereas $\breve{\gamma}\setminus x$ is the measure such that $n_{\breve{\gamma}\setminus x}(x) = n_{\breve{\gamma}}(x) - 1$ and $n_{\breve{\gamma}\setminus x}(y) = n_{\breve{\gamma}}(y)$ for $y \neq x$.

It is known, see, e.g., Zessin (1983), page 397, that there exists a collection $\{v_s\}_{s\in\mathbb{N}} =: \mathcal{V} \subset\subset C^+_{cs}(X)$ of suitable functions such that the metric

$$d(\breve{\gamma},\breve{\gamma}) = \sum_{b\in\mathbb{N}} \frac{2^{-s} d_s(\breve{\gamma},\breve{\gamma}')}{1+d_s(\breve{\gamma},\breve{\gamma}')}, \qquad d_s(\breve{\gamma},\breve{\gamma}') := \Big|\sum_{x\in\breve{\gamma}} v_s(x) - \sum_{x\in\breve{\gamma}'} v_s(x)\Big|. \tag{5.14}$$

is complete and consistent with the weak-hash topology of $\breve{\Gamma}$. In the sequel, we will always mean this topology of $\breve{\Gamma}$. Obviously, we can and will assume that $\mathcal{V}$ contains also the following functions.

Let δ be a complete metric of X and X' a countable dense subset of X. Each $x' \in X'$ has a countable base of compact neighborhoods, which we denote by $D(x')$. Each $\Delta \in D(x')$ contains balls

$$B_q(x') = \{x \in X : \delta(x,x') < q\}$$

with compact closures, where q is a rational number satisfying $q \leq q'$ for a δ-specific $q' \in Q$. For $x' \in X'$, $\delta \in D(x')$, $q \leq q'$ and $\varsigma \in (0,1)\cap Q$, let $v \in C^+_{cs}(X)$ be such that:

(a) $v(x) \equiv \varsigma$ for $x \in B_q(x')$;

(b) $v(x) = 0$ for $x \in X \setminus \Delta$.

The countable set of all such functions is supposed to be a part of $\mathcal{V}$, and hence they are taken into account in (5.14). Since each vs has compact support, for each compact $\lambda \subset X$ and any two configurations, $d_s(\breve{\gamma}\cap\Lambda, \breve{\gamma}'\cap\Lambda) > 0$ only for finitely many s. Here $\breve{\gamma}\cap\Lambda := (p(\breve{\gamma})\cap\Lambda, n)$.

For $\gamma \in \Gamma$, let $a : \gamma \to A$ be a map, for which we denote

$$|a(x)| = \sum_{\alpha\in a(x)} n_{a(x)}(\alpha). \tag{5.15}$$

Then the pair $\hat{\gamma} = (\gamma, a)$ is a marked configuration whose ground configuration is γ and the mark map is a. By writing $\hat{x} = (x, \alpha) \in \hat{\gamma}$ we will mean that $x \in \gamma$ and $\alpha \in a(x)$. The configuration of marks $a(x) = \{\alpha_1, \dots \alpha_{|a(x)|}\}$ yields the ages of the particles located at $x \in \gamma$, whereas $|a(x)|$ is the total number of such particles. In some cases, we write $a_{\hat{\gamma}}$ to indicate that a is defined on a given $\hat{\gamma}$. Let $\hat{\Gamma}$ denote the set of all marked configurations $\hat{\gamma}$. Let also $\breve{p} : \hat{\Gamma} \to \breve{\Gamma}$ be the map such that $\breve{p}(\gamma, a) = (\gamma, |a|)$, where $|a|(x) = |a(x)|$ see (5.15). Then $p \circ \breve{p}$ maps $\hat{\gamma} = (\gamma, a)$ into its ground configuration γ. For brevity, by writing $p(\hat{\gamma})$ we will mean $(p \circ \breve{p})(\hat{\gamma})$. Our aim now is to equip $\hat{\Gamma}$ with a complete metric. Define

$$\kappa(\hat{\gamma}, \hat{\gamma}') = \sum_{s,k,n \in \mathbb{N}} \frac{2^{-(s+k+n)} \kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}')}{1 + \kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}')}, \tag{5.16}$$

$$\kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}') := \Big| \sum_{x \in p(\hat{\gamma})} v_s(x) \sum_{\alpha \in a_{\hat{\gamma}(x)}} w_{k,n}(\alpha) - \sum_{x \in p(\hat{\gamma}')} v_s(x) \sum_{\alpha \in a_{\hat{\gamma}'(x)}} w_{k,n}(\alpha) \Big|.$$

Note that the latter can also be written as, cf. (5.13),

$$\kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}') = \Big| \sum_{x \in \hat{\gamma}} g_{s,k,n}(\hat{x}) - \sum_{x \in \hat{\gamma}'} g_{s,k,n}(\hat{x}) \Big|. \tag{5.17}$$

$$g_{s,k,n}(x, \alpha) := v_s(x) w_{k,n(\alpha)}.$$

For a compact $\Lambda \subset X$, we write $\hat{\gamma} \cap \Lambda = (p(\hat{\gamma}) \cap \Lambda, a)$, where a is the restriction of a from $p(\hat{\gamma})$ to $p(\hat{\gamma}) \cap \Lambda$.

Proposition 5.1.6. *For each $\varepsilon > 0$, one may find a compact $\Lambda_\varepsilon \subset X$ such that, for any two configurations, the following holds*

$$|\kappa(\hat{\gamma}, \hat{\gamma}') - \kappa(\hat{\gamma} \cap \Lambda_\varepsilon, \hat{\gamma}' \cap \Lambda_\varepsilon)| < \varepsilon. \tag{5.18}$$

Proof. Fix $\varepsilon > 0$ and then pick $s_* \in \mathbb{N}$ such that $2^{s_*} > \frac{1}{\varepsilon}$. Now let Λ_ε be covered by the supports of v_s with $s \leq s_*$. For such s and all $k, n \in \mathbb{N}$, we have $\kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}') = \kappa_{s,k,n}(\hat{\gamma} \cap \Lambda_\varepsilon, \hat{\gamma}' \cap \Lambda_\varepsilon)$, see (5.16). This clearly yields (5.18). □

Since $\sigma = 0$ is in Σ, by (5.8) and (5.14) we have that

$$d(\breve{p}(\hat{\gamma}), \breve{p}(\hat{\gamma}')) \leq \kappa(\hat{\gamma}, \hat{\gamma}') \tag{5.19}$$

Proposition 5.1.7. *The metric space $(\hat{\Gamma}, \kappa)$ is complete.*

Proof. We begin by pointing out the following evident fact

$$\tilde{\kappa}_s(\hat{\gamma}, \hat{\gamma}') := \sum_{k,n \in \mathbb{N}} \frac{2^{-(s+k+n)} \kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}')}{1 + \kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}')} \leq \kappa(\hat{\gamma}, \hat{\gamma}'), \tag{5.20}$$

holding for all $s \in \mathbb{N}$ and $\hat{\gamma}, \hat{\gamma}'$. Let now $\{\hat{\gamma}_m = (\gamma_m, a_m)\}_{m\in\mathbb{N}} \subset \hat{\Gamma}$ be a κ-Cauchy sequence. By (5.19) the sequence $\{\breve{p}(\hat{\gamma}_m)\}_{m\in\mathbb{N}} \subset \breve{\Gamma}$ converges to some $\breve{\gamma}$. Take now $x \in p(\breve{\gamma})$ and then pick a compact $\Delta \subset X$ such that $\Delta \cap p(\breve{\gamma}) = \{x\}$. For this Δ, we then set

$$n_m(x) = \sum_{y\in p(\breve{\gamma}_m)\cap\Delta} n_{\breve{\gamma}\cap\Lambda}(y), \qquad m \in \mathbb{N}.$$

From the convergence of $\{\breve{p}(\hat{\gamma}_m)\}_{m\in\mathbb{N}}$ to $\breve{\gamma}$, it follows that $n_m(x) \to n(x)$; hence, there exists $m_* \in \mathbb{N}$ such that $n_m(x) = n(x)$ for all $m > m_*$. Now we pick $x' \in X'$ and $q \in Q$ such that $x \in B_{\frac{q}{2}}(x')$ and the closure of $B_q(x')$ lies in Δ. Let now $v_s \in V$ be such that $v_s(y) = \varsigma \in (0,1) \cap Q$, $y \in B_{\frac{q}{2}}(x')$, and $v_s(y) = 0$ for $y \in X \setminus B_q(x')$. For these m_* and s, $\{\hat{\gamma}_m\}_{m\geq m_*+1}$ is also a $\tilde{\kappa}_s$-Cauchy sequence, see (5.20), for which we have

$$\kappa_{s,k,n}(\hat{\gamma}_m, \hat{\gamma}_{m+l}) = \varsigma \Big| \sum_{y\in p(\hat{\gamma}_m)\cap\Delta} \sum_{\alpha\in a_{\hat{\gamma}_m(y)}} w_{\sigma_k,n}(\alpha) - \sum_{y\in p(\hat{\gamma}_{m+l})\cap\Delta} \sum_{\alpha\in a_{\hat{\gamma}_{m+l}(y)}} w_{\sigma_k,n}(\alpha) \Big|,$$

holding for all $k, n \in \mathbb{N}$, $m > m_*$ and $l \in \mathbb{N}$.
Let us enumerate $\hat{x} = (x, \alpha) \in \hat{\gamma}_m \cap \Delta$ in such a way that $\alpha_{p,m} \leq \alpha_{p+1,m}$ for all p. This yields $\hat{\gamma}_m \cap \Delta = \{(x_{1,m}, \alpha_{1,m}), \ldots, (x_{n,m}, \alpha_{n,m})\}$ with $n = n_m(x) = n(x)$. Similarly, we have

$$\hat{\gamma}_{m+l} \cap \Delta = \{(x_{1,m+l}, \alpha_{1,m+l}), \ldots, (x_{n,m+l}, \alpha_{n,m+l})\}$$

with the same n. Then $\{\alpha_{1,m}, ...\alpha_{n,m}\} =: a_m \in \mathcal{A}$, and also $\{\alpha_{1,m+l}, ...\alpha_{n,m+l}\} =: a_{m+l} \in \mathcal{A}$, and the latter equality can be rewritten as follows

$$\kappa_{s,k,n}(\hat{\gamma}_m, \hat{\gamma}_{m+l}) = \varsigma \Big| \sum_{p=1}^{n} w_{\sigma_k,n}(\alpha_{p,m}) - \sum_{p=1}^{n} w_{\sigma_k,n}(\alpha_{p,m+l}) \Big| = \varsigma \rho_{k,n}(a_m, a_{m+l}),$$

see (5.12). By (5.20) and (5.12) we then get

$$\rho(a_m, a_{m+l}) \leq \frac{2^s}{\varsigma} \tilde{\kappa}_s(\hat{\gamma}_m, \hat{\gamma}_{m+l}).$$

By Proposition 5.1.5 this yields the convergence of $\{a_m\}_{m>m_*}$ to some $a(x) \in \mathcal{A}$, which holds for each $x \in \breve{\gamma}$. This defines the map $a : p(\breve{\gamma}) \to \mathcal{A}$, and hence the configuration $\hat{\gamma} = (p(\breve{\gamma}), a)$. Our aim now is to prove that $\kappa(\hat{\gamma}_m, \hat{\gamma}) \to 0$ as $m \to +\infty$.

