Maria Curie - Skłodowska University
in Lublin
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Dissertation

Coalescence processes in continuum with applications

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Lublin, 2021

In memory of Dr Igor Omelyan, who untimely passed away.
Without His help and mentorship, the numerical aspects of this work would never arise.

I wish to express my gratitude to Professor Jurij Kozicki
for His continuous and unbounded support $\mathcal{C}_{ubs}$.
Professor, every time I was about to stray from the path leading here,
You were there, calmly pointing the way.

Finally, I would like to express my thanks to friends and family
for all Your kindness, understanding and faith in me.

Contents

1. Introduction

There exists a broad spectrum of publications dedicated to studying dynamics of large systems of various kinds performed at different levels of mathematical sophistication, see, e.g., [8, 11, 12, 28, 35]. Their common feature, however, is that the size of the system under consideration – as well as the complexity of interactions persistent therein – predetermine the statistical/probabilistic character of the theory developed.

The random motion of infinite systems in the course of which the constituents can merge, attracts considerable attention. The Arratia flow introduced in [1] provides an example of this kind. In recent years, it has been extensively studied, see [6, 16, 17, 25] and the references therein. In Arratia's model, an infinite number of Brownian particles move in $\mathbb{R}$ independently up to their collision, then merge and move together as single particles. Correspondingly, the description of this motion (and its modifications) is performed in terms of stochastic (diffusion) processes. In this work, an alternative look at this kind of motion is proposed, basing on the Kawasaki model [4, 7, 24], in which point particles perform random walks (jumps) in $\mathbb{R}^d$, $d \geq 1$ with repulsion. One of the main aims of the present work is developing and studying similar models that describe this kind of walks accompanied by coalescence. It is done by introducing an individual-based model of an infinite particle system placed in $\mathbb{R}^d$, in which two point particles, located at x and y, merge into a particle, located at $z \neq x, y$, with intensity $c_1(x, y; z)$. Thereafter, the new particle participates in the motion of this kind. In some cases, we also include the dependence of the intensity on the rest of the particles. This is realized in Sections 3 and 4.

The phase space of such a system is the set Γ of all locally finite configurations $\gamma \subset \mathbb{R}^d$, see [4, 7, 14, 18, 19]. In our context, it is introduced in Section 2.1. As is usual in the approach we follow, the states of the system are probability measures on Γ, the set of which will be denoted as $\mathcal{P}(\Gamma)$. The considerations are restricted to the case of sub-Poissonian states, which are defined in Section 2.2. The description of their evolution $\mu_0 \to \mu_t$ is based on the relation $\mu_t(F_0) = \mu_0(F_t)$ where $F_0 : \Gamma \to \mathbb{R}$ is supposed to belong to a measure-defining class of functions, $\mu(F) := \int F d\mu$ and the evolution $F_0 \to F_t$ is defined by the corresponding Kolmogorov equation (2.20). The precise form of the proposed and studied model is given by (3.1) in Section 3.

A similar individual-based approach can be found e.g. in [33, 34] for describing phytoplankton dynamics. There, however, only finite particle systems are considered and no interaction between the particles is directly taken into account – its description is done in a mean-field like way by employing aggregated parameters. Similar models with merging in a single type population are used to describe predation in marine ecology [9]. The model studied in this work fits also in the framework of the coagulation-fragmentation theory, see e.g. [3].

In view of the mentioned applications, my second aim here was to prepare the proposed model to possible modifications as well as developing the corresponding numerical setting. As is typical for theories of this kind, the microscopic (individual-based) description provides a kind of general picture based on existential results obtained by analytic methods in suitable Banach spaces, whereas more detailed information can be obtained only by numerical tools. Most of them are tailored to treat classical integro-differential equations of various kinds, and thus are barely applicable in infinite-dimensional Banach spaces. Therefore, it might be quite natural to pass to the mesoscopic description based on kinetic equations, which rigorously can be done by a scaling procedure, cf. [2, 32]. For our model, it is performed in Section 5, resulting in the corresponding kinetic equation (5.5). For this equation, existence and uniqueness of solutions is also proven, but more importantly, it provides a numerically treatable approximation of the dynamics of the original system. In the last section, an algorithm for finding numerical solutions to this equation is developed, complemented by the analysis of the results of the performed simulations. They shed some light on the details of the system behaviour, like existence of non-trivial steady states or emergence of propagating spatial inhomogeneities.

The dissertation consists of four parts. First, in Section 2 some basic facts and tools are provided. Its first part consists mainly of discussing how metrics can be introduced on configuration spaces. Relationship between the Prohorov metric and Euclidean-type ones is given. Most of the facts proven there are the effect of recent work and are presented here for the first time. In this part, spaces of simple configurations and multi-configurations are precisely distinguished, which is not the case for the rest of the work, as this distinction does not play crucial role in further considerations, which is explained in the next part. There, basic notions and tools are given, which are then used in the next sections. It is a compilation of facts that can be found in articles treating similar models like [14, 7, 4, 21, 18], see also [19] for more detailed background. The same holds true for the last part of the first section, where an introduction to the general framework of constructing dynamics, which is used within the work, is given.

Sections 3 and 4 are devoted to the construction of microscopic dynamics of the model of coalescing repulsive jumps. The contents of these two sections are published in [23]. In the first part, the model is defined using the framework described in Section 2.3. It is done by precising the exact form of operator L in equation (2.20). Then, the corresponding equation for correlation functions of states is derived. In the second part, the main results corresponding to the dynamics of the system studied are given in the form of two theorems. Theorem 3.2 covers existence and uniqueness of the classical solution to a corresponding equation on the level of correlation functions (2.23) for a finite time horizon in a scale of adequate Banach spaces. The most technically involved and definitely the most laborious result presented within this dissertation is Theorem 3.3, which shows that the result from Theorem 3.2 can be uniquely identified with a state of the system. The proof of this theorem is quite complicated, it involves an additional auxiliary model, as well as its pre-dual and local evolution that are used to prove a required positivity property (4.1). Section 4 is entirely devoted to this proof.

In Section 5 a scaling from micro- to mesoscopic level of description is performed. It results in a corresponding kinetic equation which describes the evolution of density of Poisson state that approximate the actual sub-Poissonian state of the system at the microscopic level. The main result of this part is Theorem 5.4, which proves the continuity of the performed scaling in corresponding scale of Banach spaces. This result was communicated in [22] and also in [31] for a special case of coalescence kernels. However, the proof was not given there, and a different formulation that utilizes a notion of Poisson approximability was used, that in the view of Remark 5.1 should be reconsidered. Next part discuss the similar results for an extension of model discussed, where coalescence is endowed with a repulsion term similar as jumps. It actually was studied earlier in [30]. The existence and uniqueness of local in time solution to kinetic equation is also proven. This result covers the case of corresponding equation obtained in the first part, for the main model studied within this work.

The last part of the dissertation, Section 6, describes the algorithm devised for solving the kinetic equation derived in Section 5 for a specific choice of jump and coalescence intensities. It is based on applying adequate boundary conditions, automatically adjusting system size and using Runge-Kutta method for numerical integration, some details are given in Section 6.1. The contents of this section are the results of cooperation with Dr. Igor Omelyan in years 2018-19. The most interesting results of performed simulations, including emergence and propagation of spatial inhomogeneity, as well as possible existence of non-trivial stationary states, are presented in Section 6.2. Most of the results of Section 6 are published in [22] and [29]. Some additional, unpublished results are given as well.

To summarize, my results presented in this work are:

(1) comparing Prohorov and Euclidean-based metrics on configurations spaces, proving some basic metric properties (Section 2.1),
(2) introducing the model of coalescing random jumps (Section 3.1),
(3) proving the existence and uniqueness of local in time microscopic dynamics (Sections 3 and 4),
(4) passing to the mesoscopic level by continuous scaling (Section 5.1),
(5) introducing extension of the model (Section 5.2),
(6) proving the existence and uniqueness of local in time solutions to corresponding kinetic equations for both models (Section 5.2),
(7) elaborating numerical algorithm for finding solutions to the kinetic equation for a special case of the coalescence kernel (Section 6.1),
(8) performing numerical simulations of the system dynamics in several interesting cases and analysis of the results (Section 6.2).

2. Configuration spaces

The basic notion used in this work is the configuration space. Its elements are called configurations. It allows one to deal with systems of many (usually infinitely many) particles.

This section is devoted to providing some introductory information regarding configuration spaces. The first subsection introduces the space of configurations and discusses how it can be metrized. In the next part, the measures which are required later, as well as some useful technical tools are introduced. In the last subsection, the general idea of introducing dynamics on the configuration space is given for better understanding of steps taken in Section 3.

2.1. Metric properties of configuration spaces. Let (X, d) be a topological metric space.

Definition 2.1. The n-element configuration space $\Gamma^{(n)}(X)$ is the family of all subsets $\gamma \subset X$ of cardinality $|\gamma| = n$.

Let Σ_n stand for the set of all permutations of $\{1, \ldots, n\}$. Consider metrics d_n and D_n on $\Gamma^{(n)}(X)$ given by

$$d_n(\xi, \eta) = \min_{\sigma \in \Sigma_n} \sum_{k=1}^{n} d(x_{\sigma(k)}, y_k),$$

where $\xi = \{x_1, \ldots, x_n\}$ and $\eta = \{y_1, \ldots, y_n\}$, and

$$D_n = \frac{d_n}{1 + d_n}.$$

Lemma 2.2. *If (X, d) is locally compact, then $(\Gamma^{(n)}(X), D_n)$ is locally compact as well.*

Proof. Take an arbitrary $\eta \in \Gamma^{(n)}(X)$. Denote $\eta = \{y_1, \ldots, y_n\}$. As X is locally compact, there exist compact closed balls $K(y_i, r_i), i = 1, \ldots, n$. For simplicity, assume that they are disjoint. If not, we take closed balls of smaller radii instead, e.g. $\delta/3$, where δ is the smallest distance between elements of η. They are compact as closed subsets of compact sets.

We claim that

$$K_\eta = \Big\{\{x_1, \dots, x_n\} : \forall i\ x_i \in K_i\Big\}$$

is a compact neighbourhood of η in $\Gamma^{(n)}(X)$.

The set K_η contains an open ball $B(\eta, \frac{r}{1+r})$, where $r = \min\limits_{i=1,\dots,n} r_i$, and therefore is a neighbourhood of η. It remains to show that K_η is compact. We will show sequential compactness, as in metric spaces it is an equivalent notion.

Take an arbitrary sequence (η_k) of elements of K_η. We have

$$\eta_k = \{y_1^{(k)}, \dots, y_n^{(k)}\}, \quad \forall i\ y_i^{(k)} \in K_i.$$

Next, choose from (η_k) a subsequence (η_{k_j}) such that for each i

$$y_i^{(k_j)} \xrightarrow{j\to\infty} x_i \in K_i$$

for some $\xi = \{x_1, \dots, x_n\}$. We can do it, as K_i are disjoint and compact (first we choose a subsequence such that elements lying in K_1 converge, then a subsubsequence such that elements lying in K_2 converge and so on).

We have for each i

$$d(y_i^{(k_j)}, x_i) \xrightarrow{j\to\infty} 0,$$

so that

$$d_n(\eta_{k_j}, \xi) \xrightarrow{j\to\infty} 0$$

and therefore

$$D_n(\eta_{k_j}, \xi) \xrightarrow{j\to\infty} 0.$$

It means that K_η is compact and therefore $(\Gamma^{(n)}(X), D_n)$ is locally compact. □

Lemma 2.3. *If (X, d) is σ-compact, then $(\Gamma^{(n)}(X), D)$ is σ-compact as well.*

Proof. Let $X = \bigcup\limits_{k=1}^{\infty} X_k$, where each X_k is compact. Let A_k be their countable dense subsets. For $a \in A_k$ and $r > 0$ define sets

$$C_k(a, r) = K(a, r) \cap X_k,$$

where $K(a, r)$ denotes closed ball centered at a with radius r. Sets C_k are compact as closed subsets of compact sets.

Consider sets

$$C\big(r; (k_1, a_1), \dots, (k_n, a_n)\big) = \begin{cases} \Big\{\{x_1, \dots, x_n\} : x_i \in C_{k_i}(a_i, r)\Big\}, \\ \qquad \text{if } C_{k_i}(a_i, r) \text{ are pairwise disjoint,} \\ \emptyset \text{ otherwise,} \end{cases}$$

where $r > 0$ and $\forall i\ k_i \in \mathbb{N}, a_i \in A_{k_i}$.

Then

$$\Gamma^{(n)}(X) = \bigcup_{N=1}^{\infty} \bigcup_{k_1\in\mathbb{N}} \dots \bigcup_{k_n\in\mathbb{N}} \bigcup_{a_1\in A_{k_1}} \dots \bigcup_{a_n\in A_{k_n}} C\Big(\frac{1}{N}; (k_1, a_1), \dots, (k_N, a_N)\Big).$$

Indeed, for each $\eta \in \Gamma^{(n)}(X)$ we can always choose $N \in \mathbb{N}$ such that the smallest distance between elements of η is bigger than $\frac{4}{N}$. Each element y_k of η lies in some X_k and lies closer than $\frac{1}{N}$ to some $a_k \in A_k$ (as A_k are dense in X_k). Closed balls $K(a_k, \frac{1}{N})$ are pairwise disjoint (because the smallest distance between every two y_k is

$\frac{4}{N}$), so that $C_k(a_k, \frac{1}{N})$ are also pairwise disjoint. $y_k \in C_k(a_k, \frac{1}{N})$, as $y_k \in K(a_k, \frac{1}{N})$ and $y_k \in X_k$.