Fix $\varepsilon > 0$ and then pick a compact $\Lambda_\varepsilon \subset X$ such that (5.18) holds with $\frac{\varepsilon}{3}$ in the right-hand side. Let Λ^o_ε be its interior. Then pick compact mutually disjoint $\Delta_x \subset \Lambda^o$, $x \in p(\breve{\gamma}) \cap \Lambda^o$ such that $p(\breve{\gamma}) \cap \Delta_x = \{x\}$. As Λ_ε is compact, $p(\breve{\gamma}) \cap \Lambda^o$ is finite. Let $\{x_j\}_{j\leq J}$ be an enumeration of it. For brevity, we will write Δ_j in place of Δ_{x_j}, $j = 1, \ldots, J$. Similarly as above, by the convergence of $\{\breve{p}(\hat{\gamma}_m)\}_{m\in\mathbb{N}}$ to $\breve{\gamma}$, one finds m_* such that $p(\breve{\gamma}) \cap \Delta_j$ is a singleton and $|\breve{p}(\hat{\gamma}_m) \cap \Delta_j| =: n_m(x_j) = n(x_j)$, holding for all $m > m_*$ and $j \leq J$. Now we repeat the construction just made in each of Δ_j. That is, we enumerate

$$\hat{\gamma}_m \cap \Delta_j = \{(x^j_{1,m}, \alpha^j_{1,m}), \ldots, (x^j_{n(x_j),m}, \alpha^j_{n(x_j),m})\},$$

and then set $a^j_m = \{\alpha^j_{1,m}, \ldots, \alpha^j_{n(x_j),m}\}$. Then we set $\hat{\gamma}_{*,m} = (\gamma_{*,m}, a_{*,m})$, where $\gamma_{*,m} = p(\breve{\gamma}) \cap \Lambda^o_\varepsilon = \{x_1, \ldots, x_J\}$ and $a_{*,m}(x_j) = a^j_m$. In other words, the ground configuration of $\hat{\gamma}_{*,m}$ is the part of the limiting configurations $p(\hat{\gamma})$ contained in Λ^o_ε, whereas the marks are taken from the corresponding part of $\hat{\gamma}_m$. By the triangle inequality we then have

$$\kappa(\hat{\gamma}, \hat{\gamma}_m) \leq \kappa(\hat{\gamma} \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) + \varkappa(\hat{\gamma}_m \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) + \frac{\varepsilon}{3}. \tag{5.21}$$

By (5.16), for each $s, k, n \in \mathbb{N}$, we have

$$\begin{aligned} \kappa_{s,k,n}((\hat{\gamma} \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) &= \Big| \sum_{j=1}^J v_s(x_j) \Big[\sum_{\alpha \in a^j_m} w_{\sigma_k,n}(\alpha) - \sum_{\alpha \in a(x_j)} w_{\sigma_k,n}(\alpha) \Big] \Big| \\ &\leq J \max_{j \leq J} \rho_{k,n}(a^j_m, a(x_j)). \end{aligned}$$

Likewise,

$$\begin{aligned} \kappa_{s,k,n}((\hat{\gamma} \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) &= \Big| \sum_{j=1}^J \Big(\sum_{p=1}^{n(x_j)} v_s(x^j_{p,m}) \Big) \sum_{\alpha \in a^j_m} w_{\sigma_k,n}(\alpha) \\ &- \sum_{j=1}^J v_s(x_j) \sum_{\alpha \in a^j_m} w_{\sigma_k,n}(\alpha) \Big| \\ &\leq d(\breve{\gamma}_m \cap \Lambda^o_\varepsilon, \breve{\gamma} \cap \Lambda^o_\varepsilon). \end{aligned}$$

Both latter estimates yield

$$\kappa(\hat{\gamma} \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) \leq J \max_{j \leq J} \rho(a^j_m, a(x_j))$$

$$\kappa(\hat{\gamma}_m \cap \Lambda^o_\varepsilon, \hat{\gamma}_{*,m}) \leq d(\breve{\gamma}_m, \breve{\gamma}).$$

By the aforementioned convergence $\breve{\gamma}_m \to \breve{\gamma}$ and $a^j_m \to a(x_j)$, one can find $m_\varepsilon > m_*$ such that the first two commands in (5.21) are smaller than ε for $m > m_\varepsilon$, which completes the proof.

□

5.2 Measures and functions on configuration spaces

For v_s and $w_{k,n}$ as in (5.16) we set

$$\theta_{s,k,n}(x,\alpha) = \exp(-v_s(x)w_{k,n}(\alpha)) - 1 = \exp(-g_{s,k,n}(x,\alpha)) - 1,$$

see (5.17). Then $\theta_{s,k,n}(x,\alpha) \in C_{cs}(\hat{X})$ and $\theta_{s,k,n}(x,\alpha) \in (-1,0]$. Let Θ be the subset of $C_{cs}(\hat{X})$ consisting of

$$\theta(\hat{x}) = e^{-g(\hat{x})} - 1, \qquad g(x,\alpha) = \sum_j v'_{s_j}(x) w_{k_j,n_j}(\alpha), \tag{5.22}$$

where the latter sum runs over a finite subset of $\mathbb{N}^3$. That is, each such g is a finite sum of $g_{s,k,n}$ defined in (5.17). Note that Θ is countable and closed under the map $(\theta,\theta') \to (\theta * \theta')$, where

$$(\theta * \theta')(\hat{x}) = \theta(\hat{x}) + \theta'(\hat{x}) + \theta(\hat{x})\theta'(\hat{x}) = -1 + (1+\theta(x))(1+\theta'(x)). \tag{5.23}$$

Moreover, by (5.6), (5.8) and (5.22) it follows that

$$g(x,\alpha) \leq g(x,0), \tag{5.24}$$

holding for all $\alpha \geq 0$ and $x \in X$. Now for $\theta \in \Theta$, we set

$$F^\theta(\hat{\gamma}) = \prod_{\hat{x}\in\hat{\gamma}} (1+\theta(\hat{x})) = \exp \sum_{\hat{x}\in\hat{\gamma}} \Big(- g(\hat{x}) \Big), \qquad \hat{\gamma} \in \hat{\Gamma} \tag{5.25}$$

Then $F^\theta(\hat{\gamma}) \in (0,1]$ for all $\hat{\gamma} \in \hat{\Gamma}$, and hence $F^\theta \in C_b(\hat{\Gamma})$. The set of all such functions will be denoted by F^θ. For $\mu \in \mathcal{P}(\hat{\Gamma})$, we then have

$$\mu(F^\theta) = \int_{\hat{\Gamma}} F^\theta(\hat{\gamma})\mu(d\hat{\gamma}) \leq 1.$$

The Poisson measure π_ρ with intensity measure ρ satisfies

$$\pi_\rho(F^\theta) = \exp\left(\rho(\theta)\right) = \exp\Big(\int_{\hat{X}} \theta(\hat{x})\rho(d\hat{x}) \Big). \tag{5.26}$$

For $\mu_1, \mu_2 \in \mathcal{P}(\hat{\Gamma})$, their convolution is defined by the expression

$$(\mu_1 \star \mu_2)(F) = \int_{\hat{\Gamma}^2} F(\hat{\gamma}_1 \cup \hat{\gamma}_2)\mu_1(d\hat{\gamma}_1)\mu_2(d\hat{\gamma}_2), \tag{5.27}$$

that ought to hold for all $F \in B_b(\hat{\Gamma})$. For F^θ as in (5.25), it takes the form

$$(\mu_1 \star \mu_2)(F^\theta) = \mu_1(F^\theta)\mu_2(F^\theta). \tag{5.28}$$

Recall that a set $\mathcal{F} \subset C_b(\hat{\Gamma})$ is called convergence determining if $\mu_n(F) \to \mu(F)$, $n \to +\infty$, implies $\mu_n \Rightarrow \mu$, holding for each $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\hat{\Gamma})$. It is known, see Ethier and Kurtz (1986), [13], Theorem 4.5, page 113, that such F enjoys this property if it is closed under pointwise multiplication and is strongly separating. The latter means that, for each $\hat{\gamma} \in \hat{\Gamma}$ and $\epsilon > 0$, there exists a finite family $\{F_j\} \subset \mathcal{F}$ such that

$$\inf_{\hat{\gamma}'\in\hat{C}_\epsilon} \max_j |F_j(\hat{\gamma}) - F_j(\hat{\gamma}')| > 0, \qquad \hat{C}_\epsilon := \hat{\Gamma} \setminus \hat{B}_\epsilon(\hat{\gamma}). \tag{5.29}$$

Here

$$\hat{B}_\epsilon(\hat{\gamma}) = \{\hat{\gamma}' : \kappa(\hat{\gamma}, \hat{\gamma}') < \epsilon\},$$

see (5.16). Note that taking $\epsilon \geq 1$ does not make sense as $\kappa(\hat{\gamma}, \hat{\gamma}') < 1$ for all configurations.

Proposition 5.2.1. *The set F^θ is strongly separating and thus convergence determining.*

Proof. By the very definition of Θ, cf. (5.23), F^θ is closed under pointwise multiplication. To prove (5.29), we note that, see (5.25) and (5.17),

$$|F^{\theta_{s,k,n}}(\hat{\gamma}) - F^{\theta_{s,k,n}}(\hat{\gamma}')| \geq \min\{F^{\theta_{s,k,n}}(\hat{\gamma}); F^{\theta_{s,k,n}}(\hat{\gamma}'))\}\kappa_{s,k,n}(\hat{\gamma}; \hat{\gamma}') \tag{5.30}$$

holding for all $\hat{\gamma}, \hat{\gamma}' \in \hat{\Gamma}$. Now we fix $\hat{\gamma}$ and $\epsilon \in (0,1)$ and then take m such that $2^{-m} < \frac{\epsilon}{2}$. For this m and any $\hat{\gamma} \in \hat{C}_\epsilon$, by (5.16) we readily conclude that

$$\max_{(s,k,n):s+k+n\leq m} \kappa_{s,k,n}(\hat{\gamma}, \hat{\gamma}') > 0,$$

which by (5.30) yields the proof. □

The important properties of the family F^θ are summarized in the following statement.

Proposition 5.2.2. *The following is true:*

(i) $\mathcal{B}(\hat{\Gamma}) = \sigma\{F^\theta\}$*;*

(ii) $B_b(\hat{\Gamma})$ *is the bp-closure of the linear span of* F^θ*;*

(iii) F^θ *is separating;*

(iv) F^θ *is convergence determining.*

The proof of (i) and (ii) is standard, see Dawson (1993), Lemma 3.2.5 and Theorem 3.2.6, page 43, [11]. The proof of (iv) was done above, whereas (iii) is a direct consequence of (iv), cf. Ethier and Kurtz (1986), [13].

Proposition 5.2.3. *The Kolmogorov operator introduced in (5.1) has the property* $L : F^\theta \to C_b(\hat{\Gamma})$*.*

Proof. We consider each of the summands in (5.1) – denoted by $L_i, i = 1, 2, 3$ separately. Thus,

$$(L_3 F^\theta)(\hat{\gamma}) = F^\theta(\hat{\gamma}) \int_X \theta(x, 0)\chi(dx), \tag{5.31}$$

i.e., it is just the multiplication operator by a θ-dependent constant. Next,

$$(L_2 F^\theta)(\hat{\gamma}) = -\sum_{\hat{x}\in\hat{\gamma}} m(\hat{x})\theta(\hat{x})F^\theta(\hat{\gamma}\setminus\hat{x}) = F^\theta(\hat{\gamma})\Psi_2(\hat{\gamma}), \tag{5.32}$$

where

$$\Psi_2(\hat{\gamma}) = \sum_{\hat{x}\in\hat{\gamma}} \psi_2(\hat{x}) = \sum_{\hat{x}\in\hat{\gamma}} m(\hat{x})(e^{g(\hat{x})} - 1). \tag{5.33}$$

Let us consider the following function

$$\phi_\tau(\hat{x}) = g(\hat{x}) - \tau m(\hat{x})(e^{g(\hat{x})} - 1), \tau \geq 0.$$

Since each $g_{s,k,n}(\hat{x}) < 1$, see (5.17), it follows that $g(\hat{x}) \leq J_\theta$ where J_θ is just the number of summands in the sum in (5.22). At the same time, $m(\hat{x}) \leq m_* < \infty$. Taking this into account, we set

$$\tau_* = \frac{1}{m_* e^{J_\theta}}. \tag{5.34}$$

Then

$$\phi_{\tau_*}(\hat{x}) > 0, \qquad \hat{x} \in \hat{X}.$$

Now by the simple inequality $\beta e^{-\tau\beta} \leq \frac{1}{e\tau}$, $\tau, \beta > 0$, we have, see (5.32) and (5.33),

$$(L_2 F^\theta)(\hat{\gamma}) \leq \tau_*^{-1} \exp\Big(-1 - \sum_{\hat{x}\in\hat{\gamma}} \phi_{\tau_*}(\hat{x})\Big) \leq m_* e^{J_\theta - 1}, \tag{5.35}$$

which yields the boundedness in question. The continuity of $L_2 F^\theta$ follows by the continuity of Ψ_2, which in turn follows by the fact that $\psi_2 \in C_{cs}(\hat{X})$. Finally,

$$(L_1 F^\theta)(\hat{\gamma}) = \Big(-\sum_{\hat{x}\in\hat{\gamma}} g'(\hat{x})\Big) F^\theta(\hat{\gamma}) =: \Psi_1(\hat{\gamma}) F^\theta(\hat{\gamma}).$$

By (5.22) we have

$$g'(x, \alpha) = \sum_j v_{s_j}(x) w'_{k_j, n_j}(\alpha),$$

which yields the continuity of Ψ_1. At the same time, by (5.9) we have

$$|g'(x, \alpha)| \leq \sum_j v_{s_j}(x) |w'_{k_j, n_j}(\alpha)| \leq \bar{\sigma} c g(x, \alpha), \tag{5.36}$$

which yields

$$|g'(x, \alpha)| \leq \bar{\sigma} c J_\theta, \tag{5.37}$$

and also

$$|(L_1 F^\theta)(\hat{\gamma})| \leq \frac{\bar{\sigma} c}{e}$$

This completes the proof. □

We summarize the estimates obtained above in the following

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |(LF^\theta)(\hat{\gamma})| \leq \chi(|\theta(\cdot, 0)|) + m_* e^{J_\theta - 1} + \frac{\bar{\sigma} c}{e}. \tag{5.38}$$

Note that

$$\chi(|\theta(x,\alpha)|) := \int_X |\theta(x,\alpha)|\chi(dx) = \int_X (1 - e^{-g(x,\alpha)})\chi(dx) \leq \int_X g(x,0)\chi(dx), \tag{5.39}$$

see (5.24).

5.3 The Kolmogorov equation

5.3.1 Notions and useful estimates

For $\theta \in \Theta$, see (5.22), we set

$$\theta_t(x,\alpha) = \theta(x, \alpha + t) \exp\Big(M(x,\alpha) - M(x, \alpha + t)\Big), \tag{5.40}$$

$$M(x,\alpha) = \int_0^\alpha m(x,\beta) d\beta.$$

Let $\hat{x} \in \hat{X}$, then $\hat{x} \to \theta_t(\hat{x})$ is continuous and compactly supported for all $t \geq 0$. Moreover, both maps $t \to \theta_t(\hat{x})$ and $\alpha \to \theta_t(x,\alpha)$ are continuously differentiable and the following holds

$$\frac{\partial}{\partial t}\theta_t(x,\alpha) = \frac{\partial}{\partial \alpha}\theta_t(x,\alpha) - m(x,\alpha)\theta_t(x,\alpha) \tag{5.41}$$

Note that

$$(\theta_t)_s(\hat{x}) = \theta_{t+s}(\hat{x}). \tag{5.42}$$

Next, we define, cf. (5.22),

$$g_t(\hat{x}) = -\log\left(1 + \theta_t(\hat{x})\right). \tag{5.43}$$

By (5.40) it follows that

$$|\theta_t(x,\alpha)| \leq |\theta(x,\alpha + t)| = 1 - e^{-g(x,\alpha+t)} \leq 1 - e^{-J_\theta},$$

where J_θ is the same as in (5.34). By (5.43) this yields

$$g_t(\hat{x}) \leq J_\theta, \quad t \geq 0, \quad \hat{x} \in \hat{X}. \tag{5.44}$$

By (5.40) we also have

$$|\theta_t'(x,\alpha)| \leq 2m_* + \theta'(x,\alpha+t)| \leq 2m_* + J_\theta,$$

where m_* is as in (5.34) and the estimate

$$|\theta'(x,\alpha)| \leq |g'(x,\alpha)| \leq J_\theta$$

was used, see (5.22) and (5.37).

Now we define

$$F_t^\theta(\hat{\gamma}) = \exp\Big[\int_0^t \Big(\int_X \theta(x,\alpha)e^{-M(x,\alpha)}\chi(dx)\Big)d\alpha\Big]F^{\theta_t}(\hat{\gamma}, \tag{5.45}$$

with θ_t as in (5.40). Clearly, $F_t^\theta \in C_b(\hat{\Gamma})$ for all $t > 0$ and $\theta \in \Theta$, and

$$0 < F_t^\theta(\hat{\gamma}) \leq 1, \qquad \hat{\gamma} \in \hat{\Gamma}. \tag{5.46}$$

Furthermore, for all $t, s \geq 0$, the following holds, see (5.42),

$$F_{t+s}^\theta = \exp\Big(\int_0^s \int_X \theta_u(x,0)\chi(dx)du\Big)F_t^{\theta_s}. \tag{5.47}$$

Let us prove that $F_t^\theta \in C_b(\hat{\Gamma})$ for all $\theta \in \Theta$ and $t \geq 0$. As in the proof of Proposition 5.2.3 we divide L into three parts. Similarly as in (5.31) we have

$$(L_3F_t^\theta)(\hat{\gamma}) \leq \chi(|\theta(\cdot,t)|) \leq \chi(g(\cdot,0)),$$

see (5.39) and (5.46). Since $g_t(\hat{x})$ satisfies (5.44) for all $t \geq 0$, it follows that

$$|(L_2F_t^\theta)(\hat{\gamma})| \leq m_* e^{J^\theta - 1},$$

holding for all $t > 0$, see (5.35). The estimate of $|(L_1F_t^\theta)(\hat{\gamma})|$ is obtained as follows. Denote

$$q_t(x,\alpha) = \exp\Big(-\int_\alpha^{\alpha+t} m(x,\beta)d\beta\Big).$$

Then by (5.40) we have

$$\Phi(g_t(x,\alpha)) = q_t(x,\alpha)\Phi(g(x,\alpha+t)), \quad \Phi(b) := 1 - e^{-b}, \quad b \geq 0, \tag{5.48}$$

by which we get that $g_t(x,\alpha) \leq g(x,\alpha+t)$. Let us prove that

$$g_t(x,\alpha)e^{-g_t(x,\alpha)} \geq q_t(x,\alpha)g(x,\alpha+t)e^{-g(x,\alpha+t)}. \tag{5.49}$$

By (5.48) this is equivalent to the fact that the function $b \to \frac{b}{e^b-1}$ is decreasing, which is obviously the case. Now we take the α-derivative from both sides of (5.48) and obtain

$$\begin{aligned} g_t'(x,\alpha) &= q_t(x,\alpha)g'(x,\alpha+t)\exp\Big(g_t(x,\alpha) - g(x,\alpha+t)\Big) \\ &+ m(x,\alpha+t)(\exp(g_t(x,\alpha)) - 1) - m(x,\alpha)(\exp(g_t(x,\alpha)) - 1), \end{aligned}$$

that can be estimated as follows

$$|g_t'(x,\alpha)| \; = \; \bar{\sigma} c q_t(x,\alpha) g'(x,\alpha+t) \exp\Big(g_t(x,\alpha) - g(x,\alpha+t)\Big) \qquad (5.50)$$

$$+ \; 2m_*(\exp(g_t(x,\alpha)) - 1) \leq g_t(x,\alpha) e^{J_\theta}(\bar{\sigma} c + 2m_*),$$

where we used (5.36), (5.44) and (5.49). Now we proceed as in obtaining (5.38), which eventually yields

$$|(LF_t^\theta)(\hat{\gamma})| \leq \chi(g(\cdot,0)) + m_* e^{J_\theta - 1} + (\bar{\sigma} c + 2m_*) e^{J_\theta} =: l_\theta. \qquad (5.51)$$

The key property of the latter estimate is that it is uniform in t. However, it does depend on θ. Along with the estimates derived above, we will use also the following. For $\theta \in \Theta$, the corresponding g has the form as in (5.22). By (5.7) and (5.8) we have that

$$\exp\Big(-\frac{\bar{\sigma}\sqrt[3]{4}}{3}\Big) w_{k,n}(0) \leq w_{k,n}(\alpha) \leq w_{k,n}(0),$$

which means that, cf. (5.24),

$$\bar{c} g(x,0) \leq g(x,\alpha) \leq g(x,0), \qquad \bar{c} := \exp\Big(-\frac{\bar{\sigma}\sqrt[3]{4}}{3}\Big). \qquad (5.52)$$

For Φ as in (5.48), we have

$$b \geq \Phi(b) \geq b - \frac{b^2}{2}, \qquad b \geq 0,$$

which we use together with 5.52) to obtain the following

$$g_t(\hat{x}) \; \geq \; q_t(\hat{x})\Phi(\bar{c} g(x,0)) \geq \exp(-m_* t)\Phi\Big(\frac{\bar{c} g(x,0)}{J_\theta}\Big) \qquad (5.53)$$

$$\geq \; \exp(-m_* t) \frac{\bar{c} g(x,0)}{J_\theta}\Big(1 - \frac{\bar{c} g(x,0)}{2 J_\theta}\Big)$$

$$\geq \; \exp(-m_* t) \bar{c}_\theta g(x,0),$$

where, $\bar{c}_\theta := \frac{\bar{c}}{2J_\theta}$, and we have used the fact that $g(x,0) \leq J_\theta$, see (5.44).

5.3.2 The operator

We fix $\hat{\gamma} \in \hat{\Gamma}$ and calculate the t-derivative of (5.41). This yields

$$\frac{\partial}{\partial t}(LF)(\hat{\gamma}) \; = \; \Big(\int_X \theta(x,t) e^{-M(x,t)} \chi(dx)\Big) F_t^\theta(\hat{\gamma}) + \sum_{\hat{x} \in \hat{\gamma}} \frac{\partial \theta(\hat{x})}{\partial t} F_t^\theta(\hat{\gamma} \setminus \hat{x}) \qquad (5.54)$$

$$= \; \Big(\int_X \theta_t(x,0) \chi(dx)\Big) F_t^\theta(\hat{\gamma}) + \sum_{\hat{x} \in \hat{\gamma}} \frac{\partial}{\partial \alpha} F_t^\theta(\hat{\gamma})$$

$$+ \; \sum_{x \in \gamma} m(\hat{x}) \left[F_t^\theta(\hat{\gamma} \setminus \hat{x}) - F_t^\theta(\hat{\gamma})\right] = (LF_t^\theta)(\hat{\gamma}).$$

This means that we have found a solution of the Kolmogorov equation for (5.1) in the following sense. It is a map $t \to F_t \in C_b(\hat{\Gamma})$, which is pointwise in $\hat{\gamma}$ continuously t-differentiable and such that the equality in (2.4) holds. Our aim now is to solve (2.4) in a suitable Banach space. Recall that the paths $t \to \theta_t$ have the flow property (5.42), see also (5.47). Below, by saying of a property of $\theta_s, s \geq 0$, holding for all θ, we attribute this property to all θ_s given in (5.40) with θ taken from Θ.

Proposition 5.3.1. *For each $\theta \in \Theta$ and $s \geq 0$, it follows that $F_t^{\theta_s} \to F^\theta s$ as $t \to 0$ in the norm of $C_b(\hat{\Gamma})$.*

Proof. For each $\hat{\gamma}, s \geq 0$ and $\theta \in \Theta$, by (5.54) it follows that

$$\begin{aligned} F_t^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma}) &= \exp\Big(- \int_0^s \int_X \theta_u(x,0)\chi(dx)du\Big)[F_{t+s}^\theta(\hat{\gamma}) - F_s^\theta(\hat{\gamma})] \quad (5.55) \\ &= \exp\Big(- \int_0^s \int_X \theta_u(x,0)\chi(dx)du\Big) + \int_s^{s+t} (LF_u^\theta)(\hat{\gamma})du, \end{aligned}$$

which by (5.51) yields

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |F_t^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma})| \leq t l_\theta \exp\Big(- \int_0^s \int_X \theta_u(x,0)\chi(dx)du\Big). \quad (5.56)$$

This completes the proof. □

The next statement is a refinement of the one just proved.

Proposition 5.3.2. *For each $\theta \in \Theta$, $s \geq 0$ it follows that $LF_t^{\theta_s} \to LF^{\theta_s}$ as $t \to 0$ in the norm of $C_b(\hat{\Gamma})$.*

Proof. First of all we note that the equality in the first line of (5.55) allows one to obtain the property in question by showing that $LF_{t+s}^\theta \to LF_s^\theta$ in the same sense. As in the proof of Proposition 5.2.3, we split L into three parts and consider each of them separately.

Fix $\theta \in \Theta$ and then denote

$$\dot{\eta}(u) = \int_X \theta_u(x,0)\chi(dx), \qquad \eta(t) = \int_0^t \dot{\eta}(u)du. \quad (5.57)$$

Then

$$\begin{aligned} |(L_3F_{t+s}^\theta)(\hat{\gamma}) - (L_3F_s^\theta)(\hat{\gamma})| &\leq |\dot{\eta}(t+s)e^{\eta(t+s)} - \dot{\eta}(s)e^{\eta(s)}| \\ &+ |\dot{\eta}(s)|e^{\eta(s)}|F_{t+s}^\theta(\hat{\gamma}) - F_s^\theta(\hat{\gamma})| \\ &=: I_1(t) + I_2(t), \end{aligned}$$

see (5.46). Then $I_1(t) \to 0$ as $t \to 0$ since $\dot{\eta}(t)e^{\eta(t)}$ is a continuous function of t. At the same time, $I_2(t)$ can be estimated as in (5.56). This yields the proof for L_3.