It remains to show that $C\big(r; (k_1, a_1), \dots, (k_n, a_n)\big)$ are compact subsets of $(\Gamma^{(n)}(X), D_n)$.

Take an arbitrary sequence (η_i) of elements of $C\big(r; (k_1, a_1), \dots, (k_n, a_n)\big)$ and denote

$$\eta_i = \{y_1^{(i)}, \dots, y_n^{(i)}\}, \quad \forall k \; y_k^{(i)} \in C_{k_i}(a_i, r).$$

Next, choose a subsequence (η_{i_j}) such that

$$y_k^{(i_j)} \xrightarrow{j\to\infty} x_k$$

for some $x_k \in C_{k_i}(a_i, r)$ for all $k = 1, \dots, n$. It is possible, as $C_{k_i}(a_i, r)$ are disjoint and compact (first we choose a subsequence such that elements lying in $C_{k_1}(a_1, r)$ converge, then a subsubsequence such that elements lying in $C_{k_2}(a_2, r)$ converge and so on). We have $\xi = \{x_1, \dots, x_n\} \in C\big(r; (k_1, a_1), \dots, (k_n, a_n)\big)$.

For each $k = 1, \dots, n$

$$d(y_k^{(i_j)}, x_k) \xrightarrow{j\to\infty} 0,$$

so that

$$d_n(\eta_{i_j}, \xi) \xrightarrow{j\to\infty} 0$$

and therefore

$$D_n(\eta_{i_j}, \xi) \xrightarrow{j\to\infty} 0.$$

It means that

$$\eta_{i_j} \xrightarrow{j\to\infty} \xi$$

in $(\Gamma^{(n)}(X), D_n)$. Therefore $C\big(r; (k_1, a_1), \dots, (k_n, a_n)\big)$ is compact, which means that $(\Gamma^{(n)}(X), D_n)$ is indeed σ-compact. $\square$

Even if (X, d) is a complete, σ-locally-compact metric space, the space of configurations $\Gamma^{(n)}(X)$ need not be complete (for $n \geq 2$).

Example 1. Consider $X = \mathbb{R}$ with Euclidean distance $d(x, y) = |x - y|$. Take a sequence (η_n) of elements of $\Gamma^{(2)}(\mathbb{R})$ given by $\eta_n = \{-\frac{1}{n}, \frac{1}{n}\}$. Then

$$D_2(\eta_n, \eta_{n+k}) \leq d_2(\eta_n, \eta_{n+k}) = 2\left(\frac{1}{n} - \frac{1}{n+k}\right) = \frac{2k}{n(n+k)} < \frac{2}{n}$$

so that (η_n) is Cauchy. However it does not converge in $\Gamma^{(2)}(\mathbb{R})$, as the only limit point of $\bigcup_{n=1}^{\infty} \{-\frac{1}{n}, \frac{1}{n}\}$ is 0.

In view of Example 1, we may consider another metric on $\Gamma^{(n)}(X)$. For $\xi = \{x_1, \dots, x_n\}$ and $\eta = \{y_1, \dots, y_n\}$ define

$$\hat{D}_n(\eta, \xi) = D_n(\eta, \xi) + \delta_n(\eta, \xi),$$

where

$$\delta_n(\eta, \xi) = \left|\frac{1}{\delta_n(\eta)} - \frac{1}{\delta_n(\xi)}\right|$$

with $\delta_n(\eta) = \min_{i \neq j} d(y_i, y_j)$ being the smallest distance between elements of $\eta \in \Gamma^{(n)}(X)$.

Lemma 2.4. *If (X, d) is complete, then $(\Gamma^{(n)}(X), \hat{D}_n)$ is complete as well.*

Proof. Take any sequence (η_k) of elements of $(\Gamma^{(n)}(X), \hat{D}_n)$ which is Cauchy, i.e.

$$\forall \varepsilon > 0\ \exists m_0(\varepsilon) \in \mathbb{N}\ \forall m \geq m_0(\varepsilon)\ \forall k \in \mathbb{N} :\ \hat{D}_n(\eta_m, \eta_{m+k}) < \varepsilon.$$

From $\hat{D}_n(\eta_m, \eta_{m+k}) < \varepsilon$ we have both

$$D_n(\eta_m, \eta_{m+k}) < \varepsilon$$

which means that (η_k) is Cauchy in $(\Gamma^{(n)}(X), D_n)$, and

$$\left| \frac{1}{\delta_n(\eta_m)} - \frac{1}{\delta_n(\eta_{m+k})} \right| < \varepsilon,$$

which means that $\big(1/\delta_n(\eta_k)\big)$ is Cauchy in $\mathbb{R}$ with Euclidean metric and therefore converges to $\frac{1}{\delta} > 0$. It cannot converge to 0, as it would mean that $(\delta_n(\eta_k))$ is unbounded so that (η_k) could not be Cauchy in $(\Gamma^{(n)}(X), D_n)$. Indeed, take any $\varepsilon > 0$ and m such that for all $k \in \mathbb{N}$ we have $D_n(\eta_m, \eta_{m+k}) < \varepsilon < 1$. Then $d_n(\eta_m, \eta_{m+k}) < \frac{\varepsilon}{1-\varepsilon}$ and therefore for each $y \in \eta_m$ there exists $y' \in \eta_{m+k}$ which lies in $B(y, \frac{\varepsilon}{1-\varepsilon})$. It means that $\delta(\eta_{m+k}) < \delta(\eta_m) + \frac{2\varepsilon}{1-\varepsilon}$ so that $(\delta_n(\eta_k))$ is bounded.

From convergence of $\big(1/\delta_n(\eta_k)\big)$ we have, in particular

$$\exists R > 0\ \exists M_0(R) \in \mathbb{N}\ \forall m > M_0(R) :\ \delta_n(\eta_m) > R.$$

Take an arbitrary

$$\varepsilon < \frac{R}{4+R}.$$

and

$$m_0 = \max(m_0(\varepsilon), M_0(R)).$$

Denote

$$\eta_{m_0} = \{x_1, \ldots, x_n\}.$$

Then for each $m > m_0$ exactly one element $y_m^{(i)}$ of η_m lies in each ball $B(x_i, \frac{\varepsilon}{1-\varepsilon})$. For each $i = 1, \ldots, n$ the sequence $(y_m^{(i)})$ is Cauchy in (X, d), which we assumed to be complete. Therefore each of these sequences converges to some $x^{(i)}$ that lies in a closed ball $K(x_i, \frac{\varepsilon}{1-\varepsilon})$, each two of them distant by at least $R/2$, which in particular means that each $x^{(i)}$ is different, so that

$$\xi = \{x^{(1)}, \ldots, x^{(n)}\} \in \Gamma^{(n)}(X).$$

Notice that ξ does not depend on the choice of ε. Moreover, we have

$$D_n(\eta_m, \xi) \leq d_n(\eta_m, \xi) = \sum_{i=1}^{n} d\left(y_m^{(i)}, x^{(i)}\right) \leq \frac{n\varepsilon}{1-\varepsilon}$$

and

$$\begin{aligned}\delta_n(\eta_m, \xi) = \left| \frac{1}{\delta_n(\eta_m)} - \frac{1}{\delta_n(\xi)} \right| &\leq \left| \frac{1}{\delta_n(\eta_m)} - \frac{1}{\delta} \right| + \left| \frac{1}{\delta} - \frac{1}{\delta_n(\xi)} \right| \\ &\leq \varepsilon |\delta - \delta_n(\xi)| \frac{1}{\delta \cdot \delta_n(\xi)} \leq \frac{4\varepsilon}{R^2(1-\varepsilon)},\end{aligned}$$

as $\delta \geq R$, $\delta_n(\xi) \geq R/2$ and $|\delta - \delta_n(\xi)| \leq \frac{2\varepsilon}{1-\varepsilon}$. It means that for any sufficiently small $\varepsilon > 0$ we can pick m_0 such that for all $m > m_0$ we have

$$\hat{D}_n(\eta_m, \xi) \leq \frac{n\varepsilon}{1-\varepsilon} + \frac{4\varepsilon}{R^2(1-\varepsilon)} \xrightarrow{\varepsilon \to 0} 0.$$

Therefore (η_m) converges to ξ in $(\Gamma^{(n)}(X), \hat{D}_n)$. □

Instead of changing the metric, we can consider the completion of $(\Gamma^{(n)}(X), D_n)$.

Definition 2.5. By $(\hat{\Gamma}^{(n)}(X), D_n)$ we denote the completion of $(\Gamma^{(n)}(X), D_n)$.

We can identify $\hat{\Gamma}^{(n)}(X)$ with symmetrization of X^n, i.e. the set of all equivalence classes $\{[x_1, \ldots, x_n] : x_i \in X\}$, where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ belong to the same class if and only if there exists a permutation $\sigma \in \Sigma_n$ for which $x_i = y_{\sigma(i)}$. The metrics d_n and D_n are defined in the same way as previously

$$d_n([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = \min_{\sigma \in \Sigma_n} \sum_{k=1}^{n} d(x_{\sigma(k)}, y_k)$$

and

$$D_n = \frac{d_n}{1 + d_n}.$$

If we consider the set of all classes $[x_1, \ldots, x_n]$ having $x_i \neq x_j$ for $i \neq j$, we obtain the representation of $\Gamma^{(n)}(X)$. The isometrism is given trivially by $[x_1, \ldots, x_n] \mapsto \{x_1, \ldots, x_n\}$.

We can identify elements of $\hat{\Gamma}^{(n)}(X)$ with multisets of cardinality n on X. The multisets having at least one element of multiplicity bigger than 1 can be identified with the corresponding classes of divergent Cauchy sequences in $(\Gamma^{(n)}(X), D_n)$. In view of the above, we will call the elements of $\hat{\Gamma}^{(n)}(X)$ multi-configurations in contrast to (simple) configurations of $\Gamma^{(n)}(X)$.

From now on, assume that (X, d) is a complete separable metric space (so that the corresponding topological space is Polish). An important idea is to identify the elements of $\hat{\Gamma}^{(n)}(X)$ with measures on $(X, \mathcal{B}(X))$. Each $\hat{\mu} = [x_1, \ldots, x_n] \in \hat{\Gamma}^{(n)}(X)$ corresponds to a measure

$$\hat{\mu} = \sum_{i=1}^{n} \delta_{x_i}, \tag{2.1}$$

where δ_x is a Dirac δ-measure centered at x, i.e. for all measurable $A \subset X$

$$\hat{\mu}(A) = \sum_{i=1}^{n} I_A(x_i)$$

with I_A being the indicator function of A. For a measurable $f : X \to \mathbb{R}$ and $\hat{\xi} = [x_1, \ldots, x_n]$ we will use notation

$$\hat{\xi}(f) = \int_X f d\hat{\xi} = \sum_{i=1}^{n} f(x_i).$$

Each measure of the form (2.1) corresponds to an element of $\hat{\Gamma}^{(n)}(X)$ and each such measure with distinct x_i corresponds to an element of $\Gamma^{(n)}(X)$. In view of the above measure representation of configurations, we can equip $\Gamma^{(n)}(X)$ and its completion with Prohorov metric. We define it as follows (see [10], A2.5).

Definition 2.6. Let $\hat{\xi}, \hat{\eta} \in \hat{\Gamma}^{(n)}(X)$. The Prohorov distance between $\hat{\xi}$ and $\hat{\eta}$ is given by

$$\pi_n(\hat{\xi}, \hat{\eta}) = \inf\{\varepsilon \geq 0 : \ \hat{\xi}(F) \leq \hat{\eta}(F^\varepsilon) + \varepsilon \text{ and } \hat{\eta}(F) \leq \hat{\xi}(F^\varepsilon) + \varepsilon \text{ for all closed } F \subset X\},$$

where

$$A^\varepsilon = \bigcup_{x \in A} B(x, \varepsilon), \varepsilon \geq 0,$$

denotes ε-neighbourhood of set A.

Notice that for $\hat{\xi}, \hat{\eta} \in \hat{\Gamma}^{(n)}(X)$, straightforwardly from the above definition we have

$$\pi_n(\hat{\xi}, \hat{\eta}) \leq n, \tag{2.2}$$

as $\hat{\xi}(X) = \hat{\eta}(X) = n$ so that for any $F \subset X$ we have $\hat{\xi}(F), \hat{\eta}(F) \leq n$.

We will show that metrics D_n and π_n are strongly equivalent, but let us start with two technical lemmas.

LEMMA 2.7. *Let $n \in \mathbb{N}$, $\hat{\xi} = [x_1, \ldots, x_n] \in \hat{\Gamma}^{(n)}$ and $A_1, \ldots, A_n \subset X$ be such that*

$$\forall k \leq n, k \in \mathbb{N} \ \forall \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} : \ \hat{\xi}\Big(\bigcup_{j=1}^{k} A_{i_j}\Big) \geq k.$$

Then, there exists a permutation $\sigma \in \Sigma_n$ such that $x_i \in A_{\sigma(i)}$ for each $i = 1, \ldots, n$.

PROOF. We will prove this fact by mathematical induction. The base case $n = 1$ is trivial. Suppose it is valid for case $n = N \in \mathbb{N}$. Suppose the assumptions are satisfied for case $N+1$. Then x_{N+1} must belong to some A_k. Let $B_i = A_i$ for $i = 1, \ldots, k-1$ and $B_i = A_{i+1}$ for $i = k, \ldots, N$. Then by the induction hypothesis in the case N, there exists $\sigma \in \Sigma_N$ such that $x_i \in B_{\sigma(i)}$, which means that there exists a bijection $s : \{1, \ldots, N\} \to \{1, \ldots, k-1, k+1, \ldots, N+1\}$ such that for $i = 1, \ldots, N$ we have $x_i \in A_s(i)$. Take $\sigma' = s \cup \{(N+1, k)\}$, which is the desired permutation from Σ_{N+1}. $\square$

LEMMA 2.8. *Let $\hat{\xi} = [x_1, \ldots, x_n]$ and $\hat{\eta} = [y_1, \ldots, y_n]$. Let $s \in \Sigma_n$ be such that*

$$d_n(\hat{\xi}, \hat{\eta}) = \sum_{k=1}^{n} d(x_{s(k)}, y_k).$$

Then

$$\text{(a)} \ \max_k \ d(x_{s(k)}, y_k) \leq 1 \implies \pi_n(\xi, \eta) \geq \frac{d_n(\hat{\xi}, \hat{\eta})}{n},$$

$$\text{(b)} \ \max_k \ d(x_{s(k)}, y_k) > 1 \implies \pi_n(\hat{\xi}, \hat{\eta}) \geq \frac{1}{n}.$$

PROOF. Take $\hat{\xi} = [x_1, \ldots, x_n]$ and $\hat{\eta} = [y_1, \ldots, y_n]$.

Denote $\varepsilon = \pi_n(\hat{\xi}, \hat{\eta})$. Suppose in case (a) that

$$\varepsilon < \frac{d_n(\hat{\xi}, \hat{\eta})}{n}$$

and in case (b) that

$$\varepsilon < \frac{1}{n}.$$

In both cases it makes $\varepsilon < 1$. Choose any $k \leq n$, $i_1, \ldots, i_k \in \mathbb{N}$ and pick a closed set $F = \{x_{i_1}, \ldots, x_{i_k}\}$ in Definition 2.6 of $\pi_n(\hat{\xi}, \hat{\eta})$. Taking into account $\varepsilon < 1$ and that $\hat{\xi}, \hat{\eta}$ take integer values only, we have

$$k \leq \hat{\xi}(F) \leq \hat{\eta}(F^\varepsilon) = \hat{\eta}\Big(\bigcup_{j=1}^{k} B(x_{i_j}, \varepsilon)\Big).$$

By Lemma 2.7 it means that there exists $\sigma \in \Sigma_n$ such that each $y_i \in B(x_{\sigma(i)}, \varepsilon)$, which means that

$$d_n(\hat{\xi}, \hat{\eta}) < n\varepsilon,$$

which is contrary to the assumption we made in case (a). In case (b) on the other hand, it means that

$$d_n(\hat{\xi}, \hat{\eta}) < 1,$$

which makes $\max_k d(x_{s(k)}, y_k) > 1$ impossible. $\square$

LEMMA 2.9. *Metrics D_n and π_n are strongly equivalent.*

PROOF. Take arbitrary $\hat{\xi} = [x_1, \ldots, x_n]$ and $\hat{\eta} = [y_1, \ldots, y_n]$ enumerated in such way that

$$d_n(\hat{\xi}, \hat{\eta}) = \sum d(x_i, y_i)$$

and denote $R = \max\{d(x_i, y_i) : i = 1, \ldots, n\}$.

If $R > 1$, then

$$\frac{D_n(\hat{\xi}, \hat{\eta})}{n} \leq \frac{1}{n} \leq \pi_n(\hat{\xi}, \hat{\eta}) \leq n \leq 2nD_n(\hat{\xi}, \hat{\eta}),$$

where the first inequality is obvious due to $D_n \leq 1$, the second comes by Lemma 2.8(b) and the third is just inequality (2.2). The last inequality is an effect of assumption $R > 1$, in which case $d_n(\hat{\xi}, \hat{\eta}) \geq 1$, so that $D_n(\hat{\xi}, \hat{\eta}) \geq \frac{1}{2}$.

If $R \leq 1$, then

$$\frac{D_n(\hat{\xi}, \hat{\eta})}{n} \leq \frac{d_n(\hat{\xi}, \hat{\eta})}{n} \leq \pi_n(\hat{\xi}, \hat{\eta}) \leq R \leq d_n(\hat{\xi}, \hat{\eta}) \leq (n+1)D_n(\hat{\xi}, \hat{\eta}) \leq 2nD_n(\hat{\xi}, \hat{\eta}).$$

The first inequality comes from the definition of D_n, the second holds true by Lemma 2.8(a). The third inequality is valid, as

$$\hat{\xi}\Big(\bigcup_{y\in\hat{\eta}} K(y, R)\Big) = \hat{\eta}\Big(\bigcup_{x\in\hat{\xi}} K(x, R)\Big) = n.$$

The fourth inequality comes just from our definition of R, the fifth from the definition of D_n and the fact that $d_n(\hat{\xi}, \hat{\eta}) \leq n$ if $R \leq 1$. The last, sixth inequality is trivial, as $n \geq 1$.

Finally, merging both cases, we obtain $\frac{1}{n}D_n(\hat{\xi}, \hat{\eta}) \leq \pi_n \leq 2nD_n(\hat{\xi}, \hat{\eta})$. $\square$

DEFINITION 2.10. The space of finite configurations $\Gamma_0(X)$ is the family of all finite subsets $\gamma \subset X$.

Notice that each nonempty $\gamma \in \Gamma_0(X)$ has its finite cardinality $|\gamma| = n \in \mathbb{N}$ and therefore can be treated as an element of $\Gamma^{(n)}(X)$. It allows us to write

$$\Gamma_0(X) = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \Gamma^{(n)}(X) \tag{2.3}$$

and equip $\Gamma_0(X)$ with disjoint union topology. It can be metrized by

$$D(\xi, \eta) = \begin{cases} D_{|\xi|}(\xi, \eta), & |\xi| = |\eta|, \\ 1, & |\xi| \neq |\eta|, \end{cases}$$

where $\xi, \eta \in \Gamma_0(X)$ and $D_0 = 0$. Of course, Γ_0 with this metric is not complete. As previously, we can also consider the space of all finite multi-configurations.

DEFINITION 2.11. By $\Big(\hat{\Gamma}_0(X), D\Big)$ we denote the completion of $\Big(\Gamma_0(X), D\Big)$.

Notice that for each Cauchy sequence in $\Gamma_0(X)$, there exists $n \in \mathbb{N}$ for which almost all of the elements of the given sequence belong to $\Gamma^{(n)}(X)$. It means that each element of $\hat{\Gamma}_0(X)$ can be identified with an element of a certain $\hat{\Gamma}^{(n)}(X)$, so that

$$\hat{\Gamma}_0(X) = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \hat{\Gamma}^{(n)}(X).$$

Each element of $\hat{\Gamma}_0(X)$ corresponds to a finite measure as in (2.1). Therefore, we can consider also the Prohorov metric π on $\hat{\Gamma}_0(X)$ given exactly as π_n in Definition 2.6. Namely, for $\hat{\xi}, \hat{\eta} \in \hat{\Gamma}_0(X)$

$$\pi(\hat{\xi}, \hat{\eta}) = \inf\{\varepsilon \geq 0 : \ \hat{\xi}(F) \leq \hat{\eta}(F^{\varepsilon}) + \varepsilon \text{ and } \hat{\eta}(F) \leq \hat{\xi}(F^{\varepsilon}) + \varepsilon \text{ for all closed } F \subset X\}.$$

It is known (see e.g. [10], A2.5) that π generates the topology of weak convergence on the space of finite measures $\mathcal{M}(X)$ on X and therefore also on $\hat{\Gamma}_0(X)$, which can be identified with a subset of $\mathcal{M}(X)$.

As shown in [13], the same topology can be generated by metric D_{BL}, where for $\hat{\xi}, \hat{\eta} \in \hat{\Gamma}_0(X)$ we define

$$D_{BL}(\hat{\xi}, \hat{\eta}) = \sup_{f:||f||_{BL} \leq 1} \left| \hat{\xi}(f) - \hat{\eta}(f) \right|,$$

where $f : X \to \mathbb{R}$ are bounded Lipschitzian functions and

$$||f||_{BL} = ||f||_L + ||f||_\infty$$

with

$$||f||_\infty = \sup_{x \in X} |f(x)|$$

and

$$||f||_L = \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We will show that the metric D generates the topology of weak convergence as well.

Lemma 2.12. *Topologies on $\hat{\Gamma}_0(X)$ generated by metrics D and D_{BL} are equal.*

Proof. First, we will show that convergence in D implies convergence in D_{BL}. Take an arbitrary $\hat{\eta} = [y_1, \ldots, y_N] \in \hat{\Gamma}_0$ and suppose that

$$D(\hat{\eta}_n, \hat{\eta}) \xrightarrow{n \to \infty} 0.$$

Then $\hat{\eta}_n$ has N elements for sufficiently big $n \in \mathbb{N}$ and for such n also

$$d_N(\hat{\eta}_n, \hat{\eta}) \xrightarrow{n \to \infty} 0.$$

For arbitrary $\varepsilon > 0$ pick $n_0 \in \mathbb{N}$ such that for $n > n_0$ cardinality of $\hat{\eta}_n$ is N and

$$d_N(\hat{\eta}_n, \hat{\eta}) < \frac{\varepsilon}{N}.$$

For $n > n_0$ denote $\hat{\eta}_n = [\tilde{x}_{n,1}, \ldots, \tilde{x}_{n,N}]$. Let $x_{n,i} = \tilde{x}_{n,\sigma_n(i)}$, where $\sigma_n \in \Sigma_N$ is such that

$$d_N(\hat{\eta}_n, \hat{\eta}) = \sum_{i=1}^{N} d(\tilde{x}_{n,\sigma_n(i)}, y_i).$$

It immediately follows that $d(x_{n,i}, y_i) < \frac{\varepsilon}{N}$ for $n > n_0$ and each $i = 1, \ldots, N$. Now for arbitrary $f : X \to \mathbb{R}$ such that $||f||_{BL} \leq 1$ (and therefore $||f||_L \leq 1$) we have

$$|\hat{\eta}_n(f) - \hat{\eta}(f)| \leq \sum_{i=1}^{N} |f(x_{n,i}) - f(y_i)| \leq \sum_{i=1}^{N} d(x_{n,i}, y_i) \leq N \frac{\varepsilon}{N} = \varepsilon$$

Therefore $D_{BL}(\hat{\eta}_n, \hat{\eta}) \leq \varepsilon$. As ε was picked arbitrarily, it means that

$$D_{BL}(\hat{\eta}_n, \hat{\eta}) \xrightarrow{n \to \infty} 0.$$

To show the implication in the opposite direction, again take an arbitrary $\hat{\eta}$ and assume that $D_{BL}(\hat{\eta}_n, \hat{\eta})$ converges to 0 with $n \to \infty$.

If $\hat{\eta} = \emptyset$, it follows that for almost all $n \in \mathbb{N}$ also $\hat{\eta}_n = \emptyset$. If only there exists an element y of $\hat{\eta}_n$ we take

$$f(x) = \frac{1}{2} \max\Big(0, 1 - d(x, y)\Big),$$

for which $||f||_{BL} = 1$ but $|\hat{\eta_n}(f) - 0| \geq \frac{1}{2}$, which is contrary to our assumption. If $\hat{\eta}_n$ and $\hat{\eta}$ are empty configurations, then trivially $D(\hat{\eta}_n, \hat{\eta}) = 0$.

Suppose now that $\hat{\eta} = [y_1, \ldots, y_N]$ is non-empty and let $x_1, \ldots, x_M$ denote all its distinct elements and $N_i = \hat{\eta}(\{x_i\})$ their multiplicity. If $M = 1$, then let $R = 1$. Otherwise, denote

$$r = \min_{i \neq j} d(x_i, x_j) > 0$$

and take $R = \min(\frac{r}{3}, 1)$. Introduce for $y \in X$ function $f_y : X \to \mathbb{R}$ by

$$f_y(x) = \frac{1}{2} \max\Big(0, R - d(x, y)\Big).$$

Then, as $R \leq 1$, each such f_y satisfies $||f_y||_\infty, ||f_y||_L \leq \frac{1}{2}$ and therefore $||f_y||_{BL} \leq 1$. Additionally, it is non-negative, takes positive values only inside the ball $B(y, R)$ and the maximum value $\frac{R}{2}$ at y.

Take an arbitrary $0 < \varepsilon < NR$ and n_0 such that for $n > n_0$

$$D_{BL}(\hat{\eta}_n, \hat{\eta}) < \frac{\varepsilon}{2N} < \frac{R}{2}.$$

Denote $\hat{\eta}_n = [\tilde{y}_{n,1}, \ldots, \tilde{y}_{n,k_n}]$. Each $\tilde{y}_{n,j}$ lies inside one of the balls $B_i = B(x_i, R)$, where $i = 1, \ldots, M$. Indeed, suppose that one of them, say y, does not belong to any B_i. Then for f_y we have $\hat{\eta}_n(f_y) \geq \frac{R}{2}$ but $\hat{\eta}(f_y) = 0$ which is contrary to $D_{BL}(\hat{\eta}_n, \hat{\eta}) < \frac{R}{2}$.