Now we proceed to L_2, for which it follows that

$$(L_2 F_t^{\theta_s})(\hat{\gamma}) - (L_2 F^{\theta_s})(\hat{\gamma}) = J_t(\hat{\gamma}) + K_t(\hat{\gamma}), \tag{5.58}$$

$$J_t(\hat{\gamma}) = \sum_{\hat{x}\in\hat{\gamma}} \frac{-m(\hat{x})\theta_{t+s}(\hat{x})}{1+\theta_{t+s}(\hat{x})}[F_t^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma})]$$

$$K_t(\hat{\gamma}) = F^{\theta_s}(\hat{\gamma}) \sum_{\hat{x}\in\hat{\gamma}} \frac{m(\hat{x})(\theta_s(\hat{x}) - \theta_{t+s}(\hat{x}))}{(1+\theta_{t+s}(\hat{x}))(1+\theta_s(\hat{x}))}.$$

By (5.45) and (5.57) we have

$$\begin{aligned} F_t^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma}) &= (e^{\eta(t+s)-\eta(s)} - 1)F_{t+s}^{\theta}(\hat{\gamma}) \\ &+ F^{\theta_{t+s}}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma}) =: \Upsilon_1(t,\hat{\gamma}) + \Upsilon_2(t,\hat{\gamma}). \end{aligned} \tag{5.59}$$

By (5.39) and (5.40), (5.57) it follows that

$$|e^{\eta(t+s)-\eta(s)} - 1| \le t\chi(g(\cdot,0)),$$

which then yields

$$\Upsilon_1(t,\hat{\Gamma}\hat{\gamma}) \le t\chi(g(\cdot,0)) \exp\Big(- \sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x})\Big). \tag{5.60}$$

To estimate Υ_2 we write

$$h_t(\hat{x}) = \min\{g_{t+s}(\hat{x}); g_s(\hat{x})\}.$$

Then by (5.50) and (5.53) we get

$$\begin{aligned} |\Upsilon_2(t,\hat{\gamma})| &\le \exp\Big(- \sum_{\hat{x}\in\hat{\gamma}} h_t(\hat{x})\Big) \sum_{\hat{x}\in\hat{\gamma}} |g_{t+s}(\hat{x}) - g_s(\hat{x})| \\ &\le \exp(J_\theta)(\bar{\sigma}c + 2m_*) \exp\Big(- \sum_{\hat{x}\in\hat{\gamma}} h_t(\hat{x})\Big) \sum_{\hat{x}\in\hat{\gamma}} \int_s^{t+s} g_u(x,\alpha)du \\ &\le \exp(J_\theta)(\bar{\sigma}c + 2m_*) \exp\Big(- e^{-m_* t}\bar{c}_\theta \sum_{\hat{x}\in\hat{\gamma}} g(x,0)\Big) \sum_{\hat{x}\in\hat{\gamma}} tg(x,0) \\ &=: tC_\theta e^{-\bar{c}_\theta(t)\Psi_0(\hat{\gamma})}\Psi_0(\hat{\gamma}), \end{aligned} \tag{5.61}$$

$$\bar{c}_\theta(t) := e^{-m_* t}\bar{c}_\theta,$$

$$\Psi_0(\hat{\gamma}) := \sum_{\hat{x}\in\hat{\gamma}} g(x,0).$$

At the same time,

$$0 \leq \sum_{\hat{x}\in\hat{\gamma}} \frac{-m(\hat{x})\theta_{t+s}(\hat{x})}{1+\theta_{t+s}(\hat{x})} \leq m_* e^{J_\theta} \sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x}) =: m_* e^{J_\theta}\Psi_1(t,\hat{\gamma}).$$

Thereafter, we have

$$\begin{aligned} |J_t(\hat{\gamma})| &\leq tm_* e^{J_\theta}\chi(g(\cdot,0))\Psi_1(t,\hat{\gamma})e^{-\Psi_1(t,\hat{\gamma})} \\ &+ tC_\theta m_* e^{J_\theta}\Psi_1(t,\hat{\gamma})\Psi_0(\hat{\gamma})e^{-\bar{c}_\theta(t)\Psi_0(\hat{\gamma})} \\ &:= \Pi_1(t,\hat{\gamma}) + \Pi_2(t,\hat{\gamma}), \end{aligned}$$

where $C_\theta, \bar{c}_\theta(t)$ and Ψ_0 are as in (5.61). Then

$$\Pi_1(t,\hat{\gamma}) \leq tm_* e^{J_\theta - 1}\chi(g(\cdot,0)) \to 0, \qquad t \to +\infty.$$

Let $t_\theta > 0$ be the (unique) solution of

$$e^{-m_* t}\bar{c}_\theta = 2\sqrt[3]{t}.$$

Then for t_θ, we have

$$\begin{aligned} \Pi_2(t,\hat{\gamma}) &\leq tC_\theta m_* e^{J_\theta}[\Psi_0(\hat{\gamma})]^2 e^{-\bar{c}_\theta(t)\Psi_0(\hat{\gamma})} \\ &\leq \sqrt[3]{t}C_\theta m_* e^{J_\theta - 2}\exp\Big(-(\bar{c}_\theta(t) - 2\sqrt[3]{t})\Psi_0(\hat{\gamma})\Big) \\ &\leq \sqrt[3]{t}C_\theta m_* e^{J_\theta - 2}, \end{aligned} \tag{5.62}$$

which yields the convergence

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |J_t(\hat{\gamma})| \to 0, \qquad t \to 0.$$

Now we turn to $K_t(\hat{\gamma})$. First, by (5.41) we have

$$|\theta_s(\hat{x}) - \theta_{t+s}(\hat{x})| = \Big|\int_s^{t+s}\Big(\frac{\partial}{\partial\alpha}\theta_u(x,\alpha) - m(x,\alpha)\theta_u(x,\alpha)\Big)du\Big|. \tag{5.63}$$

Next, by (5.43) and (5.40) it follows that

$$\begin{aligned} \Big|\frac{\partial}{\partial\alpha}\theta_u(x,\alpha)\Big| &= e^{-g_u(x,\alpha)}|g_u'(x,\alpha)| \\ &\leq e^{J_\theta}(\bar{\sigma}c + 2m_*)g_u(x,\alpha) \\ &\leq e^{J_\theta}(\bar{\sigma}c + 2m_*)g(x,0), \end{aligned}$$

and also

$$|m(x,\alpha)\theta_u(x,\alpha)| \leq m_* g_u(x,\alpha) \leq m_* g(x,0).$$

The latter two estimates yield

$$\mathrm{LHR}(5.63) \leq e^{J_\theta}(\bar{c} + 3m_*)g(x,0).$$

By (5.52) this yields

$$\begin{aligned} K_t(\hat{\gamma}) &\leq tm_*(\bar{\sigma} + 3m_*)e^{3J_\theta}\Psi_0(\hat{\gamma})e^{-\bar{c}\Psi_0(\hat{\gamma})} \\ &\leq tm_*(\bar{\sigma} + 3m_*)\frac{e^{3J_\theta - 1}}{\bar{c}} \to 0, t \to +\infty. \end{aligned}$$

By (5.58) this completes the proof for L_2. Next, we write

$$\begin{aligned} (L_1 F_t^{\theta_s})(\hat{\gamma}) - (L_1 F^{\theta_s}\theta_s)(\hat{\gamma}) &= Q_t(\hat{\gamma}) + R_t(\hat{\gamma}), \\ Q_t(\hat{\gamma}) &= \sum_{\hat{x}\in\hat{\gamma}} \theta'_{t+s}(\hat{x})\Big[F_t^{\theta_s}(\hat{\gamma}\ \hat{x}) - F^{\theta_s}(\hat{\gamma}\setminus\hat{x})\Big], \\ R_t(\hat{\gamma}) &= \sum_{\hat{x}\in\hat{\gamma}} \Big[\theta'_{t+s}(\hat{x}) - \theta'_s(\hat{x})\Big]F^{\theta_s}(\hat{\gamma}\setminus\hat{x}). \end{aligned} \tag{5.64}$$

Then

$$\begin{aligned} Q_t(\hat{\gamma}) &= -\sum_{\hat{x}\in\hat{\gamma}} g'_{t+s}(\hat{x})\Big[F_t^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma})\Big] \\ &+ F^{\theta_s}(\hat{\gamma})\sum_{\hat{x}\in\hat{\gamma}} g'_{t+s}(\hat{x})\Big[e^{g_s(\hat{x}) - g_{t+s}(\hat{x})} - 1\Big] \\ &=: Q_t^{(1)}(\hat{\gamma}) + Q_t^{(2)}(\hat{\gamma}). \end{aligned}$$

By (5.50) and then by (5.59), (5.60), (5.61) we get

$$\begin{aligned} |Q_t^{(1)}(\hat{\gamma})| &\leq e^{J_\theta}(\bar{\sigma}c + 2m_*)(\Upsilon_1(t,\hat{\gamma}) + \Upsilon_2(t,\hat{\gamma}))\sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x}) \\ &\leq t\chi(g(\cdot,0))e^{J_\theta}(\bar{\sigma}c + 2m_*)\Big(\sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x})\Big)\exp\Big(-\sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x})\Big) \\ &+ tC_\theta e^{J_\theta}(\bar{\sigma}c + 2m_*)e^{-\bar{c}_\theta(t)\Psi_0(\hat{\gamma})}[\Psi_0(\hat{\gamma})]^2 \\ &=: \Xi_t^{(1)}(\hat{\gamma}) + \Xi_t^{(2)}(\hat{\gamma}). \end{aligned}$$

Then

$$\Xi_t^{(1)}(\hat{\gamma}) \leq t\chi(g(\cdot,0))e^{J_\theta - 1}(\bar{\sigma}c + 2m_*) \to 0, \qquad t \to 0,$$

and also, cf. (5.62),

$$\begin{aligned} \Xi_t^{(2)}(\hat{\gamma}) &\leq \sqrt[3]{t}C_\theta e^{J_\theta - 2}(\bar{\sigma}c + 2m_*)\exp\Big(-(\bar{c}_\theta(t) - 2\sqrt[3]{t})\Psi_0(\hat{\gamma})\Big) \\ &\leq \sqrt[3]{t}C_\theta m_* e^{J_\theta - 2}(\bar{\sigma}c + 2m_*), \end{aligned}$$

for $t \leq t_\theta$. The latter two estimates yield

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |Q_t^{(1)}(\hat{\gamma})| \to 0, \qquad t \to 0. \tag{5.65}$$

Next, by (5.44), (5.50) and (5.52) we have

$$\begin{aligned} |Q_t^{(2)}(\hat{\gamma})| &\leq e^{J_\theta} F^{\theta_s}(\hat{\gamma}) \sum_{\hat{x}\in\hat{\gamma}} |g'_{t+s}(\hat{x})||g_{t+s}(\hat{x}) - g_s(\hat{x})| \\ &\leq e^{3J_\theta}(\bar{\sigma}c + 2m_*)^2 F^{\theta_s}(\hat{\gamma}) \sum_{\hat{x}\in\hat{\gamma}} g_{t+s}(\hat{x}) \int_s^{t+s} g_u(\hat{x})du \\ &\leq tJ_\theta e^{3J_\theta}(\bar{\sigma}c + 2m_*)^2 \Psi_0(\hat{\gamma}) \exp\left(-\bar{c}\Psi_0(\hat{\gamma})\right) \\ &\leq tJ_\theta e^{3J_\theta} \frac{(\bar{\sigma}c + 2m_*)^2}{\bar{c}} \to 0, \qquad t \to 0, \end{aligned} \tag{5.66}$$

which together with (5.65) yields

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |Q_t(\hat{\gamma})| \to 0, \qquad t \to 0. \tag{5.67}$$

Now we turn to estimating R_t. By (5.40) we have

$$\begin{aligned} |\theta'_{t+s}(x,\alpha) - \theta'_s(x,\alpha)| &\leq |\theta'(x,\alpha+t+s)||q_{t+s}(x,\alpha) - q_s(x,\alpha)| \\ &+ |\theta'(x,\alpha+t+s) - \theta'(x,\alpha+s)| \\ &+ |m(x,\alpha+t+s) - m(x,\alpha+s)| \cdot |\theta_{t+s}(x,\alpha)| \\ &+ |m(x,\alpha+s) - m(x,\alpha)||\theta_s(x,\alpha)| \\ &=: \delta_1(t,\hat{x}) + \delta_2(t,\hat{x}) + \delta_3(t,\hat{x}) + \delta_4(t,\hat{x}). \end{aligned} \tag{5.68}$$

By (5.50) we have

$$\begin{aligned} \delta_1(t,\hat{x}) &\leq e^{-g(x,\alpha+t+s)}|g'(x,\alpha+t+s)| \int_{\alpha+s}^{\alpha+t+s} m(x,\beta)d\beta \\ &\leq tm_* e^{J_\theta}(\bar{\sigma}c + 2m_*)g(x,0). \end{aligned} \tag{5.69}$$

To estimate δ_2, by (5.43) we first get

$$\theta'(\hat{x}) = -g'(\hat{x})e^{-g(\hat{x})},$$

by which we then obtain

$$\begin{aligned}\delta_2(t,\hat{x}) &\leq |g'(x,\alpha+t+s)-g'(x,\alpha+s)|e^{-g(x,\alpha+t+s)}\\ &+ |g'(x,\alpha+s)||e^{-g(x,\alpha+t+s)}-e^{-g(x,\alpha+s)}|\\ &\leq |g'(x,\alpha+t+s)-g'(x,\alpha+s)|\\ &+ |g'(x,\alpha+s)||g(x,\alpha+t+s)-g(x,\alpha+s)|\\ &=: \delta_{2,1}(t,\hat{x})+\delta_{2,2}(t,\hat{x}).\end{aligned}$$

Now we recall that $g(x,\alpha)$ is as in (5.22) with $w_{k,n}$ defined in (5.8). Thus, we can write

$$\delta_{2,1}(t,\hat{x}) \leq \sum_j v_{s_j}(x)\int_s^{t+s}|w''k_j,n_j(\alpha+u)|du. \tag{5.70}$$