Now, consider functions $g_y : X \to \mathbb{R}$, $y \in \mathbb{R}$, given by

$$g_y(x) = \frac{1}{2} \max\Big(0, \min\big(R, 2R - d(x, y)\big)\Big).$$

Clearly, $||g_y||_{BL} \leq 1$. Additionally, it is non-negative, takes positive values only inside the ball $B(y, 2R)$ and the maximum value $\frac{R}{2}$ uniformly on the closed ball $K(y, R)$. Notice that for $i \neq j$ balls $B(x_i, 2R)$ and $B_j = B(x_j, R)$ are disjoint, as $R \leq \frac{r}{3}$.

For $i = 1, \ldots, M$ we have

$$|\hat{\eta}_n(g_{x_i}) - \hat{\eta}(g_{x_i})| = \left|\Big(\hat{\eta}_n(B_i) - N_i\Big)\frac{R}{2}\right| < \frac{\varepsilon}{2N},$$

from which we conclude that $\hat{\eta}_n(B_i) = N_i$. Therefore, there exists $\sigma \in \Sigma_N$ such that $y_{n,i} = \tilde{y}_{n,\sigma(i)} \in B(y_i, R)$ for all $i = 1, \ldots, N$.

For $i = 1, \ldots, M$ we have

$$|\hat{\eta}_n(f_{x_i}) - \hat{\eta}(f_{x_i})| = \left|\sum_y f_{x_i}(y) - N_i\frac{R}{2}\right| = \sum_y \left|f_{x_i}(y) - \frac{R}{2}\right| < \frac{\varepsilon}{2N},$$

where the sum is taken over all N_i elements y of $\hat{\eta}_n$ lying inside B_i. In particular, for each such $y = y_{n,j}$ we have

$$\left|f_{x_i}(y_{n,j}) - \frac{R}{2}\right| = |f_{x_i}(y_{n,j}) - f_{x_i}(y_j)| < \frac{\varepsilon}{2N},$$

which, taking into account the form of f_{x_i} means that

$$d(y_{n,j}, y_j) < \frac{\varepsilon}{N},$$

holding for each $j = 1, \dots, N$. This, in turn, gives

$$d_N(\hat{\eta}_n, \hat{\eta}) \le \sum_{j=1}^{N} d(y_{n,j}, y_j) < \varepsilon,$$

which implies $D(\hat{\eta}_n, \hat{\eta}) < \varepsilon$. As ε can be chosen arbitrarily small, we conclude that

$$D(\hat{\eta}_n, \hat{\eta}) \xrightarrow{n\to\infty} 0.$$

$\square$

DEFINITION 2.13. The configuration space $\Gamma(X)$ consists of all subsets $\gamma \subset X$ which are locally finite, i.e. $\gamma \cap \Lambda \in \Gamma_0(X)$ for any compact $\Lambda \subset X$.

We can equip $\Gamma(X)$ with vague (or locally-weak) topology.

DEFINITION 2.14. The vague topology is the weakest topology on $\Gamma(X)$ that makes continuous the mappings

$$\Gamma(X) \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}.$$

for all continuous, compactly supported functions $f : X \to \mathbb{R}$.

The vague topology is metrizable, e.g. by

$$\pi^{\#}(\mu, \nu) = \int_0^\infty e^{-r} \frac{\pi(\mu \cap B_r, \nu \cap B_r)}{1 + \pi(\mu \cap B_r, \nu \cap B_r)} dr,$$

see [10, A2.6], where B_r denotes an open ball centered at the origin (some fixed point of X) with radius r.

As metrics D and π are topologically equivalent (and $D \le 1$), the vague topology can be metrized also by

$$D^{\#}(\mu, \nu) = \int_0^\infty e^{-r} D(\mu \cap B_r, \nu \cap B_r) dr,$$

with B_r as above.

$\Gamma(X)$ with metric $\pi^{\#}$ (or $D^{\#}$) is not complete, see Example 1 and Lemma 2.9.

By $(\hat{\Gamma}(X), D^{\#})$ we denote the completion of $(\Gamma(X), D^{\#})$. Its elements $\hat{\gamma} \in \hat{\Gamma}(X)$ satisfy $\hat{\gamma} \cap \Lambda \in \hat{\Gamma}_0(X)$ for any compact $\Lambda \subset X$ and therefore can be identified with locally finite multisets.

It is known, see [26], [37] that the corresponding topological space is a Polish space and that the vague topology on $\Gamma(X)$ can be completely metrized as well.

2.2. Measures on configuration spaces. From now on, we restrict our considerations to the case of simple configurations Γ. It is justified by the fact that measures we deal with, ignore multi-configurations, see (2.15).

By $\mathcal{B}(\Gamma)$ we will denote the σ-field generated by the vague topology on Γ.

For a compact set $\Lambda \subset \mathbb{R}^d$, by Γ_Λ we denote the family of all configurations contained within Λ, i.e.

$$\Gamma_\Lambda = \{\gamma \cap \Lambda : \gamma \in \Gamma\}.$$

These sets of spatially bounded configurations can be equipped with topologies induced from the vague topology of Γ, so that

$$\mathcal{B}(\Gamma_\Lambda) = \{A \cap \Gamma_\Lambda : A \in \mathcal{B}(\Gamma)\}.$$

The set of finite configurations can be equipped with the weak-hash topology induced from Γ. In the case of finite configurations, it is equal to the weak topology. Borel σ-field defined by this topology will be denoted $\mathcal{B}(\Gamma_0)$. From the fact that $\Gamma_0 \in \mathcal{B}(\Gamma)$ we have

$$\mathcal{B}(\Gamma_0) = \{A \cap \Gamma_0 : A \in \mathcal{B}(\Gamma)\} = \{A \in \mathcal{B}(\Gamma) : A \subset \Gamma_0\}.$$

The set of finite configurations Γ_0 can be written down as the union of sets of n-element configurations, recall (2.3), that allows one to endow it with the disjoint union topology and corresponding Borel σ-field.

In view of the above, each function $G : \Gamma_0 \to \mathbb{R}$ can be represented by a collection of $G^{(n)}$ indexed by $n \in \mathbb{N}$, such that $G^{(0)}$ is a real constant $G^{(0)} = G(\emptyset)$ and for $n \in \mathbb{N}$ its elements are symmetric real-valued functions with n arguments, each in $\mathbb{R}^d$, i.e. $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ such that

$$G^{(n)}(x_1, \ldots, x_n) = G(\{x_1, \ldots, x_n\}), \quad x_i \neq x_j \text{ if } i \neq j. \tag{2.4}$$

Each such sequence determines the unique function $G : \Gamma_0 \to \mathbb{R}$, but not vice-versa, as $G^{(n)}$ may take arbitrary values whenever $x_i = x_j$ for some $i \neq j$. This fact will be negligible, as the set of arguments on which $G^{(n)}$ is not uniquely determined by G is a zero-measure set with respect to the n-dimensional Lebesgue measure $l^{(n)}$, see (2.15).

It can be shown, cf. [14], that a function $G : \Gamma_0 \to \mathbb{R}$ is measurable if and only if there exists a collection of symmetric and measurable functions $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ satisfying (2.4) for any $n \in \mathbb{N}$.

Definition 2.15. We say that a function $G : \Gamma_0 \to \mathbb{R}$ has a bounded support if there exist a compact set $\Lambda \subset \mathbb{R}^d$ (spatial support) and an integer $N \in \mathbb{N}$ (quantitative bound) such that

$$G(\eta) = 0 \text{ whenever } \eta \cap \Lambda^c \neq \emptyset \text{ or } |\eta| > N.$$

By $\mathcal{B}_{bs}$ we denote the set of all bounded and measurable functions $G : \Gamma_0 \to \mathbb{R}$ with bounded supports.

It is worth noting that $\mathcal{B}_{bs}$ is a measure-defining set of functions, which means that two measures defined on $(\Gamma_0, \mathcal{B}(\Gamma_0))$, say μ and ν, are equal if and only if for every $G \in \mathcal{B}_{bs}$ the equality for corresponding integrals holds:

$$\int_{\Gamma_0} G(\eta)\mu(d\eta) = \int_{\Gamma_0} G(\eta)\nu(d\eta).$$

Consider as an example a function, which will further play an important role:

$$e(f, \eta) = \prod_{x \in \eta} f(x), \tag{2.5}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is given. Considering $e : \Gamma_0 \to \mathbb{R}$, we can easily make it spatially bounded by picking f to have compact support $\Lambda \subset \mathbb{R}^d$. However it is not quantitatively bounded, as for any $N \in \mathbb{N}$ we can choose $\eta \in \Gamma_0$ which is a subset of Λ but have more than N elements. Therefore, even for compactly supported functions f, in general e does not have bounded support.

Let us introduce the following transformation, called K-transform, see [19] for a broader discussion around it.

Definition 2.16. K-transform is defined by the formula

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta),$$

where $\eta \in \Gamma_0$, $\gamma \in \Gamma$ and $G : \Gamma_0 \to \mathbb{R}$.

Symbol "$\Subset$" used above means that the sum is taken over all finite subsets η of γ. Obviously, the $K-$transform is linear. It acts to the set of all measurable cylinder functions $F : \Gamma \to \mathbb{R}$, that is satisfying $F(\gamma) = F(\gamma_\Lambda)$ for some compact $\Lambda \subset \mathbb{R}^d$, where $\gamma_\Lambda = \gamma \cap \Lambda$, see [19] for more details. For $G_1, G_2 \in \mathcal{B}_{bs}$, it is known that

$$(KG_1) \cdot (KG_2) = K(G_1 \star G_2), \tag{2.6}$$

where the 'convolution' $G_1 \star G_2$ is defined as

$$(G_1 \star G_2)(\eta) = \sum_{\xi \subset \eta} G_1(\xi) \sum_{\zeta \subset \xi} G_2(\eta \backslash \xi \cup \zeta) \in \mathcal{B}_{bs}. \tag{2.7}$$

One of the important properties of the K-transform is that

$$(Ke(f, \cdot))(\gamma) = e(1 + f, \gamma), \tag{2.8}$$

recall (2.5).

We will use the following combinatorial fact for changing the order of summation for $G, H : \Gamma_0 \to \mathbb{R}$

$$\sum_{\eta \Subset \gamma} \sum_{\xi \Subset \gamma \backslash \eta} H(\eta) G(\xi) = \sum_{\xi \Subset \gamma} \sum_{\eta \subset \xi} H(\eta) G(\xi \backslash \eta), \tag{2.9}$$

where $\gamma \in \Gamma$. Actually, it holds true even if we limit η to be element of $\Gamma^{(n)}$ for a given $n \in \mathbb{N}$, for example in the case $n = 1$ with $G : \Gamma_0 \to \mathbb{R}, H : \mathbb{R}^d \to \mathbb{R}$ we have

$$\sum_{x \in \gamma} \sum_{\xi \Subset \gamma \backslash x} H(x) G(\xi) = \sum_{\xi \Subset \gamma} \sum_{x \in \xi} H(x) G(\xi \backslash x) \tag{2.10}$$

or in the case $n = 2$ with $G : \Gamma_0 \to \mathbb{R}$ and symmetric $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we obtain

$$\sum_{\{x,y\} \subset \gamma} \sum_{\xi \Subset \gamma \backslash \{x,y\}} H(x, y) G(\xi) = \sum_{\xi \Subset \gamma} \sum_{\{x,y\} \subset \xi} H(x, y) G(\xi \backslash \{x, y\}) \tag{2.11}$$

Consider a projection $p_\Lambda : \Gamma \to \Gamma_\Lambda$ given by formula $p_\Lambda(\gamma) = \gamma \cap \Lambda$. For any measure μ on $(\Gamma, \mathcal{B}(\Gamma))$, using p_Λ we can define projection μ^Λ of μ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by

$$\mu^\Lambda(A) = \mu(p_\Lambda^{-1}(A)), \qquad A \in \mathcal{B}(\Gamma_\Lambda). \tag{2.12}$$

The set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ will be denoted by $\mathcal{P}(\Gamma)$. For a measure $\mu \in \mathcal{P}(\Gamma)$ define a Bogoliubov functional as

$$B_\mu(\theta) = \int_\Gamma F_\theta(\gamma) \mu(d\gamma), \tag{2.13}$$

where

$$F_\theta(\gamma) = \prod_{x \in \gamma} (1 + \theta(x))$$

and $\theta \in \Theta$ with Θ being a convenient set of real-valued functions defined on Γ. For example, we can choose Θ to be the set of all compactly supported, continuous functions $\theta : \mathbb{R}^d \to (-1, 0]$. Then the above functional is bounded by 1 for any probability measure μ.

An important role for us will play the Poisson measure, see e.g. [15]. It distributes points independently over $\mathbb{R}^d$ with given density and corresponds to the most chaotic state in the system. For the homogeneous Poisson measure π_ρ (i.e. with constant density $\rho \in \mathbb{R}$), the Bogoliubov functional takes the form

$$B_{\pi_\rho}(\theta) = \exp\Big(\rho \int\limits_{\mathbb{R}^d} \theta(x)dx\Big).$$

In the case of Poisson measure, the above functional can be naturally extended to $L^1(\mathbb{R}^d)$. We define $\mathcal{P}_{\exp} \subset \mathcal{P}(\Gamma)$ as the set of all probability measures $\mu \in \mathcal{P}(\Gamma)$ for which it is possible to extend the Bogoliubov functional (2.13) to an exponential type entire function on $L^1(\mathbb{R}^d)$, which means that it can be written as

$$B_\mu(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n) \prod_{i=1}^{n} \theta(x_i) dx_1 \dots dx_n \tag{2.14}$$

with symmetric $k_\mu^{(n)} \in L^\infty\big((\mathbb{R}^d)^n\big)$, which for every $n \in \mathbb{N}$ undergoes the estimation $||k_\mu^{(n)}|| \leq C^n$ for some constant $C > 0$. The elements of $\mathcal{P}_{\exp}$ are called sub-Poissonian measures. In view of (2.4) we introduce the correlation function of measure $\mu \in \mathcal{P}_{\exp}$ as follows.