For each k and n, we have

$$\begin{aligned}|w''_{k,n}(\alpha)| &= |-\sigma_k u''_n(\alpha)e^{-\sigma_k u_n(\alpha)}+[\sigma_k u'_n(\alpha)]^2 e^{-\sigma_k u_n(\alpha)}|\\ &\leq \bar{\sigma}|u''_n(\alpha)|+|\bar{\sigma}u'_n(\alpha)^2|\leq \bar{C},\end{aligned}$$

holding for some $\bar{C}>0$ that is independent of k,n and α. The latter conclusion follows by (5.9) and the fact that $|u''_n(\alpha)|\leq 2\phi(n\alpha^3)$ with

$$\phi(\beta)=\frac{1+\beta^2}{(1+\beta)^3}, \qquad \beta\geq 0.$$

Then by (5.70) we get

$$\delta_{2,1}(t,\hat{x})\leq t\bar{C}g(x,0). \tag{5.68}$$

At the same time, by (5.36) and (5.37) it follows that

$$\delta_{2,2}(t,\hat{x})\leq |g'(x,\alpha+s)|\int_s^{t+s}|g'(x,\alpha+u)|du\leq t(\bar{\sigma}c)^2 J_\theta g(x,0),$$

which together with (5.3.2) yields

$$\delta_2(t,\hat{x})\leq t[\bar{C}+(\bar{\sigma}c)^2 J_\theta]g(x,0). \tag{5.68}$$

Finally, (5.2) and (5.43), (3.13) we have

$$\delta_3(t,\hat{x})\leq \varkappa(t)g_t(\hat{x})\leq \varkappa(t)g(x,0).$$

The same estimate holds true also for $\delta_4(t,\hat{x})$. Then by (5.68) and (5.69), (5.3.2) we have that

$$|\theta'_t(x,\alpha)-\theta'(x,\alpha)|\leq \omega(t)g(x,0), \quad \omega(t)\to 0, \quad t\to 0.$$

holding for some continuous function ω and all $\hat{x} \in \hat{X}$. Now we use this in (5.64) and obtain, cf.(5.66)

$$|R_t(\hat{\gamma})| \leq \omega(t)e^{J_\theta}F^\theta(\hat{\gamma})\Psi_0(\hat{\gamma}) \leq \omega(t)e^{J_\theta}\Psi_0(\hat{\gamma})e^{-\bar{c}\Psi_0(\hat{\gamma})} \leq \frac{\omega(t)e^{J_\theta}}{\bar{c}}, \tag{5.68}$$

which together with (5.67) yields

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} |(L_1 F^{\theta_t})(\hat{\gamma}) - (L_1 F^\theta)(\hat{\gamma})| \to 0, t \to 0.$$

This completes the whole proof. □

5.3.3 The domain

We recall that Θ consists of the functions as in (5.22) and the countable collection $F^\theta \subset C_b(\hat{\Gamma})$ consists of the functions introduced in (5.25). It has a number of useful properties established in Propositions 5.2.1 and 5.2.2. Let $\mathcal{C}_0$ be the linear span with rational coefficients of the set $F^{\theta_s} : s \in \mathbb{Q}_+, \theta \in \Theta$, i.e., each $F \in \mathcal{C}_0$ is a finite linear combination of F^{θ_s}, with positive rational s and θ_s given in (5.40) with all possible choices of $\theta \in \Theta$.

Remark 5.3.3. The set $\mathcal{C}_0$ is countable. It enjoys all the properties mentioned in Proposition 5.2.2.

Now we set

$$\mathcal{C} = \mathcal{C}_0, \tag{5.69}$$

i.e., $\mathcal{C}$ is the closure of $\mathcal{C}_0$ in the norm of $C_b(\hat{\Gamma})$, which we denote $\|\cdot\|$. With this norm it is then a separable Banach space.

For $\lambda > 0$ and $\theta \in \Theta$ and $s \geq 0$, we define, cf. (5.45),

$$\begin{aligned} F_{\lambda,\theta_s}(\hat{\gamma}) &= \int_0^\infty e^{-\lambda t} F_t^{\theta_s}(\hat{\gamma})dt \qquad (5.70) \\ &= \int_0^\infty \exp\Big[-\lambda t + \int_0^t \Big(\int_X \theta_s(x,\alpha)e^{-M(x,\alpha)}\chi(dx)\Big) d\alpha\Big] F^{\theta_{t+s}}(\hat{\gamma})dt. \end{aligned}$$

Since $\theta_s(x,\alpha) \leq 0$ and $F_t^{\theta_s}$ satisfies (5.46), the above integral converges for each $\hat{\gamma}$. By the dominated convergence theorem and the boundedness $F_t^{\theta_s}(\hat{\gamma}) \leq 1$ it follows that $F_{\lambda,\theta_s} \in C_b(\hat{\Gamma})$. Moreover, it can also be understood as the Bochner integral in the latter Banach space. Therefore, $F_{\lambda,\theta}$ can be approximated in $\|\cdot\|$ by the Riemann integral sums centered at rational t, which means that

$$F_{\lambda,\theta_s} \in \mathcal{C}, \text{ for all } s \geq 0 \text{ and } \lambda > 0. \tag{5.71}$$

At the same time, we also have

$$0 < F_{\lambda,\theta_s}(\hat{\gamma}) < \frac{1}{\lambda}, \quad \hat{\gamma} \in \hat{\Gamma}. \tag{5.72}$$

Let $\alpha \in (0,+\infty)$. The continuous differentiability of the map

$$\alpha \to F_{\lambda,\theta_s}(\hat{\gamma} \setminus \hat{x} \cup (x,\alpha)) \in \mathbb{R}$$

for each $\hat{\gamma}$ and $\hat{x} \in \hat{\gamma}$ follows from the dominated convergence theorem.

Lemma 5.3.4. *For each $\lambda > 0, s \geq 0$ and $\theta \in \Theta$, the following holds*

$$LF_{\lambda,\theta_s} = \lambda F_{\lambda,\theta_s} - F^{\theta_s} \tag{5.73}$$

Proof. By (5.45) and (5.70) we have

$$\begin{aligned} LF_{\lambda,\theta_s} &= L\int_0^{+\infty} e^{-\lambda t} F_t^{\theta_s} dt = \int_0^{+\infty} e^{-\lambda t} LF_t^{\theta_s} dt \\ &= \int_0^{+\infty} e^{-\lambda t} \frac{\partial}{\partial t} F_t^{\theta_s} dt = -F^{\theta_s} + \lambda F_{\lambda,\theta_s}, \end{aligned} \tag{5.74}$$

where we have taken into account the upper bound in (5.72). The commutation $L\int = \int L$ can be justified by means of the Lebesgue dominated convergence theorem. □

Lemma 5.3.5. *For each $\theta \in \Theta$ and $s \geq 0$, it follows that $\|\lambda F_{\lambda,\theta_s} - F^{\theta_s}\| \to 0$ and $\|\lambda LF_{\lambda,\theta_s} - LF^{\theta_s}\| \to 0$ as $\lambda \to +\infty$.*

Proof. In view of (5.72), $\{\lambda F_{\lambda,\theta_s} : \lambda > 0\}$ is bounded. By (5.70) we have

$$\lambda F_{\lambda,\theta_s}(\hat{\gamma}) = \int_0^{+\infty} \exp\Big(-t + \int_0^{\varepsilon t} \Big(\int_X \theta_{\alpha+s}(x,0)\chi(dx)\Big) d\alpha\Big) F^{\theta_{\varepsilon t+s}}(\hat{\gamma}) dt,$$

where $\varepsilon := \lambda - 1$.

Then by (5.56) it follows that

$$|\lambda F_{\lambda,\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma})| \leq \int_0^{+\infty} e^{-t} |F_{\varepsilon t}^{\theta_s}(\hat{\gamma}) - F^{\theta_s}(\hat{\gamma})| dt \leq \frac{l_\theta}{\lambda},$$

which yields that $\|\lambda F_{\lambda,\theta_s} - F^{\theta_s}\| \to 0$ as $\lambda \to +\infty$. In the same way, by Proposition 5.3.2 we have, cf. (5.3.2), that

$$\|\lambda LF_{\lambda,\theta_s} - LF^{\theta_s}\| \leq \tilde{\omega}\Big(\frac{1}{\lambda}\Big) \to 0, \lambda \to +\infty,$$

holding for an appropriate continuous $\tilde{\omega}$ such that $\tilde{\omega}(\varepsilon) \to 0$ as $\varepsilon \to 0$. □

Let $\mathcal{D}_0(L)$ denote the linear span of the set $\{F_{\lambda,\theta_s} : \lambda > 0, s \geq 0, \theta \in \Theta\}$. By Lemma 5.3.5 and (5.69) it follows that

$$\mathcal{C}_0 \subset \mathcal{D}_0(L), \tag{5.75}$$

i.e., $\mathcal{C}_0$ is contained in the closure of $\mathcal{D}_0(L)$ in the norm of $C_b(\hat{\Gamma})$.
Hence, $\mathcal{D}_0(L)$ is a dense subset of the Banach space $\mathcal{C}$, see (5.69). Define

$$\|F\|_L = \|F\| + \|LF\|, \qquad F \in \mathcal{D}_0(L), \tag{5.76}$$

where as above $\|\cdot\|$ is the norm of $C_b(\hat{\Gamma})$.

Definition 5.3.6. *By the domain of the Kolmogorov operator L, denoted by $\mathcal{D}(L)$, we mean the closure of $\mathcal{D}_0(L)$ in the graph-norm introduced in (5.76).*

Lemma 5.3.7. *The operator $(L, \mathcal{D}(L))$ is closed and densely defined in $\mathcal{C}$. Its resolvent set contains $(0, +\infty)$ and $\mathcal{C}_0 \subset \mathcal{D}(L)$.*

Proof. In view of (5.71) and (5.75), the $\|\cdot\|-$closure of $\mathcal{D}(L)$ is $\mathcal{C}$. The closedness of $(L, \mathcal{D}(L))$ is immediate, and the inclusion $\mathcal{C}_0 \subset \mathcal{D}(L)$ follows by the second part of Lemma 5.3.5. By (5.73) it follows that the resolvent of L, denoted $R_\lambda(L)$, has the property

$$R_\lambda(L)F^{\theta_s} = F_{\lambda,\theta_s}, \quad \lambda > 0, \quad s \geq 0, \quad \theta \in \Theta,$$

by which and (5.72) we also have that the operator norm of $R_\lambda(L)$ satisfies

$$\|R_\lambda(L)\| \leq \frac{1}{\lambda}$$

as F^{θ_s} form a dense subset of $\mathcal{C}$. This completes the whole proof. □

5.3.4 Solving the Kolmogorov equation.

The result obtained in Lemma 5.3.7 allows one to solve the Kolmogorov equation (2.4) in the following sense.

Theorem 5.3.8. *Let $(L, \mathcal{D}(L))$ and $\mathcal{C}$ be as in Lemma 5.3.7. Then, for each $F \in \mathcal{D}(L)$, there exists a unique continuously differentiable map*

$$[0, +\infty) \ni t \to F_t \in \mathcal{D}(L) \subset C,$$

which solves (2.4) with $F_0 = F$. In particular, for $F = F^{\theta_s}$, with F^{θ_s} as in (5.40), $s \geq 0$ and $\theta \in \Theta$, the solution has the following explicit form, cf. (5.45)

$$F_t(\hat{\gamma}) = \exp\Big(\int_0^t \Big(\int_X \theta_s(x,\alpha)\chi(dx)\Big)d\alpha\Big)F^{\theta_{t+s}}(\hat{\gamma}). \tag{5.77}$$

Proof. By the celebrated Hille-Yosida theorem, see, e.g, Pazy (1983), page 8, [32], $(L, \mathcal{D}(L))$ is the generator of a $\mathcal{C}_0-$semigroup of bounded linear operators $S(t) : \mathcal{C} \to \mathcal{C}$ such that $\|S(t)\| = 1$ and the solution in question is $F_t = S(t)F$, the uniqueness of which is also a standard fact, see Pazy (1983), Theorem 1.3, page 102, [32]. The validity of (5.77) follows by the calculations as in (5.54). □

5.4 The result

5.4.1 The martingale problem

We begin by recalling, see Proposition 5.2.2, that the class of functions $\mathcal{F}_\Theta$, see (5.25), is separating, i.e., if $\mu_1(F) = \mu_2(F)$ for all $F \in \mathcal{F}_\Theta$, then $\mu_1 = \mu_2$, that holds for each pair $\mu_1, \mu_2 \in \mathcal{P}(\hat{\Gamma})$. Next, for $t \geq 0, \mu \in \mathcal{P}(\hat{\Gamma})$ and $\theta \in \Theta$, we determine $\mu^t \in \mathcal{P}(\hat{\Gamma})$ like in definition 4.1.2 by the relation

$$\mu^t(F^\theta) = \mu(F^{\theta_t}). \tag{5.78}$$

Recall that, for a positive Radon measure ϱ on $\hat{X}$, the Poisson measure π with intensity measure ϱ is defined in (5.26). For $t \geq 0$, we then introduce the Poisson measure $\pi_t = \pi_{\varrho_t}$ by defining its intensity measure

$$\varrho_t(d\hat{x}) = \mathbf{1}_{[0,t)}(\alpha) \exp\left(-M(\hat{x})\right)\chi(dx)d\alpha, \tag{5.79}$$

where χ is the same as in (5.1), $d\alpha$ is the Lebesgue measure on $\mathbb{R}_+$, M is as in (5.40), and $\mathbf{1}_{[0,t)}(\cdot)$ in (2.1). Note that $\pi_0(\{\varnothing\}) = 1$ since $\pi_0(F^\theta) = 1$ for all θ. Then we set, see (5.27),

$$\mu_t = \pi_t \star \mu^t, \qquad t \geq 0. \tag{5.80}$$

Let $\delta_{\hat{\gamma}}$ be the Dirac measure centered at a given $\hat{\gamma} \in \hat{\Gamma}$. Then

$$p_t^{\hat{\gamma}} := \pi_t \star \delta_{\hat{\gamma}}^t \tag{5.81}$$

is a transition function, cf. Ethier and Kurtz (1986), page 156, [13]. Indeed, $p_t^{\hat{\gamma}} \in \mathcal{P}(\hat{\Gamma})$, $p_0^{\hat{\gamma}} = \delta_{\hat{\gamma}}$ and the measurability of the map $(t, \hat{\gamma}) \to p_t^{\hat{\gamma}}(B) \in \mathbb{R}, B \in \mathcal{B}(\hat{\Gamma})$, follows by the measurability of $(t, \hat{\gamma}) \to \delta_{\hat{\gamma}}^t(B) \in \mathbb{R}$ and the continuity of $t \to \pi_t(B) \in \mathbb{R}$. In view of the separating property of F^θ, see Proposition 5.2.2, item (iii), the flow property of $\{p_t^{\hat{\gamma}}\}_{t\geq 0}$ can be obtained by showing that

$$p_{t+s}^{\hat{\gamma}}(F^\theta) = \int_{\hat{\Gamma}} p_t^{\hat{\gamma}'}(F^\theta) p_s^{\hat{\gamma}}(d\hat{\gamma}'), \quad t, s \geq 0, \quad \theta \in \Theta. \tag{5.82}$$

By (5.27) we have

$$\begin{aligned} p_t^{\hat{\gamma}}(F^\theta) &= \exp\Big(\int_X \int_0^t \theta_\alpha(x,0)d\alpha\chi(dx)\Big)\delta_{\hat{\gamma}'}^t(F^\theta) \\ &= \exp\Big(\int_X \int_0^t \theta_\alpha(x,0)d\alpha\chi(dx)\Big)F^{\theta_t}(\hat{\gamma}'), \end{aligned}$$

see also (5.78) and (5.40). By the latter formula and (5.79), (5.81) we then get

$$\begin{aligned} \text{RHS}(5.82) &= \exp\Big(\int_X \int_0^t \theta_\alpha(x,0)d\alpha\chi(dx) + \int_X \int_0^s \theta_{t+\alpha}(x,0)d\alpha\chi(dx)\Big)\delta_{\hat{\gamma}'}^s(F^{\theta_t}) \\ &= \exp\Big(\int_X \int_0^{t+s} \theta_\alpha(x,0)d\alpha\chi(dx)\Big)F^{\theta_{t+s}}(\hat{\gamma}) = \text{LHS}(5.82). \end{aligned}$$

As is known, cf. Ethier and Kurtz (1986), Theorem 1.2, page 157 [13], the transition function (5.81) determines a Markov process $\mathcal{X}$ with values in $\hat{\Gamma}$, the finite-dimensional distributions of which are given by the following formula

$$P(\mathcal{X}(s_1) \in B_1, \ldots, \mathcal{X}(s_n) \in B_n) = \int_{\hat{\Gamma}} \int_{B_1} \cdots \int_{B_{n-1}} p^{\hat{\gamma}_{n-1}}_{s_n - s_{n-1}}(B_n) p^{\hat{\gamma}_{n-2}}_{s_{n-1}-s_{n-2}}(d\hat{\gamma}_{n-1}) \times \ldots \times p^{\hat{\gamma}_1}_{s_2 - s_1}(d\hat{\gamma}_2) p^{\hat{\gamma}}_{s_1}(d\hat{\gamma}_1)\mu(d\hat{\gamma}), \tag{5.83}$$

holding for all $n \in \mathbb{N}, 0 < s_1 \leq s_2 \leq \ldots \leq s_n$ and $B_i \in \mathcal{B}(\hat{\Gamma})$. Here $\mu \in \mathcal{P}(\hat{\Gamma})$ is the initial distribution of $\mathcal{X}$. Our aim is to show that such a process is unique up to modifications.

5.4.2 The statement

The process determined by (5.83) describes the stochastic evolution of the population which we consider. To verify whether it is the only one, we have to specify which processes of this kind can be associated to the model defined by the Kolmogorov operator (5.1). As is standard, the corresponding specification is made by their martingale property, see Ethier and Kurtz (1986), Chapt. 4, [13].

Definition 5.4.1. *Let $\mathcal{X}$ be a measurable process on some probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ with values in $\hat{\Gamma}$. Let also $\{\mathfrak{F}_t\}_{t\geq 0}$ be a filtration such that $\mathcal{X}(t)$ and*

$$\int_0^t G(\mathcal{X}(u))du$$

are $\mathfrak{F}_t-$measurable for all t and $G \in B(\hat{\Gamma})$. We say that $\mathcal{X}$ is a solution of the martingale problem for $(L, \mathcal{D}(L))$ if for each $F \in \mathcal{D}(L)$,

$$M(t) := F(X(t)) - \int_0^t (LF)(\mathcal{X}(u))du$$

is a $\mathfrak{F}_t-$martingale. If there exists a solution of the martingale problem for $(L, \mathcal{D}(L))$ and uniqueness holds, we say that the problem is well-posed. In the same way, we define the martingale problem for $(L, \mathcal{D}(L), \mu)$ if the initial distribution $\mu \in \mathcal{P}(\hat{\Gamma})$ is specified.

The process related to the transition function (5.81) solves the martingale problem for $(L, \mathcal{D}(L))$. Its uniqueness will be shown by proving that all other solutions have the same finite-dimensional marginals, i.e., they coincide with those defined in (5.83). We are going also to show that the solution is temporarily ergodic.

Definition 5.4.2. *Let the martingale problem for $(L, \mathcal{D}(L), \mu)$ be well-posed. Then $\mu \in \mathcal{P}(\hat{\Gamma})$ is said to be a stationary distribution if for each n and $0 < s_1 < s_2 < \ldots < s_n$, the $n-$dimensional marginals introduced in (5.83) corresponding to $t + s_1, \ldots, t + s_n$ are independent of $t \geq 0$.*

Definition 5.4.3. *By a solution of (2.3) we understand a map $\mathbb{R}_+ \ni t \to \mu_t \in \mathcal{P}(\hat{\Gamma})$ possessing the following properties:*

(a) for each $F \in C_b(\hat{\Gamma}$, the map $\mathbb{R}_+ \ni t \to \mu_t(F) \in \mathbb{R}$ is measurable;

(b) the equality in (2.3) holds for all $F \in \mathbb{D}(L)$.

If P is as in (5.83), then μ is stationary if and only if

$$\mu = \int_{\hat{\Gamma}} p_s^{\hat{\gamma}} \mu(d\hat{\gamma}),$$

holding for all $t > 0$. Now we can formulate our result.

Theorem 5.4.4. *The martingale problem for $(L, \mathcal{D}(L))$ is well-posed in the sense of Definition 5.4.1. Its solution is defined by finite-dimensional marginals, see (5.83), with the transition function defined in (5.81). If the departure function satisfies $m(\hat{x}) \geq m_0 > 0$, holding for all $\hat{x}$ and some m_0, then there exists a unique stationary distribution $\mu = \pi_\varrho$, which is the Poisson measure with intensity measure*

$$\varrho(d\hat{x}) = \exp\left(-M(\hat{x})\right)\chi(dx)d\alpha. \tag{5.84}$$

Moreover, in this case the solution $\mathcal{X}$ of the martingale problem for $(L, \mathcal{D}(L), \mu)$ is temporarily ergodic in the following sense. Let $\mu_t \in \mathcal{P}(\hat{\Gamma})$ be the law of $\mathcal{X}(t)$, $t \geq 0$. Then $\mu_t \Rightarrow \pi_\varrho$ as $t \to +\infty$.

Proof. The proof of Theorem 5.4.4 is divided into the following steps:

(a) Proving uniqueness.

(b) Showing the stationarity and ergodicity if $m(\hat{x}) \geq m_* > 0$.

The realization of (a) is based on the Fokker-Planck equation (2.3) for this for L. It turns out that its solutions can be obtained explicitly. We are going to do this now.

Lemma 5.4.5. *For each $\mu_0 \in \mathcal{P}(\hat{\Gamma})$, the map $t \to \mu_t$ defined in (5.80) is a unique solution of (2.3).*

Proof. For each $\theta \in \Theta$, the map $t \to \mu_t(F^\theta) = \mu_0(F^{\theta_t})$ is continuous by the dominated convergence theorem and hence measurable. By (5.28) we then have

$$\begin{aligned}
\mu_t(F^\theta) &= \pi_t(F^\theta)\mu_t(F^\theta) = \pi_t(F^\theta)\mu_0(F^{\theta_t}) \\
&= \exp\Big(\int_0^t \Big[\int_X \theta(x,\alpha)e^{-M(x,\alpha)}\chi(dx)\Big]d\alpha\Big)\mu_0(F^{\theta_t}).
\end{aligned}$$

Thus, the map $t \to \mu_t(F^\theta)$ is continuous and hence measurable. Then the measurability of $t \to \mu_t(F)$ for all $F \in C_b(\hat{\Gamma})$ follows by claim (ii) of Proposition 5.2.2. Now we turn to proving the equality in (2.3) for $F = F_{\lambda,\theta}, \theta \in \Theta$, see Definition5.3.6. By (5.80) and (5.45) for $s, t \geq 0$ and $\theta \in \Theta$, we have

$$\begin{aligned}
\mu_t(F^\theta s) &= \exp\Big(\int_0^s \Big(\int_X \theta_\alpha(x,0)\chi(dx)\Big)d\alpha\Big)\pi_t(F^{\theta_s})\mu^t(F^{\theta_s}) \\
&= \exp\Big(\int_0^s \Big(\int_X \theta_\alpha(x,0)\chi(dx)\Big)d\alpha \\
&+ \int_0^t \Big(\int_X \theta_{s+\alpha}(x,0)\chi(dx)\Big)d\alpha\Big)\mu_0\Big(F^{\theta_{s+t}}\Big) \\
&= \exp\Big(\int_0^{s+t} \Big(\int_X \theta_\alpha(x,0)\chi(dx)\Big)d\alpha\Big)\mu_0\Big(F^{\theta_{s+t}}\Big) = \mu_0\Big(F^\theta_{s+t}\Big).
\end{aligned}$$

Then by Fubini's theorem and the latter fact we get

$$\begin{aligned}
\mu_t(F_{\lambda,\theta}) - \mu_0(F_{\lambda,\theta}) &= \int_0^{+\infty} e^{-\lambda s}\Big[\mu_t(F^\theta_s) - \mu_0(F^\theta_s)\Big]ds \\
&= \int_0^{+\infty} e^{-\lambda s}\mu_0\Big(F^\theta_{t+s} - F^\theta_s\Big)ds \\
&= \int_0^{+\infty} e^{-\lambda s}\int_0^t \frac{\partial}{\partial u}\mu_0\Big(F^\theta_{s+u}\Big)dsdu \\
&= \int_0^{+\infty} e^{-\lambda s}\int_0^t \frac{\partial}{\partial s}\mu_u(F^\theta_s)dsdu \\
&= \int_0^t \mu_u\Big(\int_0^{+\infty} e^{-\lambda s}\frac{\partial}{\partial s}F^\theta_s ds\Big)du = \int_0^t \mu_u(LF_{\lambda,\theta})du,
\end{aligned}$$

where we have used also (5.74). Now we prove uniqueness by applying arguments similar to those used in Costantini and Kurtz (2015), Lemma 2.11. Assume that a map $t \to \mu_t$ satisfies condition (a) Definition 5.4.3 and F, $G \in C_b(\hat{\Gamma})$ are such that

$$\mu_t(F) - \mu_0(F) = \int_0^t \mu_s(G)ds,$$

holding for all $t \geq 0$. Then the map $t \to \mu_t(F)$ is almost everywhere differentiable and

$$d\mu_t(F) = \mu_t(G)dt.$$

Then integrating by parts we get

$$-\lambda\int_0^t e^{-\lambda s}\mu_s(F)ds = e^{-\lambda t}\mu_t(F) - \mu_0(F) - \int_0^t e^{-\lambda s}\mu_s(G)ds,$$

which yields

$$\mu_0(F) = e^{-\lambda t}\mu_t(F) + \int_0^t e^{-\lambda s}\Big[\lambda\mu_s(F) - \mu_s(G)\Big]ds,$$

holding for all $t, \lambda > 0$. Passing here to the limit $t \to +\infty$, for $F = F_{\lambda,\theta}$ and $G = LF_{\lambda,\theta}$, see (2.3), we arrive at

$$\mu_0(F_{\lambda,\theta}) = \int_0^{+\infty} e^{-\lambda s}\mu_s(\lambda F_{\lambda,\theta} - LF_{\lambda,\theta})ds = \int_0^{+\infty} e^{-\lambda s}\mu_s(F^\theta)ds, \tag{5.85}$$

see (5.73). Assume now that (2.3) has two solutions, μ_t and $\tilde{\mu}_t$, satisfying the same initial condition $\mu_t|_{t=0} = \tilde{\mu}_t|_{t=0} = \mu_0$. By (5.85) the Laplace transforms of both maps $t \to \mu_t(F^\theta)$ and $t \to \tilde{\mu}_t(F^\theta)$ coincide, which yields $\mu_t(F^\theta) = \tilde{\mu}_t(F^\theta)$ holding for each t and all F^θ, $\theta \in \Theta$. Then the uniqueness in question follows by Proposition 5.2.2. This completes the whole proof. □