Definition 2.17. The correlation function of measure $\mu \in \mathcal{P}_{\exp}$ is a function $k_\mu : \Gamma_0 \to \mathbb{R}$ such that $k_\mu(\emptyset) = 1$ and $k_\mu(\eta) = k_\mu^{(n)}(x_1, \dots, x_n)$ for $\eta = \{x_1, \dots, x_n\}$, where $k_\mu^{(n)}$ is as in (2.14).

One can define the Lebesgue-Poisson measure, a correlation measure of a homogeneous Poisson measure with density $\rho \equiv 1$ (see [19] for details) in terms of integrals for functions from $\mathcal{B}_{bs}$, recall Definition 2.15.

Definition 2.18. The Lebesgue-Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined as a measure satisfying

$$\int\limits_{\Gamma_0} G(\eta)\lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^d)^n} G^{(n)}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

for all $G \in \mathcal{B}_{bs}$.

Note that for the n-dimensional Lebesgue measure $l^{(n)}$ we have

$$l_n(\{(x_1, \dots, x_n) : x_i = x_j \text{ for some } i \neq j\}) = 0, \tag{2.15}$$

which justifies the restriction we made at the beginning of this section.

One can rewrite (2.14) in terms of the Lebesgue-Poisson integral

$$B_\mu(\theta) = \int\limits_{\Gamma_0} k_\mu(\eta) \prod_{x \in \eta} \theta(x) \lambda(d\eta), \qquad \theta \in L^1(\mathbb{R}^d),$$

obtaining a dependency, recall (2.13),

$$\int\limits_{\Gamma} F_\theta(\gamma)\mu(d\gamma) = \int\limits_{\Gamma_0} k_\mu(\eta) \prod_{x \in \eta} \theta(x) \lambda(d\eta). \tag{2.16}$$

For $\mu \in \mathcal{P}_{\exp}$ and bounded $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, the projection μ^Λ given by (2.12) is absolutely continuous with respect to Lebesgue-Poisson measure λ, recall Definition 2.18.

We write R^Λ_μ for density (Radon-Nikodym derivative) of μ^Λ w.r.t. λ. It is connected with the restriction of correlation function k_μ to Γ_Λ by the relation

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R^\Lambda_\mu(\eta \cup \xi)\lambda(d\xi), \quad \eta \in \Gamma_\Lambda. \tag{2.17}$$

For each sub-Poissonian measure $\mu \in \mathcal{P}_{\text{exp}}$ and $G \in \mathcal{B}_{bs}$ the following very useful equality holds, see e.g. [4] equation (2.13):

$$\int_{\Gamma} (KG)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} k_\mu(\eta)G(\eta)\lambda(d\eta), \tag{2.18}$$

recall Definitions 2.16, 2.17 and 2.18.

The following fact (see e.g. Proposition 2.3 in [4]) allows to show that a given function is a correlation function. It is the basis of the proof of Theorem 3.3, one of the main results presented in this work.

Proposition 2.19. *A function $k : \Gamma_0 \to \mathbb{R}$ is the correlation function of a unique measure $\mu \in \mathcal{P}_{\text{exp}}$ if it satisfies the conditions:*

$$\begin{aligned} &k(\emptyset) = 1, \\ &k(\eta) \le C^{|\eta|} \quad \text{for a constant } C > 0, \\ &\int_{\Gamma_0} Gk \; d\lambda \ge 0 \quad \text{for all } G \in \mathcal{B}^*_{bs}, \end{aligned}$$

where $\mathcal{B}^*_{bs} = \{G \in \mathcal{B}_{bs} : (KG)(\gamma) \ge 0, \gamma \in \Gamma\}$, recall Definition 2.16 of K-transform.

A technical lemma repeatedly used throughout this work is so-called Minlos lemma (see e.g. [14] for a more general version).

Lemma 2.20. *For measurable $G : \Gamma_0 \to \mathbb{R}, H : \Gamma_0 \times \Gamma_0 \to \mathbb{R}$*

$$\int_{\Gamma_0}\int_{\Gamma_0} G(\eta \cup \xi)H(\eta, \xi)\lambda(d\eta)\lambda(d\xi) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \backslash \xi)\lambda(d\eta).$$

Proof. By the basic properties of integral, it is enough to consider positive functions G, H. By Definition 2.18 we have

$$RHS = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G(\{x_1, \dots, x_n\}) \sum_{\xi \subset \{x_1,\dots,x_n\}} H(\xi, \{x_1, \dots, x_n\} \backslash \xi) dx_1 \dots dx_n =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \int_{(\mathbb{R}^d)^{n-k}} \int_{(\mathbb{R}^d)^k} G(\{x_1, \dots, x_n\})H(\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_n\}) dx_1 \dots dx_n.$$

Reordering the obtained double sum, we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\binom{n+k}{k}}{(n+k)!} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^k} G(\{x_1, \dots, x_{n+k}\})H(\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_{n+k}\}) dx_1 \dots dx_{n+k} =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} G(\{x_1, \dots, x_{n+k}\})H(\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_{n+k}\}) dx_1 \dots dx_{n+k}$$

which is exactly the LHS. □

We will also use two special cases of Lemma 2.20, which can be proven in a straightforward way by taking

$$H(\eta_1,\eta_2)=\begin{cases} H(x,\eta_2), & \eta_1=\{x\}\\ 0, & |\eta_1|\neq 1\end{cases}$$

and

$$H(\eta_1,\eta_2,\eta_3)=\begin{cases} H(x,y,\eta_3), & \eta_1=\{x\},\eta_2=\{y\}\\ 0, & |\eta_1|\neq 1 \text{ or } |\eta_2|\neq 1,\end{cases}$$

respectively, combined with the use of Definition 2.18.

Lemma 2.21. *For positive and measurable $G:\Gamma_0\to\mathbb{R}, H:\mathbb{R}^d\times\Gamma_0\to\mathbb{R}$*

$$\int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)H(x,\eta)dx\lambda(d\eta)=\int_{\Gamma_0}\sum_{x\in\eta}G(\eta)H(x,\eta\backslash x)\lambda(d\eta).$$

Lemma 2.22. *For positive and measurable $G:\Gamma_0\to\mathbb{R}, H:\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_0\to\mathbb{R}$*

$$\frac{1}{2}\int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(\eta\cup\{x,y\})H(x,y,\eta)dxdy\lambda(d\eta)=$$

$$\int_{\Gamma_0}\sum_{\{x,y\}\subset\eta}G(\eta)H(x,y,\eta\backslash\{x,y\})\lambda(d\eta).$$

2.3. Dynamics on the configuration space. In the approach we follow, the states of the system are probabilistic measures on Γ, that is elements of $\mathcal{P}(\Gamma)$. The dynamics of the system is described by the corresponding Fokker-Planck equation (or Kolmogorov forward equation)

$$\frac{d}{dt}\mu_t=L^*\mu_t,\qquad \mu_{t=0}=\mu_0, \tag{2.19}$$

where μ_0 is the initial state of the system. However, in many cases it is more convenient to introduce the dynamics of the system by the means of corresponding (backward) Kolmogorov equation

$$\frac{d}{dt}F_t=LF_t,\qquad F_{t=0}=F_0, \tag{2.20}$$

which specifies the evolution $\mu_0\to\mu_t$ of states by the relation

$$\mu_t(F_0)=\mu_0(F_t),$$

where, by convention (that will be used also further in the work),

$$\mu(F)=\int_{\Gamma}F(\gamma)\mu(\gamma).$$

The exact form of the operator L in (2.20) specifies the model. One can pass from the Kolmogorov (2.20) to the Fokker-Planck (2.19) equation by relation

$$\int_{\Gamma}LF_t\ d\mu_0=\int_{\Gamma}F_0\ d(L^*\mu_t). \tag{2.21}$$

For infinite systems, it is difficult to deal directly with Kolmogorov equation (2.20) or Fokker-Planck equation (2.19). In the case of studying sub-Poissonian states, to avoid imposing artificial restrictions, we may pass to the problem for their correlation functions, recall Definition 2.17. It is performed by passing action of operator L from

(2.20) to operator L^{Δ} acting on the level of correlation functions. Having in mind (2.16), we can describe it by the duality

$$\int_{\Gamma}(LF_{\theta})(\gamma)\mu(d\gamma)=\int_{\Gamma_0}(L^{\Delta}k_{\mu})(\eta)\prod_{x\in\eta}\theta(x)\lambda(d\eta), \tag{2.22}$$

where F_{θ} is defined as previously:

$$F_{\theta}(\gamma)=\prod_{x\in\gamma}(1+\theta(x)).$$

Therefore, instead of (2.20) or (2.19), we can study a problem posed on the level of correlation functions, namely

$$\frac{d}{dt}k_t(\eta)=L^{\Delta}k_t(\eta), \qquad k_{t=0}=k_0, \tag{2.23}$$

where k_0 is the correlation function of the initial state of the system.

Due to the properties of correlation functions, the appropriate Banach spaces in which we consider equation (2.23) are of L^{∞} type. We define the following scale of Banach spaces.

Definition 2.23. For $\theta\in\mathbb{R}$, by $\mathcal{K}_{\theta}$ we denote the Banach space

$$\mathcal{K}_{\theta}=\{k:\Gamma_0\to\mathbb{R}:||k||_{\theta}<\infty\}$$

with the norm

$$||k||_{\theta}=\operatorname*{ess\,sup}_{\eta\in\Gamma_0}\left(e^{-\theta|\eta|}|k(\eta)|\right),$$

where ess sup is taken with respect to the Lebesgue-Poisson measure.

For $\theta'>\theta$ we have $\mathcal{K}_{\theta}\hookrightarrow\mathcal{K}_{\theta'}$, that is $\mathcal{K}_{\theta}$ is continuously embedded in $\mathcal{K}_{\theta'}$. Directly from Definition 2.23, any $k\in\mathcal{K}_{\theta}$ undergoes the estimate

$$|k(\eta)|\le e^{\theta|\eta|}||k||_{\theta}. \tag{2.24}$$

3. Microscopic dynamics

In this section, the exact form of the studied model of coalescing random jumps is given. The model is based on the one describing repulsive jumps, which was studied e.g. in [7] or [4]. An additional term responsible for coalescence is considered, see (3.1) below, that makes the analysis of the system dynamics more challenging. This model in a slightly more general form was first introduced in [30]. Later, in the form presented here, it was studied in more details in [23]. This section is devoted to the presentation of the results published in the latter article.

3.1. The model.

The discussed model is specified by the operator $L=L_1+L_2$ involved in corresponding Kolmogorov equation (2.20). Its action on observable $F:\Gamma\to\mathbb{R}$ is defined as

$$(L_1F)(\gamma)=\sum_{\{x,y\}\subset\gamma}\int_{\mathbb{R}^d}c_1(x,y;z)\Big(F\big(\gamma\backslash\{x,y\}\cup z\big)-F(\gamma)\Big)dz \tag{3.1}$$
$$(L_2F)(\gamma)=\sum_{x\in\gamma}\int_{\mathbb{R}^d}\tilde{c}_2(x;y;\gamma)\Big(F\big(\gamma\backslash x\cup y\big)-F(\gamma)\Big)dy.$$

Function c_1 denotes the intensity of coalescence – $c_1(x,y;z)$ is the intensity of action in which x and y coalesce into z. Note that c_1 does not depend on other (than x and y)

elements of configuration γ. Function $\tilde{c}_2$ describes the intensity of jumps – $\tilde{c}_2(x;y;\gamma)$ is the intensity of action in which x change into y in the presence of configuration γ. The intensity of jumps is lowered by the configuration with the jumping element excluded. Therefore, we may express it in the form

$$\tilde{c}_2(x;y;\gamma) = c_2(x;y) \prod_{u\in\gamma\setminus x} e^{-\phi(y-u)},$$

where ϕ is the repulsion potential and c_2 the jump kernel. We assume the following, quite general, properties of the parameter functions involved in the model. The functions c_1, c_2 and ϕ take real, non-negative values and additionally fulfill the following integrability and boundedness conditions:

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c_1(x_1,x_2;x_3)dx_i dx_j = \langle c_1\rangle \ &< \infty,\\
c_1^{\max} = \sup_{x,y\in\mathbb{R}^d} \int_{\mathbb{R}^d} c_1(x,y;z)dz &< \infty,\\
\int_{\mathbb{R}^d} c_2(x;y)dx = \int_{\mathbb{R}^d} c_2(x;y)dy = \langle c_2\rangle \ &< \infty,\\
\langle\phi\rangle = \int_{\mathbb{R}^d} \phi(x)dx &< \infty,\\
|\phi| = \sup_{x\in\mathbb{R}^d} \phi(x) &< \infty.
\end{aligned} \tag{3.2}$$

In view of duality relation (2.22), we pass to the corresponding operator acting on the level of correlation functions. Let us start with L_1.