The existence of a solution of the martingale problem for $(L, \mathcal{D}(L))$ was shown by the very construction of the finite-dimensional marginals of $\mathcal{X}$ in (5.83). To prove uniqueness we use the following fact, see Ethier and Kurtz (1986), Proposition 4.2, page 184. Given $\mu \in \mathcal{P}(\hat{\Gamma})$, let $\mathcal{X}$ and $\mathcal{X}'$ be solutions of the martingale problem for $(L, \mathcal{D}(L), \mu)$ whose onedimensional marginals, μ_t and $\mu_t{}'$, coincide for all $t \geq 0$. Then all their finite-dimensional marginals coincide and hence the problem is well-posed. Clearly, both μ_t and μ_t' solve the Fokker-Planck equation with the initial condition μ. Then they coincide by Lemma 5.4.5. This yields well-posedness.

Now we show the stated ergodicity. If m satisfies $m(\hat{x}) \geq m_0 > 0$, then $M(x,\alpha) \geq m_0\alpha$, see (5.40), which for θ as in (5.40) yields

$$\forall_{t>0} |\theta_t(\hat{x})| \leq e^{-tm_0}.$$

By the continuity of the map $t \to \mu(F^{\theta_t})$ we then get

$$\begin{aligned}\mu_t(F^\theta) &= \exp\Big(\int_0^t \Big[\theta_\alpha(x,0)\chi(dx)\Big]d\alpha\Big)\mu(F^{\theta_t}) \qquad (5.86)\\ &\to \exp\Big(\int_0^{+\infty}\Big[\int_X \theta_\alpha(x,0)\chi(dx)\Big]d\alpha\Big), \quad t \to +\infty.\end{aligned}$$

By claim (iv) of Proposition 5.2.2 this yields $\mu_t \Rightarrow \pi_\varrho$, see (5.84), holding for each initial $\mu \in \mathcal{P}(\hat{\Gamma})$. Clearly, $\mu_t = \pi_\varrho$ if $\mu = \pi_\varrho$, which means that π_ϱ is a stationary state. If there exists another stationary state, say μ', then (5.86) fails to hold for $\mu = \mu'$, which contradicts the convergence just established. This completes the proof of Theorem 5.4.4.

□

Chapter 6

Appendix

Here we prove that $\tilde{r}$ introduced in (5.3) satisfies (5.4). During the whole proof, we deal with the function $l(\alpha) = \alpha + \frac{1}{\alpha}$, for $\alpha > 0$. In case $\alpha_2 = 0$, and $\alpha_1, \alpha_2 \leq 1$ we have $\omega(\alpha_1) = \alpha_1$, $\omega(\alpha_2) = \alpha_2$, $r(\alpha_1, \alpha_3) = |\alpha_3 - \alpha_1|$. The triangle inequality turns into true statement:

$$\alpha_1 \leq |\alpha_3 - \alpha_1| + \alpha_3.$$

For $\alpha_3 > 1$ we can have:

$$\alpha_3 - \alpha_1 \leq \alpha_1 + \frac{1}{\alpha_3}$$

or

$$\alpha_3 - \alpha_1 > \alpha_1 + \frac{1}{\alpha_3}.$$

For the first one we get:

$$\omega(\alpha_1) = \alpha_1 \leq \alpha_3 - \alpha_1 + \frac{1}{\alpha_3} = r(\alpha_1, \alpha_3) + \omega(\alpha_3).$$

Then

$$2\alpha_1 \leq l(\alpha_3).$$

The equality holds only for $\alpha_1 = \alpha_3 = 1$. In the second case we have true inequality $\alpha_1 \leq \alpha_1 + \frac{2}{\alpha_3}$.

For $\alpha_1 > 1$, $\alpha_3 \leq 1$, we have two possibilities:

$$\alpha_1 - \alpha_3 \leq \alpha_3 + \frac{1}{\alpha_1}$$

or

$$\alpha_1 - \alpha_3 > \alpha_3 + \frac{1}{\alpha_1}.$$

In the first case we have

$$\omega(\alpha_1) = \frac{1}{\alpha_1} \leq \alpha_1 = \alpha_1 - \alpha_3 + \alpha_3,$$

which yields (5.4). In the second case we have:

$$\omega(\alpha_1) = \frac{1}{\alpha_1} \leq \alpha_3 + \frac{1}{\alpha_1} + \alpha_3.$$

It remained to consider the situation when $\alpha_3 \geq \alpha_3 > 1$ and $1 < \alpha_1 < \alpha_3$.
When $\alpha_1 \geq \alpha_3 > 1$ for $\alpha_1 - \alpha_3 \leq \frac{1}{\alpha_1} + \frac{1}{\alpha_1}$ we have:

$$\omega(\alpha_1) = \frac{1}{\alpha_1} \leq \frac{1}{\alpha_3} \leq \alpha_1 - \alpha_3 + \frac{1}{\alpha_3},$$

which proves (5.4) for this case.
When $\alpha_3 \geq \alpha_3 > 1$ for $\alpha_1 - \alpha_3 > \frac{1}{\alpha_1} + \frac{1}{\alpha_1}$ we have:

$$\omega(\alpha_1) = \frac{1}{\alpha_1} \leq \frac{1}{\alpha_3} \leq \alpha_1 + \frac{1}{\alpha_3} + \frac{1}{\alpha_3},$$

which yields (5.4).
When $1 < \alpha_1 < \alpha_3$ for $\alpha_1 - \alpha_3 \leq \frac{1}{\alpha_1} + \frac{1}{\alpha_1}$ we have to prove :

$$\frac{1}{\alpha_1} \leq \alpha_3 - \alpha_1 + \frac{1}{\alpha_3},$$

which is equivalent to $l(\alpha_1) \leq l(\alpha_3)$. The latter follows by $\alpha_3 \geq \alpha_1 > 1$ as l is increasing. For $\alpha_3 - \alpha_1 > \frac{1}{\alpha_1} + \frac{1}{\alpha_3}$ the proof of (5.4) is immediate.

Now we consider the case $0 < \alpha_1 < \alpha_2$. If $\alpha_2 \leq 1$ and $\alpha_3 = 0$, then

$$\alpha_2 - \alpha_1 \leq \omega(\alpha_1) + \omega(\alpha_2) = \alpha_1 + \alpha_2.$$

For $\alpha_3 \in (0, 1]$, it follows that $\tilde{r}(\alpha_3, \alpha_i) = |\alpha_3 - \alpha_i|$. Then (5.4) turns into the triangle inequality for $|\cdot|$.
The same is true also for $\alpha_3 > 1$ such that $\alpha_3 - \frac{1}{\alpha_3} \leq 2\alpha_1$. For

$$2\alpha_1 < \alpha_3 - \frac{1}{\alpha_3} \leq 2\alpha_2,$$

the right-hand side of (5.4) is $\alpha_3 - \alpha_2 + \alpha_1 + \frac{1}{\alpha_3}$. Then $\alpha_2 - \alpha_1 \leq \text{RHS}(5.4)$ turns into $2(\alpha_2 - \alpha_1) \leq l(\alpha_3)$, which holds since $2(\alpha_2 - \alpha_1) \leq 2 < l(\alpha_3)$ for $\alpha_3 > 1$. For $2\alpha_2 < \alpha_3 - \frac{1}{\alpha_3}$, the right-hand side of (5.4) is $\alpha_1 + \alpha_2 + 2/\alpha_3$, which is bigger than $\alpha_2 - \alpha_1$.

Consider now $0 < \alpha_1 \leq 1 < \alpha_2$ and $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1$. The latter means that

$$\tilde{r}(\alpha_2, \alpha_1) = \alpha_2 - \alpha_1.$$

For $\alpha_3 = 0$, the right-hand side of (5.4) is $\alpha_1 + \frac{1}{\alpha_2}$, and the latter turns into $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1$, which holds in this case. The same is true for $2\alpha_3 \leq \alpha_2 - \frac{1}{\alpha_2}$. For $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_3 \leq 2\alpha_1$, the right-hand side of (5.4) is $\alpha_2 - \alpha_3 + \alpha_1 - \alpha_3$, which is

bigger than $\alpha_2 - \alpha_1$ as $\alpha_3 \leq \alpha_1$.
For $\alpha_1 \leq \alpha_3 \leq 1$, the right-hand side of (5.4) is

$$\alpha_2 - \alpha_3 + \alpha_3 - \alpha_1 = \text{LHS}(5.4),$$

Next, consider $1 < \alpha_3 \leq \alpha_2$, where

$$\alpha_3 - \frac{1}{\alpha_3} \leq \alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1.$$

For $\alpha_2 - \frac{1}{\alpha_2} \leq l(\alpha_3)$, the right-hand side of (5.4) is

$$\alpha_2 - \alpha_3 + \alpha_3 - \alpha_1 = \text{RHS}(5.4).$$

The case of $\alpha_2 - \frac{1}{\alpha_2} > l(\alpha_3) > 2$ is impossible since $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1 \leq 2$. For $\alpha_3 > \alpha_2$ such that $\alpha_3 - \frac{1}{\alpha_3} \leq 2\alpha_1$ and $\alpha_3 - \frac{1}{\alpha_3} \leq l(\alpha_2)$, we have $\tilde{r}(\alpha_3, \alpha_1) = \alpha_3 - \alpha_1$ and $\tilde{r}(\alpha_3, \alpha_2) = \alpha_3 - \alpha_2$. Then the right-hand side of (5.4) is

$$\alpha_3 - \alpha_1 + \alpha_3 - \alpha_2 = 2\alpha_3 - (\alpha_2 + \alpha_1),$$

which is bigger than $\alpha_2 - \alpha_1$. The case of $\alpha_3 - \frac{1}{\alpha_3} > l(\alpha_2)$ is impossible for $\alpha_3 - \frac{1}{\alpha_3} \leq 2\alpha_1$. For $\alpha_3 > \alpha_2$ such that $\alpha_3 - \frac{1}{\alpha_3} > 2\alpha_1$ and $\alpha_3 - \frac{1}{\alpha_3} \leq l(\alpha_2)$, we have $\tilde{r}(\alpha_3, \alpha_1) = \alpha_1 + \frac{1}{\alpha_3}$ and $\tilde{r}(\alpha_3, \alpha_2) = \alpha_3 - \alpha_2$. Then the right-hand side of (2.2) is

$$l(\alpha_3) - \alpha_2 + \alpha_1,$$

which yields (5.4) in the form $2(\alpha_2 - \alpha_1) \leq l(\alpha_3)$. By $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1$, $\alpha_2 > 1$, we have that $2(\alpha_2 - \alpha_1) \leq 2\sqrt{1 + \alpha_1^2}$, whereas $\alpha_3 - \frac{1}{\alpha_3} > 2\alpha_1$ yields $l(\alpha_3) > l(\alpha_*) = 2\sqrt{1 + \alpha_1^2}$, which proves (5.4) in this case. Here α_* is the positive solution of $\alpha - \frac{1}{\alpha} = 2\alpha_1$.
For $\alpha_3 > \alpha_2$ such that $\alpha_3 - \frac{1}{\alpha_3} > 2\alpha_1$ and $\alpha_3 - \frac{1}{\alpha_3} > l(\alpha_2)$, we have $\tilde{r}(\alpha_3, \alpha_1) = \alpha_1 + \frac{1}{\alpha_3}$ and $\tilde{r}(\alpha_3, \alpha_2) = \frac{1}{\alpha_3} + \frac{1}{\alpha_2}$. Then (5.4) turns to

$$\alpha_2 - \alpha_1 \leq \alpha_1 + 2/\alpha_3 + \frac{1}{\alpha_2},$$

which holds as $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1$.

Consider now $0 < \alpha_1 \leq 1 < \alpha_2$ and $\alpha_2 - \frac{1}{\alpha_2} > 2\alpha_1$. The latter means that $\tilde{r}(\alpha_1, \alpha_2) = \alpha_1 + \frac{1}{\alpha_2}$. For $\alpha_3 \in [0, \alpha_1]$, we have that $\tilde{r}(\alpha_1, \alpha_3) = \alpha_1 - \alpha_3$ and $\tilde{r}(\alpha_2, \alpha_3) = \alpha_3 + \frac{1}{\alpha_2}$. Hence, (5.4) turns into equality.
For $\alpha_3 \in (\alpha_1, 1]$ such that $\alpha_2 - \frac{1}{\alpha_2} > 2\alpha_3$, we have the right-hand side of (5.4) in the following form

$$\alpha_3 - \alpha_1 + \alpha_3 + \frac{1}{\alpha_2},$$

which is bigger that $\tilde{r}(\alpha_1, \alpha_2)$ since $\alpha_1 < \alpha_3$.
For $\alpha_3 \in (\alpha_1, 1]$ such that $\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_3$, we have that the right-hand side of (5.4) is

$$\alpha_3 - \alpha_1 + \alpha_2 - \alpha_3 = \alpha_2 - \alpha_1 > \tilde{r}(\alpha_1, \alpha_2).$$

Consider now $\alpha_3 > 1$ such that $\alpha_3 - \frac{1}{\alpha_3} \le 2\alpha_1$, which means that $\alpha_3 < \alpha_2$ and $\tilde{r}(\alpha_1, \alpha_3) = \alpha_3 - \alpha_1$. For $\tilde{r}(\alpha_2, \alpha_3) = \alpha_2 - \alpha_3$, the right-hand side of (5.4) is $\alpha_2 - \alpha_3 + \alpha_3 - \alpha_1 \ge \tilde{r}(\alpha_2, \alpha_3)$. For $\tilde{r}(\alpha_2, \alpha_3) = \frac{1}{\alpha_3} + \frac{1}{\alpha_2}$, the right-hand side of (5.4) is

$$\alpha_3 - \alpha_1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_3};$$

hence, (5.4) turns into $2\alpha_1 \le l(\alpha_3)$, which holds since $\alpha_1 \le 1$ and $l(\alpha_3) \ge 2$. For $\alpha_3 > 1$ such that $\alpha_3 - \frac{1}{\alpha_3} > 2\alpha_1$, we have that $\tilde{r}(\alpha_1, \alpha_3) = \alpha_1 + \frac{1}{\alpha_3}$. Then (5.4) turns into

$$\frac{1}{\alpha_2} \le \frac{1}{\alpha_3} + \tilde{r}(\alpha_2, \alpha_3), \tag{6.1}$$

which clearly holds for $\alpha_3 \le \alpha_2$, and also for $\alpha_3 > \alpha_2$, where for $\tilde{r}(\alpha_2, \alpha_3) = \alpha_3 - \alpha_2$ it turns into $l(\alpha_2) \le l(\alpha_3)$ – which is true as $l(\alpha)$ is increasing for $\alpha > 1$. For $\tilde{r}(\alpha_2, \alpha_3) = \frac{1}{\alpha_3} + \frac{1}{\alpha_2}$, the validity (6.1) is immediate.

Let us consider now the case of $1 < \alpha_1 < \alpha_2$ and $l(\alpha_1) \le \alpha_2 - \frac{1}{\alpha_2}$, where $\tilde{r}(\alpha_2, \alpha_1) = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$. For $\alpha_3 \le 1$ such that $2\alpha_3 \le \alpha_1 - \frac{1}{\alpha_1}$, we have $\tilde{r}(\alpha_3, \alpha_i) = \alpha_3 + \frac{1}{\alpha_i}, i = 1, 2$. Then (5.4) obviously holds.
For $\alpha_3 \le 1$ satisfying $\alpha_1 - \frac{1}{\alpha_1} < 2\alpha_3 \le \alpha_2 - \frac{1}{\alpha_2}$, we have $\tilde{r}(\alpha_3, \alpha_1) = \alpha_1 - \alpha_3$ and $\tilde{r}(\alpha_3, \alpha_2) = \alpha_3 + \frac{1}{\alpha_2}$. Hence, (5.4) turns into equality in this case. The remaining case $\alpha_2 - \frac{1}{\alpha_2} < 2\alpha_3 \le 2$ is impossible since $\alpha_2 - \frac{1}{\alpha_2} \ge l(\alpha_1) > 2$.
For $\alpha_3 > 1$ such that $2\alpha_3 \le \alpha_1 - \frac{1}{\alpha_1}$, we have $\tilde{r}(\alpha_3, \alpha_i) = \frac{1}{\alpha_3} + \frac{1}{\alpha_i}, i = 1, 2$. Then (5.4) obviously holds. For $\alpha_3 \in (1, \alpha_1]$ satisfying $\alpha_1 - \frac{1}{\alpha_1} < 2\alpha_3 \le \alpha_2 - \frac{1}{\alpha_2}$, (5.4) turns into

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \le \alpha_1 - \alpha_3 + \frac{1}{\alpha_2} + \frac{1}{\alpha_3},$$

which holds for $\alpha_3 \le \alpha_1$ as the function $\alpha - \frac{1}{\alpha}$ is increasing. For $\alpha_3 \in (\alpha_1, \alpha_2]$ such that $\alpha_3 - \frac{1}{\alpha_3} \le l(\alpha_1) < l(\alpha_3) < \alpha_2 - \frac{1}{\alpha_2}$, the right-hand side of (5.4) is $\alpha_3 - \alpha_1 + \frac{1}{\alpha_3} + \frac{1}{\alpha_2}$, and hence the latter turns into $l(\alpha_1) < l(\alpha_3)$. For $\alpha_3 \in (\alpha_1, \alpha_2]$ satisfying $\alpha_3 - \frac{1}{\alpha_3} \le l(\alpha_1)$ and $\alpha_2 - \frac{1}{\alpha_2} \le l(\alpha_3)$, we have (5.4) in the form $\frac{1}{\alpha_2} + \frac{1}{\alpha_1} \le \alpha_2 - \alpha_3 + \alpha_3 - \alpha_1$, which holds as $l(\alpha_1) < \alpha_2 - \frac{1}{\alpha_2}$. For $\alpha_3 \in (\alpha_1, \alpha_2]$ satisfying $l(\alpha_1) < \alpha_3 - \frac{1}{\alpha_3}$ and $\alpha_2 - \frac{1}{\alpha_2} \le l(\alpha_3)$, the right-hand side of (5.4) is

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_3} + \alpha_2 - \alpha_3$$

which is bigger than $\tilde{r}(\alpha_1, \alpha_2)$ as $\alpha - \frac{1}{\alpha}$ is increasing. For $\alpha_3 > \alpha_2$ such that $\alpha_3 - \frac{1}{\alpha_3} \le l(\alpha_2)$, the right-hand side of (5.4) is

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_3} + \alpha_3 - \alpha_2.$$

Hence, (5.4) holds as $l(\alpha_2) < l(\alpha_3)$. For $\alpha_3 - \frac{1}{\alpha_3} > l(\alpha_2)$, (5.4) turns into

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \le \frac{1}{\alpha_1} + \frac{1}{\alpha_3} + \frac{1}{\alpha_3} + \frac{1}{\alpha_2}.$$

Finally, let us consider the case of $1 < \alpha_1 < \alpha_2$ and $l(\alpha_1) > \alpha_2 - \frac{1}{\alpha_2}$, where $\tilde{r}(\alpha_2, \alpha_1) = \alpha_2 - \alpha_1$. For $\alpha_3 \leq 1$ such that $2\alpha_3 \leq \alpha_1 - \frac{1}{\alpha_1}$, it follows that $\tilde{r}(\alpha_i, \alpha_3) = \alpha_3 + \frac{1}{\alpha_i}, i = 1, 2$. Then (5.4) turns into

$$\alpha_2 - \frac{1}{\alpha_2} \leq l(\alpha_1) + 2\alpha_3,$$

which evidently holds in this case. For $\alpha_1 - \frac{1}{\alpha_1} < 2\alpha_3 \leq \alpha_2 - \frac{1}{\alpha_2}$, we have that $\tilde{r}(\alpha_1, \alpha_3) = \alpha_1 - \alpha_3$ and $\tilde{r}(\alpha_2, \alpha_3) = \alpha_3 + \frac{1}{\alpha_2}$. Then (5.4) turns into

$$\alpha_2 - \frac{1}{\alpha_2} \leq 2\alpha_1,$$

which is the case for $2\alpha_1 > l(\alpha_1) > \alpha_2 - \frac{1}{\alpha_2}$.
For $\alpha_3 \leq 1$ satisfying $2\alpha_3 > \alpha_2 - \frac{1}{\alpha_2}$, it follows that $\tilde{r}(\alpha_i, \alpha_3) = \alpha_i - \alpha_3, i = 1, 2$. Then (5.4) turns into

$$2\alpha_3 \leq 2\alpha_1.$$

which is obviously the case.
For $\alpha_3 \in (1, \alpha_1]$ such that $l(\alpha_3) \leq \alpha_1 - \frac{1}{\alpha_1}$, we have that $\tilde{r}(\alpha_i, \alpha_3) = \frac{1}{\alpha_i} + \frac{1}{\alpha_3}, i = 1, 2$. Then (5.4) amounts to $\alpha_2 - \frac{1}{\alpha_2} \leq l(\alpha_1) + 2/\alpha_3$, which obviously holds.
For $\alpha_1 - \frac{1}{\alpha_1} < l(\alpha_3) \leq \alpha_2 - \frac{1}{\alpha_2}$, we have that $\tilde{r}(\alpha_1, \alpha_3) = \alpha_1 - \alpha_3$ and $\tilde{r}(\alpha_2, \alpha_3) = \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$. Then (5.4) amounts to

$$\alpha_1 - \alpha_3 + \alpha_1 + \frac{1}{\alpha_3} \geq \alpha_2 - \frac{1}{\alpha_2},$$

which is the case for

$$\alpha_1 - \alpha_3 + \alpha_1 + \frac{1}{\alpha_3} \geq \alpha_1 + \frac{1}{\alpha_3} \geq l(\alpha_1) > \alpha_2 - \frac{1}{\alpha_2}.$$

For $\alpha_2 - \frac{1}{\alpha_2} < l(\alpha_3)$, we have $\tilde{r}(\alpha_i, \alpha_3) = \alpha_i - \alpha_3$, $i = 1, 2$. Then (5.4) is

$$\alpha_2 - \alpha_1 \leq \alpha_2 - \alpha_3 + \alpha_1 - \alpha_3,$$

which obviously holds as $\alpha_3 \leq \alpha_1$.

Now we consider $\alpha_3 \in (\alpha_1, \alpha_2]$. For $\alpha_3 - \frac{1}{\alpha_3} \leq l(\alpha_1)$, we have $\tilde{r}(\alpha_1, \alpha_3) = \alpha_3 - \alpha_1$ and $\tilde{r}(\alpha_2, \alpha_3) = \alpha_2 - \alpha_3$, which yields equality in (5.4).
Recall that $\alpha_2 - \frac{1}{\alpha_2} < l(\alpha_1)$; hence, $\alpha_3 - \frac{1}{\alpha_3} > l(\alpha_1)$ is impossible for $\alpha_3 \leq \alpha_2$.

It remains to consider $\alpha_3 > \alpha_2$. For $\alpha_3 - \frac{1}{\alpha_3} \leq l(\alpha_1)$, we have that $\tilde{r}(\alpha_i, \alpha_3) = \alpha_3 - \alpha_i$, $i = 1, 2$. Then (5.4) takes the form

$$\alpha_2 - \alpha_1 \leq 2\alpha_3 - \alpha_1 - \alpha_2,$$

which obviously holds in this case. For $l(\alpha_1) < \alpha_3 - \frac{1}{\alpha_3} \leq l(\alpha_2)$, (5.4) amounts to $2\alpha_2 \leq l(\alpha_3) + l(\alpha_1)$, which holds as $l(\alpha_1) > \alpha_2 - \frac{1}{\alpha_2}$ (assumed) and $l(\alpha_3) > \alpha_2 + \frac{1}{\alpha_2}$

for $\alpha_3 > \alpha_2$. For $\alpha_3 - \frac{1}{\alpha_3} > l(\alpha_2)$, we have that $\tilde{r}(\alpha_3, \alpha_i) = \frac{1}{\alpha_3} + \frac{1}{\alpha i}$, $i = 1, 2$. Then (5.4) turns into

$$\alpha_2 - \alpha_1 \leq 2/\alpha_3 + \frac{1}{\alpha_2} + \frac{1}{\alpha_1},$$

which holds since $l(\alpha_1) > \alpha_2 - \frac{1}{\alpha_2}$.
This completes the whole proof of Proposition 5.1.1.

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