$$\int_\Gamma (L_1F_\theta)(\gamma)\mu(d\gamma) =$$

$$\int_\Gamma\int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\gamma} c_1(x,y;z)\Big[(1+\theta(z)) - (1+\theta(x))(1+\theta(y))\Big] \prod_{u\in\gamma\setminus\{x,y\}} (1+\theta(u))dz\mu(d\gamma).$$

Using (2.8) we can write the RHS in the form

$$\int_\Gamma\int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\gamma} c_1(x,y;z)\Big[\theta(z)-\theta(x)-\theta(y)-\theta(x)\theta(y)\Big] \sum_{\eta\Subset\gamma\setminus\{x,y\}} \prod_{u\in\eta}\theta(u)dz\mu(d\gamma).$$

Changing the order of summation (2.11) we get

$$\int_\Gamma \sum_{\eta\Subset\gamma} \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} c_1(x,y;z)\Big[\theta(z)-\theta(x)-\theta(y)-\theta(x)\theta(y)\Big] \prod_{u\in\eta\setminus\{x,y\}} \theta(u)dz\mu(d\gamma).$$

Then, by (2.18) we rewrite it as

$$\int_{\Gamma_0} k_\mu(\eta) \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} c_1(x,y;z)\Big[\theta(z)-\theta(x)-\theta(y)-\theta(x)\theta(y)\Big] \prod_{u\in\eta\setminus\{x,y\}} \theta(u)dz\lambda(d\eta).$$

Finally, using Lemmas 2.21 and 2.22 we obtain

$$\int_\Gamma (L_1F_\theta)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} (L_1^\Delta k_\mu)(\eta) \prod_{u\in\eta} \theta(u)\lambda(d\eta),$$

where

$$L_1^\Delta = L_{11}^\Delta + L_{12}^\Delta + L_{13}^\Delta + L_{14}^\Delta \tag{3.3}$$

consists of four summands

$$\begin{aligned}
L_{11}^\Delta k(\eta) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{z\in\eta} c_1(x,y;z)k(\eta\backslash z\cup\{x,y\})dxdy,\\
L_{12}^\Delta k(\eta) &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{x\in\eta} c_1(x,y;z)k(\eta\cup y)dydz,\\
L_{13}^\Delta k(\eta) &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{y\in\eta} c_1(x,y;z)k(\eta\cup x)dxdz,\\
L_{14}^\Delta k(\eta) &= -\int_{\mathbb{R}^d}\sum_{\{x,y\}\subset\eta} c_1(x,y;z)dz\ k(\eta).
\end{aligned}$$

In the case of L_2, the procedure is similar. First, by (2.22) and (2.8)

$$\int_\Gamma (L_2F_\theta)(\gamma)\mu(d\gamma) = \int_\Gamma\sum_{x\in\gamma}\int_{\mathbb{R}^d} c_2(x;y)\prod_{z\in\gamma\backslash x} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\sum_{\eta\Subset\gamma\backslash x}\prod_{u\in\eta}\theta(u)dy\mu(d\gamma).$$

Then we split the product of repulsion terms

$$\int_\Gamma\sum_{x\in\gamma}\int_{\mathbb{R}^d} c_2(x;y)\sum_{\eta\Subset\gamma\backslash x}\prod_{z\in\gamma\backslash x\backslash\eta}\Big(1+e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta}\theta(u)dy\mu(d\gamma).$$

Using (2.8) we obtain

$$\int_\Gamma\sum_{x\in\gamma}\int_{\mathbb{R}^d} c_2(x;y)\sum_{\eta\Subset\gamma\backslash x}\sum_{\xi\Subset\gamma\backslash x\backslash\eta}\prod_{z\in\xi}\Big(e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta}\theta(u)dy\mu(d\gamma).$$

Next, by (2.9) we transform it into

$$\int_\Gamma\sum_{x\in\gamma}\int_{\mathbb{R}^d} c_2(x;y)\sum_{\xi\Subset\gamma\backslash x}\sum_{\eta\subset\xi}\prod_{z\in\xi\backslash\eta}\Big(e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta}\theta(u)dy\mu(d\gamma).$$

Changing the order of summation, recall (2.10), we get

$$\int_\Gamma\sum_{\xi\Subset\gamma}\sum_{x\in\xi}\int_{\mathbb{R}^d} c_2(x;y)\sum_{\eta\subset\xi\backslash x}\prod_{z\in\xi\backslash x\backslash\eta}\Big(e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta}\theta(u)dy\mu(d\gamma).$$

Then, by (2.18)

$$\int_{\Gamma_0} k_\mu(\xi)\sum_{x\in\xi}\int_{\mathbb{R}^d} c_2(x;y)\sum_{\eta\subset\xi\backslash x}\prod_{z\in\xi\backslash x\backslash\eta}\Big(e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta}\theta(u)dy\lambda(d\xi).$$

Changing the order of summation with the use of (2.10) again, we obtain

$$\int_{\Gamma_0}\sum_{\eta\subset\xi} k_\mu(\xi)\sum_{x\in\eta}\int_{\mathbb{R}^d} c_2(x;y)\prod_{z\in\xi\backslash\eta}\Big(e^{-\phi(y-z)}-1\Big)\prod_{z\in\eta\backslash x} e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta\backslash x}\theta(u)dy\lambda(d\xi)$$

Finally, by Lemma 2.20 and then by Lemma 2.21

$$\int_{\Gamma}(L_2F_\theta)(\gamma)\mu(d\gamma)=\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{x\in\eta}k_\mu(\eta\cup\xi)c_2(x;y)\prod_{z\in\xi}\Big(e^{-\phi(y-z)}-1\Big)\times$$
$$\prod_{z\in\eta\setminus x}e^{-\phi(y-z)}\Big[\theta(y)-\theta(x)\Big]\prod_{u\in\eta\setminus x}\theta(u)dy\lambda(d\xi)\lambda(d\eta)=\int_{\Gamma_0}\prod_{u\in\eta}\theta(u)(L_2^{\Delta}k_\mu)(\eta)\lambda(d\eta),$$

where

$$L_2^{\Delta}=L_{21}^{\Delta}+L_{22}^{\Delta}\tag{3.4}$$

with

$$L_{21}^{\Delta}k(\eta)=\int_{\mathbb{R}^d}\sum_{y\in\eta}c_2(x;y)\prod_{u\in\eta\setminus y}e^{-\phi(y-u)}\ (Q_yk)(\eta\setminus y\cup x)dx,$$
$$L_{22}^{\Delta}k(\eta)=-\int_{\mathbb{R}^d}\sum_{x\in\eta}c_2(x;y)\prod_{u\in\eta\setminus x}e^{-\phi(y-u)}\ (Q_yk)(\eta)dy,$$

and

$$(Q_yk)(\eta)=\int_{\Gamma_0}k(\eta\cup\xi)\prod_{u\in\xi}(e^{-\phi(y-u)}-1)\lambda(d\xi).\tag{3.5}$$

Therefore, we can try to study the dynamics of the system in the terms of correlation functions of the states, instead of observables, recall (2.22). In the place of problem (2.20), we consider (2.23).

3.2. Dynamics in $\mathcal{K}_\theta$. In this section we study the problem (2.23) in the scale of Banach spaces $\mathcal{K}_\theta$, recall Definition 2.23. The equation of interest is

$$\frac{d}{dt}k_t(\eta)=L^{\Delta}k_t(\eta),\qquad k_{t=0}=k_0\tag{3.6}$$

with $L^{\Delta}=L_1^{\Delta}+L_2^{\Delta}$ specified in (3.3) and (3.4). The results of this section are published in [23].

First, let us notice that operator $L^{\Delta}:\mathcal{K}_\theta\to\mathcal{K}_{\theta'}$ is bounded, if $\theta'>\theta$. Indeed, we have

$$||L^{\Delta}||_{\theta\theta'}\leq\frac{\beta(\theta)}{e(\theta'-\theta)}+\frac{2c_1^{\max}}{e^2(\theta'-\theta)^2}\tag{3.7}$$

where $||\cdot||_{\theta\theta'}$ denotes the operator norm for operators acting from $\mathcal{K}_\theta$ to $\mathcal{K}_{\theta'}$ and

$$\beta(\theta)=\frac{3}{2}e^{\theta}\langle c_1\rangle+2\exp(\langle\phi\rangle e^{\theta})\langle c_2\rangle,$$

recall (3.1) for definitions of $\langle c_1\rangle$, $\langle c_2\rangle$, $\langle\phi\rangle$ and $c_1^{\max}$. The estimate follows directly by the following lemma.

Lemma 3.1. *Let $\theta'>\theta$ and $L^{\Delta}:\mathcal{K}_\theta\to\mathcal{K}_{\theta'}$ be given as above. Then*

$$||L_{11}^{\Delta}||_{\theta\theta'},||L_{12}^{\Delta}||_{\theta\theta'},||L_{13}^{\Delta}||_{\theta\theta'}\leq\frac{e^{\theta}\langle c_1\rangle}{2(\theta'-\theta)e},$$
$$||L_{14}^{\Delta}||_{\theta\theta'}\leq\frac{2c_1^{max}}{(\theta'-\theta)^2e^2},$$
$$||L_{21}^{\Delta}||_{\theta\theta'},||L_{22}^{\Delta}||_{\theta\theta'}\leq\frac{\exp(\langle\phi\rangle e^{\theta})\langle c_2\rangle}{e(\theta'-\theta)}.$$

PROOF. Using (3.3), (2.24) and (3.1) one obtains

$$|L_{11}^{\Delta}k(\eta)| \leq \frac{1}{2}||k||_{\theta}e^{\theta(|\eta|+1)}|\eta|\langle c_1\rangle.$$

Next, by inequality

$$xe^{-ax} \leq \frac{1}{ae}, \;\; a > 0, \tag{3.8}$$

we have

$$|L_{11}^{\Delta}k(\eta)|e^{-\theta'|\eta|} \leq \frac{e^{\theta}\langle c_1\rangle}{2(\theta'-\theta)e}||k||_{\theta}.$$

The latter gives

$$||L_{11}^{\Delta}||_{\theta\theta'} \leq \frac{e^{\theta}\langle c_1\rangle}{2(\theta'-\theta)e}$$

With the same arguments, one obtains identical estimates for L_{12}^{Δ} and L_{13}^{Δ}. Similarly, for L_{14}^{Δ} by

$$x^2e^{-ax} \leq \frac{4}{a^2e^2}, \;\; a > 0 \tag{3.9}$$

we have

$$||L_{14}^{\Delta}||_{\theta\theta'} \leq \frac{2c_1^{\max}}{(\theta'-\theta)^2e^2}.$$

Working with L_{21}^{Δ}, notice that by Definition 2.18, (3.1) and inequality

$$1 + x \leq e^x$$

we can estimate

$$\int_{\Gamma_0}\prod_{u\in\xi}(1-e^{-\phi(y-u)})e^{\theta|\xi|}\lambda(d\xi) = \tag{3.10}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{e^{\theta n}}{n!}\int_{(\mathbb{R}^d)^n}\prod_{i=1}^{n}(1-e^{-\phi(y-x_i)})dx_1...dx_n \leq \exp(\langle\phi\rangle e^{\theta}),$$

which together with (3.8), (2.24) and (3.1) again, yields

$$||L_{21}^{\Delta}||_{\theta\theta'} \leq \frac{\exp(\langle\phi\rangle e^{\theta})\langle c_2\rangle}{e(\theta'-\theta)}.$$

One can obtain the same estimate for L_{22}^{Δ} reasoning analogously. □

Next, we show that (3.6) has the unique solution for a finite time horizon.

THEOREM 3.2. *If $k_0 \in \mathcal{K}_{\alpha_0}$ for a given $\alpha_0 \in \mathbb{R}$, then for any $\alpha_* > \alpha_0$ equation (3.6) has a unique classical solution $k_t \in \mathcal{K}_{\alpha_*}$ on $[0, T)$ with*

$$T = T(\alpha_*, \alpha_0) = \frac{\alpha_* - \alpha_0}{\frac{3}{2}e^{\alpha_*}\langle c_1\rangle + 2\exp(\langle\phi\rangle e^{\alpha_*})\langle c_2\rangle}.$$

PROOF. The method used to obtain the result is a modification of Ovsyannikov's method, similar to one used in [18]. The "standard" Ovsyannikov's method requires the operator to have its norm estimated by an expression like the first summand of estimate (3.7), with $\theta' - \theta$ in the first power. It is not the case for the second summand and that is why a little more sophisticated method has to be used. Let us split the operator L^{Δ} into two parts:

$$A = L_{14}^{\Delta} \quad \text{and} \quad B = L_{11}^{\Delta} + L_{12}^{\Delta} + L_{13}^{\Delta} + L_{21}^{\Delta} + L_{22}^{\Delta}.$$

Note that A is a multiplication operator

$$Ak(\eta) = -\Psi(\eta)k(\eta)$$

with

$$\Psi(\eta) = \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} c_1(x,y;z)dz,$$

see (3.3). When considered as operator acting from $\mathcal{K}_\theta$ to $\mathcal{K}_{\theta'}$, its norm can be estimated by the second summand of (3.7) and the norm of B can be estimated by the "convenient" first summand.

Now, for any $\theta' > \theta$ define $S_{\theta\theta'}(t) : \mathcal{K}_\theta \to \mathcal{K}_{\theta'}$ by

$$(S_{\theta\theta'}(t)k)(\eta) = e^{-\Psi(\eta)t}k(\eta).$$

It is continuous as a function of t, as for $k \in \mathcal{K}_\theta$

$$||S_{\theta\theta'}(t)k - S_{\theta\theta'}(0)k||_{\theta'} \leq t \cdot \frac{2c_1^{\max}||k||_\theta}{(\theta'-\theta)^2 e^2} \xrightarrow{t\to 0} 0.$$

Additionally for any $\theta'' \in (\theta, \theta')$

$$\frac{d}{dt}S_{\theta\theta'}(t) = A_{\theta''\theta'}S_{\theta\theta''}(t). \tag{3.11}$$

Obviously $S_{\theta\theta'}(t)$ is a bounded operator from $\mathcal{K}_\theta$ to $\mathcal{K}_{\theta'}$ for any $t \geq 0$ (which would be also the case for $S_{\theta\theta}$). In the case $t = 0$ it is an identity operator, embedding $\mathcal{K}_\theta$ into $\mathcal{K}_{\theta'}$.

Now consider any $\alpha_* > \alpha_0$ and $q > 1$. Define the partition $a_0, \ldots, a_{2n+1}$ of the interval $[\alpha_0, \alpha_*]$ as follows:

$$\begin{aligned} a_0 &= \alpha_0 \\ a_{2k+1} &= a_{2k} + \frac{(q-1)(\alpha_* - \alpha_0)}{q(n+1)},\ 0 \leq k \leq n \\ a_{2k} &= a_{2k-1} + \frac{(\alpha_* - \alpha_0)}{qn},\ 1 \leq k \leq n, \end{aligned}$$

that is

$$\begin{aligned} a_{2k+1} &= \alpha_0 + \left(\frac{k+1}{n+1}\cdot\frac{q-1}{q} + \frac{k}{n}\cdot\frac{1}{q}\right)(\alpha_* - \alpha_0) \\ a_{2k} &= \alpha_0 + \left(\frac{k}{n+1}\cdot\frac{q-1}{q} + \frac{k}{n}\cdot\frac{1}{q}\right)(\alpha_* - \alpha_0). \end{aligned}$$

In particular $a_{2n+1} = \alpha_*$.

For $0 \leq t_n \leq t_{n-1} \leq \ldots \leq t_2 \leq t_1 \leq t$ define the operator $\pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, ..., t_n) : \mathcal{K}_{\alpha_0} \to \mathcal{K}_{\alpha_*}$ as

$$\begin{aligned} \pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, ..., t_n) = S_{a_{2n}a_{2n+1}}(t-t_1)B_{a_{2n-1}a_{2n}}S_{a_{2n-2}a_{2n-1}}(t_1 - t_2)B_{a_{2n-1}a_{2n}}\ldots \\ \ldots S_{a_2a_3}(t_{n-1} - t_n)B_{a_1a_2}S_{a_0a_1}(t_n) \end{aligned}$$

In view of (3.11) we have

$$\frac{d}{dt}\pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, ..., t_n) = A_{\alpha\alpha_*}\pi^{(n)}_{\alpha_0\alpha}(t, t_1, ..., t_n)$$

for $\alpha \in (\alpha_0, \alpha_*)$.

By Lemma 3.1, we obtain

$$||\pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, ..., t_n)|| \leq \prod_{k=1}^{n}\left(\beta(a_{2k-1})\frac{qn}{(\alpha_* - \alpha_0)e}\right) \leq \left(\frac{n}{e}\right)^n \cdot \left(\frac{q}{T(\alpha_*, \alpha_0)}\right)^n$$

with
$$T(\alpha_*, \alpha_0) = \frac{\alpha_* - \alpha_0}{\beta(\alpha_*)}.$$

Now for $n \in \mathbb{N}$ consider
$$Q^{(n)}_{\alpha_0\alpha_*}(t) = S_{\alpha_0\alpha_*}(t) + \sum_{k=1}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \pi^{(k)}_{\alpha_0\alpha_*}(t, t_1, ..., t_k) dt_1 dt_2 \dots dt_k.$$
We have
$$Q^{(n)}_{\alpha_0\alpha_*}(t) - Q^{(n-1)}_{\alpha_0\alpha_*}(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, ..., t_n) dt_1 dt_2 \dots dt_n$$
and
$$||Q^{(n)}_{\alpha_0\alpha_*}(t) - Q^{(n-1)}_{\alpha_0\alpha_*}(t)|| \le \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \left(\frac{n}{e}\right)^n \cdot \left(\frac{q}{T(\alpha_*, \alpha_0)}\right)^n dt_1 dt_2 \dots dt_n =$$
$$= \frac{1}{n!} \left(\frac{n}{e}\right)^n \cdot \left(\frac{qt}{T(\alpha_*, \alpha_0)}\right)^n.$$
For $t < T(\alpha_*, \alpha_0)$, we can pick $q > 1$ such that $qt < T(\alpha_*, \alpha_0)$, which by the above estimate together with the Stirling's formula makes $Q^{(n)}_{\alpha_0\alpha_*}(t)$ a Cauchy sequence. Denote it's limit by $Q_{\alpha_0\alpha_*}(t)$.

For $\alpha \in (\alpha_0, \alpha_*)$ we have
$$\frac{d}{dt} Q^{(n)}_{\alpha_0\alpha_*}(t) = (A_{\alpha\alpha_*} + B_{\alpha\alpha_*}) Q^{(n)}_{\alpha_0\alpha_*}(t) - B_{\alpha\alpha_*} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \pi^{(n)}_{\alpha_0\alpha}(t, t_1, \dots, t_n) dt_1 \dots dt_n.$$
Taking $t < T(\alpha_*, \alpha_0)$ and α close enough to α_*
$$\frac{d}{dt} Q_{\alpha_0\alpha_*}(t) = (A_{\alpha\alpha_*} + B_{\alpha\alpha_*}) Q_{\alpha_0\alpha}(t).$$
Therefore we obtain for $t < T(\alpha_*, \alpha_0)$ the classical solution
$$k_t = Q_{\alpha_0\alpha_*}(t) k_0.$$

Remark

By the estimations used above we obtain
$$||Q_{\alpha_0\alpha_*}(t)|| \le \frac{T(\alpha_*, \alpha_0)}{T(\alpha_*, \alpha_0) - t} \tag{3.12}$$
for $t < T(\alpha_*, \alpha_0)$.

To finish the proof we need to show the uniqueness of the solution. Suppose that $u, v \in \mathcal{K}_{\alpha_*}$ are solutions of the problem. Denote $w = u - v$. Then
$$w_t = \int_0^t e^{-(t-s)\Psi} B w_s ds$$
and iterating n times
$$w_t = \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} e^{-(t-t_1)\Psi} B \dots e^{-(t_{n-1}-t_n)\Psi} B w_{t_n} dt_1 \dots dt_n. \tag{3.13}$$

Take $\alpha^* = 2\alpha_* - \alpha_0 > \alpha_*$ and for any $q > 1$ define partition b_0, ... b_{2n} of the interval $[\alpha_*, \alpha^*]$:

$$b_{2k} = \alpha_* + \frac{k}{n}(\alpha^* - \alpha_*),\ 0 \le k \le n$$

$$b_{2k+1} = \alpha_* + \left(\frac{k+1}{n}\cdot\frac{1}{q} + \frac{k}{n}\cdot\frac{q-1}{q}\right)(\alpha^* - \alpha_*),\ 0 \le k \le n-1$$

with b_{2n} being α^*. Consider operators in (3.13) as acting from $\mathcal{K}_{b_i}$ to $\mathcal{K}_{b_{i+1}}$, $i = 0, ..., 2n-1$. Using estimates of norm of the operator B and noting that $\alpha^* - \alpha_* = \alpha_* - \alpha_0$ we obtain for $t < T(\alpha_*, \alpha_0)$:

$$\begin{aligned}||w_t||_{\alpha^*} &\le \frac{t^n}{n!}||B||_{b_0 b_1} \dots ||B||_{b_{n-2}b_{n-1}} \sup_{s\in[0,t]} ||w_s||_{\alpha_*} \\ &\le \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(\frac{qt}{T(\alpha_*, \alpha_0)}\right)^n \sup_{s\in[0,T(\alpha_*,\alpha_0)]} ||w_s||_{\alpha_*}\end{aligned}$$

and similarly as before, picking q close enough to 1

$$||w_t||_{\alpha^*} = 0, \quad t < T(\alpha_*, \alpha_0)$$

which implies

$$w_t = 0, \quad t < T(\alpha_*, \alpha_0)$$

as an element of $\mathcal{K}_{\alpha^*}$ and being the zero vector, also of $\mathcal{K}_{\alpha_*}$. Therefore $u = v$ which finishes the proof.

□

Assume that the inital state μ_0 of our system is sub-Poissonian. Then we may consider its correlation function k_0 as the initial state of equation (2.23). Theorem 3.2 guarantees the existence and uniqueness of solution k_t of this equation for some finite time horizon. The question arises, whether this evolution may be related to the evolution of states. That is, whether k_t is a correlation function for a unique measure $\mu_t \in \mathcal{P}_{\exp}$. The next statement gives an answer to this question.

THEOREM 3.3. *Suppose that k_0 is the correlation function of the initial state $\mu_0 \in \mathcal{P}_{\exp}$. Then for any $\alpha_* > \alpha_0$, k_t discussed in Theorem 3.2 is again a correlation function of a unique state $\mu_t \in \mathcal{P}_{\exp}$ for $t < T = \frac{1}{2}T(\alpha_*, \alpha_0)$, where $T(\alpha_*, \alpha_0)$ is defined as in Theorem 3.2.*

This result gives us a weak solution μ_t to the Fokker-Planck equation (2.19) for a finite time horizon T in the sense that

$$\frac{d}{dt}\int_\Gamma F_\theta(\gamma)\mu_t(d\gamma) = \int_\Gamma F_\theta(\gamma)(L^*\mu_t)(d\gamma), \quad t < T, \quad \theta \in L^1(\mathbb{R}^d),$$

see (2.21) and (2.16).

4. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3, which can be found in [23] in almost the same form. First, note that given a correlation function k_0 of the initial sub-Poissonian state μ_0 we know, recall Definition 2.17, that $k_0(\emptyset) = 1$ and that there exists $\alpha_0 \in \mathbb{R}$ such that $k_0 \in \mathcal{K}_{\alpha_0}$. Therefore, by Theorem 3.2 we obtain for $t < T$ a classical solution k_t of equation (2.23) in $\mathcal{K}_{\alpha_*}$, where $\alpha_* > \alpha$. Note that $L^\Delta = L_1^\Delta + L_2^\Delta$ given by (3.3) and (3.4) guarantees that for any k we have $L^\Delta k(\emptyset) = 0$. It means that $k_t(\emptyset)$ is constantly equal to 1. Because $k_t \in \mathcal{K}_{\alpha_*}$, there exists a constant $C = e^{\alpha_* + \ln||k||_{\alpha_*}}$ such that $k_t(\eta) \le C^{|\eta|}$. In view of Proposition 2.19, to show that k_t is a correlation function of a unique measure $\mu_t \in \mathcal{P}_{\exp}$ it remains to prove that the positivity property holds:

$$\int_{\Gamma_0} Gk_t \, d\lambda \geq 0 \quad \text{for all } G \in \mathcal{B}^*_{bs}. \tag{4.1}$$

In order to do that, we take several steps. First, in Subsection 4.1 we introduce an auxiliary model with parameter σ with altered action of L (changed operator will be denoted L^σ). For this new case, we obtain evolution in the scale of $\mathcal{K}_\theta$ spaces similarly as previously for the original model. Then we show convergence in a weak sense of the auxiliary model solution to the original one (Subsection 4.3). For this purpose we use the predual evolution (Subsection 4.2). To show that the required positivity property holds for the auxiliary evolution, first we need to consider the local evolution (Subsection 4.4), so that we can link our problem with the evolution of so-called local correlation functions (see (4.26)), for which it is easy to show required positivity. Then, in Subsection 4.4.6 we pass to the limit with parameters describing locality, obtaining the desired result.

4.1. Auxiliary model. The auxiliary model will be introduced by the alteration of operator L in (3.1). For given parameter $\sigma > 0$ define $\psi_\sigma(x) = e^{-\sigma|x|^2}$, $x \in \mathbb{R}^d$. Obviously $\psi_\sigma(x) \in [0,1]$ and

$$\int_{\mathbb{R}^d} e^{-\sigma|x|^2} dx = \left(\frac{\pi}{\sigma}\right)^{d/2}.$$

Introduce

$$\begin{aligned} L^\sigma F(\gamma) = &\sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d} \psi_\sigma(z) c_1(x,y;z) \Big(F\big(\gamma\backslash\{x,y\}\cup z\big) - F(\gamma)\Big) dz \\ &+ \sum_{x\in\gamma} \int_{\mathbb{R}^d} \psi_\sigma(y) \tilde{c}_2(x;y;\gamma) \Big(F\big(\gamma\backslash x \cup y\big) - F(\gamma)\Big) dy. \end{aligned} \tag{4.2}$$

Similarly as before, we pass its action to the operator acting on functions $k : \Gamma_0 \to \mathbb{R}$ and split it into parts $L^{\Delta,\sigma} = L_1^{\Delta,\sigma} + L_2^{\Delta,\sigma}$ with

$$\begin{aligned} L_1^{\Delta,\sigma} k(\eta) = &\ \frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{z\in\eta} \psi_\sigma(z) c_1(x,y;z) k(\eta\backslash z \cup \{x,y\}) dx dy \\ &- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \psi_\sigma(z) \sum_{x\in\eta} c_1(x,y;z) k(\eta \cup y) dy dz \\ &- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \psi_\sigma(z) \sum_{y\in\eta} c_1(x,y;z) k(\eta \cup x) dx dz \\ &- \int_{\mathbb{R}^d} \psi_\sigma(z) \sum_{\{x,y\}\subset\eta} c_1(x,y;z) dz \ k(\eta) \\ L_2^{\Delta,\sigma} k(\eta) = &\int_{\mathbb{R}^d} \sum_{y\in\eta} (Q_y k)(\eta\backslash y \cup x) \psi_\sigma(y) c_2(x;y) \prod_{u\in\eta\backslash y} e^{-\phi(y-u)} dx \\ &- \int_{\mathbb{R}^d} (Q_y k)(\eta) \psi_\sigma(y) \sum_{x\in\eta} c_2(x;y) \prod_{u\in\eta\backslash x} e^{-\phi(y-u)} dy, \end{aligned}$$

recall (3.5) The respective summands of $L_1^{\Delta,\sigma}$ we denote as $L_{11}^{\Delta,\sigma}$, $L_{12}^{\Delta,\sigma}$, $L_{13}^{\Delta,\sigma}$, $L_{14}^{\Delta,\sigma}$ and $L_{21}^{\Delta,\sigma}$, $L_{22}^{\Delta,\sigma}$ in the case of $L_2^{\Delta,\sigma}$. Additionally, we denote

$$\begin{aligned} &A^\sigma k(\eta) = L_{14}^{\Delta,\sigma} = -\int_{\mathbb{R}^d} \psi_\sigma(z) \sum_{\{x,y\}\subset\eta} c_1(x,y;z)dz\ k(\eta), \\ &B^\sigma = L^{\Delta,\sigma} - A^\sigma. \end{aligned}$$

By $\psi_\sigma \le 1$ we obtain estimates in $\mathcal{K}_\theta$ spaces identical to (3.7). It allows us to show, just as in Theorem 3.2, the existence and uniqueness of classical solution $k_t^\sigma = Q^\sigma_{\alpha_0\alpha_*}(t)k_0$ on interval $[0, T(\alpha_*, \alpha_0))$ in the space $\mathcal{K}_{\alpha_*}$, $\alpha_* > \alpha_0$ of problem

$$\frac{d}{dt}k_t^\sigma = L^{\Delta,\sigma}k_t^\sigma, \qquad k_{t=0}^\sigma = k_0^\sigma \in \mathcal{K}_{\alpha_0}. \tag{4.3}$$

4.2. Pre-dual evolution. In this subsection we consider the pre-dual evolution. It is derived by the duality

$$\int_{\Gamma_0} G_0(\eta)k_t(\eta)\lambda(d\eta) = \int_{\Gamma_0} G_t(\eta)k_0(\eta)\lambda(d\eta).$$

In order to study the evolution of functions G_t, we need to establish a proper context of Banach spaces, which is here L^1-type $\mathcal{G}_\theta$ ($\theta \in \mathbb{R}$) with the norm

$$|G|_\theta = \int_{\Gamma_0} |G(\eta)|e^{\theta|\eta|}\lambda(d\eta). \tag{4.4}$$

Obviously for $\theta' > \theta$ we have $\mathcal{G}_{\theta'} \subset \mathcal{G}_\theta$.

Notice that $G \in \mathcal{B}_{bs}$ lies in $\mathcal{G}_\theta$ with any $\theta \in \mathbb{R}$. Indeed, let M be upper bound of G, N maximum number of particles of the support and Λ its spatial bound (recall the definition 2.15 of $\mathcal{B}_{bs}$). Then we have

$$\int_{\Gamma_0} |G(\eta)|e^{\theta|\eta|} = \sum_{n=0}^{N} \frac{1}{n!}e^{\theta n} \int_{\Lambda^n} G(x_1, \dots, x_n)dx_1 \dots dx_n \le Me^{|\Lambda|e^\theta} < \infty \tag{4.5}$$

Similarly to $Q_{\alpha_0\alpha_*}(t)$ acting between $\mathcal{K}_{\alpha_0}$ and $\mathcal{K}_{\alpha_*}$, $\alpha_* > \alpha_0$, we can construct pre-dual analogue $H_{\alpha_*\alpha_0}(t)$ from $\mathcal{G}_{\alpha_*}$ to $\mathcal{G}_{\alpha_0}$ satisfying for $G \in \mathcal{G}_{\alpha_*}$ and $k \in \mathcal{K}_{\alpha_0}$

$$\int_{\Gamma_0} G(\eta)Q_{\alpha_0\alpha_*}(t)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} H_{\alpha_*\alpha_0}(t)G(\eta)k(\eta)\lambda(d\eta). \tag{4.6}$$

In order to do that, introduce $\hat{L}$ given by the duality equation

$$\int_{\Gamma_0} G(L^\Delta k)d\eta = \int_{\Gamma_0} (\hat{L}G)kd\eta.$$

It can be shown, see Lemma 5.5, that it is of the form

$$\hat{L} = \hat{L}_1 + \hat{L}_2 \tag{4.7}$$

with

$$\hat{L}_1G(\eta) = \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} c_1(x,y;z)\Big(G(\eta\backslash\{x,y\}\cup z) - G(\eta\backslash y) - G(\eta\backslash x) - G(\eta)\Big)dz,$$

$$\hat{L}_2G(\eta) = \int_{\mathbb{R}^d} \sum_{x\in\eta} c_2(x;y) \sum_{\xi\subset\eta\backslash x} \left[G(\xi\cup y) - G(\xi\cup x)\right] \sum_{\zeta\subset\xi} \prod_{u\in\eta\backslash\xi\backslash x\cup\zeta} \left(e^{-\phi(y-u)} - 1\right) dy,$$

where, as usual, each summand is denoted by $\hat{L}_{ij}$ with adequate i and j. Denote

$$\hat{A}G(\eta) = \hat{L}_{14}G(\eta) = -\Psi(\eta)G(\eta),$$
$$\hat{B}G(\eta) = \hat{L}G(\eta) - \hat{A}G(\eta).$$

Now we repeat construction similar to one used in the proof of Theorem 3.2. In the place of $\pi^{(n)}_{\alpha_0\alpha_*}(t, t_1, \dots, t_n)$ we take

$$\omega^{(n)}_{\alpha_*\alpha_0}(t, t_1, \dots, t_n) = S_{a_1 a_0}(t - t_1)\hat{B}_{a_2 a_1} S_{a_3 a_2}(t_1 - t_2)\hat{B}_{a_4 a_3}\dots$$
$$\dots S_{a_{2n-1}a_{2n-2}}(t_{n-1} - t_n)\hat{B}_{a_{2n}a_{2n-1}} S_{a_{2n+1}a_{2n}}(t_n).$$

Note that $S_{\theta'\theta}$ acting from $\mathcal{G}_{\theta'}$ to $\mathcal{G}_\theta$, $\theta' > \theta$ is continuous. Indeed,

$$|S_{\theta'\theta}(t)G - S_{\theta'\theta}(0)G|_\theta = \int_{\Gamma_0} (1 - e^{-\Psi(\eta)t})|G(\eta)|e^{\theta|\eta|}\lambda(d\eta) \le$$
$$\le t\int_{\Gamma_0} \Psi(\eta)|G(\eta)|e^{\theta|\eta|}\lambda(d\eta) \le t\frac{2c_1^{\max}}{(\theta' - \theta)^2 e^2}|G|_{\theta'} \xrightarrow{t\to 0} 0.$$

Next, define

$$H^{(n)}_{\alpha_*\alpha_0}(t) = S_{\alpha_*\alpha_0}(t) + \sum_{k=1}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \omega^{(k)}_{\alpha_*\alpha_0}(t, t_1, ..., t_k)dt_1 dt_2 \dots dt_k \qquad (4.8)$$

and the rest follows just as in the mentioned proof of Theorem 3.2. Indeed, $||\hat{L}_1||_{\alpha_*\alpha_0}$, $||\hat{L}_2||_{\alpha_*\alpha_0}$, $||\hat{L}_{14}||_{\alpha_*\alpha_0}$ – operator norms of operators acting between $\mathcal{G}_\theta$ spaces – undergo the same estimates as, respectively, $||L_1||_{\alpha_0\alpha_*}$, $||L_2||_{\alpha_0\alpha_*}$, $||L_{14}||_{\alpha_0\alpha_*}$ for operators acting between $\mathcal{K}_\theta$ spaces with interchanged indices. Therefore we may construct for $t < T(\alpha_*, \alpha_0)$ a limit $H_{\alpha_*\alpha_0}(t)$ of (4.8) which produces the unique local classical solution $G_t = H_{\alpha_*\alpha_0}(t)G_0$ in $\mathcal{G}_{\alpha_0}$ of the problem

$$\frac{d}{dt}G_t = \hat{L}G_t, \qquad G_{t=0} = G_0 \in \mathcal{G}_{\alpha_*}. \qquad (4.9)$$

Additionally, an analogue of (3.12) holds true, i.e.

$$||H_{\alpha_*\alpha_0}(t)|| \le \frac{T(\alpha_*, \alpha_0)}{T(\alpha_*, \alpha_0) - t}. \qquad (4.10)$$

for $t < T(\alpha_*, \alpha_0)$.

In order to obtain (4.6), we show that for each $n \in \mathbb{N}$, $G \in \mathcal{G}_{\alpha_*}$ and $k \in \mathcal{K}_{\alpha_0}$ the equation

$$\int_{\Gamma_0} G(\eta)Q^{(n)}_{\alpha_0\alpha_*}(t)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} H^{(n)}_{\alpha_*\alpha_0}(t)G(\eta)k(\eta)\lambda(d\eta), \qquad (4.11)$$

holds, which in the limit gives desired result.

We have

$$\int_{\Gamma_0} G(\eta)Q^{(n)}_{\alpha_0\alpha_*}(t)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} S_{\alpha_*\alpha_0}(t)G(\eta)k(\eta)\lambda(d\eta) +$$
$$+\sum_{k=1}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \int_{\Gamma_0} G(\eta)\pi^{(k)}_{\alpha_0\alpha_*}(t, t_1, ..., t_k)k(\eta)\lambda(d\eta)dt_1 dt_2 \dots dt_k$$

and

$$\int_{\Gamma_0} G(\eta)\pi^{(k)}_{\alpha_0\alpha_*}(t,t_1,...,t_k)\lambda(d\eta) =$$

$$= \int_{\Gamma_0} G(\eta) S_{a_{2k}a_{2k+1}}(t-t_1)B_{a_{2k-1}a_{2k}}S_{a_{2k-2}a_{2k-1}}(t_1-t_2)B_{a_{2k-1}a_{2k}}\dots$$

$$\dots S_{a_2a_3}(t_{k-1}-t_k)B_{a_1a_2}S_{a_0a_1}(t_k)k(\eta)\lambda(d\eta) =$$

$$= \int_{\Gamma_0} S_{a_1a_0}(t-t_1)\hat{B}_{a_2a_1}S_{a_3a_2}(t_1-t_2)\hat{B}_{a_4a_3}\dots$$

$$\dots S_{a_{2k-1}a_{2k-2}}(t_{k-1}-t_k)\hat{B}_{a_{2k}a_{2k-1}}S_{a_{2k+1}a_{2k}}(t_k)G(\eta)k(\eta)\lambda(d\eta) =$$

$$= \int_{\Gamma_0} \omega^{(k)}_{\alpha_*\alpha_0}(t,t_1,\dots,t_k)G(\eta)k(\eta)\lambda(d\eta)$$

which results in (4.11) and therefore in (4.6).

4.3. Limit $\sigma \to 0$**.** In this subsection we show for our initial $k_0 \in \mathcal{K}_{\alpha_0}$ the convergence of $Q^\sigma(t)k_0 = k_t^\sigma$ to $Q(t)k_0 = k_t$ with $\sigma \to 0$ in the weak sense, i.e. for any $G \in \mathcal{B}_{bs}$

$$\lim_{\sigma\to 0}\int_{\Gamma_0} G(\eta)[Q(t)-Q^\sigma(t)]k_0(\eta)\lambda(d\eta) = 0. \tag{4.12}$$

The above result will be shown for $t < \frac{1}{2}T(\alpha_*\alpha_0)$, where both $Q(t)$ and $Q^\sigma(t)$ act from $\mathcal{K}_{\alpha_0}$ to $\mathcal{K}_{\alpha_*}$. First, notice that for arbitrarily chosen α_1 and α_2 satisfying $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_*$ we can write

$$[Q(t)-Q^\sigma(t)]_{\alpha_0,\alpha_*}k_0 = -\int_0^t \frac{d}{ds}[Q(t-s)Q^\sigma(s)]_{\alpha_0,\alpha_*}k_0 ds =$$

$$= \int_0^t [Q(t-s)]_{\alpha_2\alpha_*}(A-A^\sigma)_{\alpha_1\alpha_2}k^\sigma(s)ds + \int_0^t [Q(t-s)]_{\alpha_2\alpha_*}(B-B^\sigma)_{\alpha_1\alpha_2}k^\sigma(s)ds, \tag{4.13}$$

as far, as $t < \min(T(\alpha_1,\alpha_0),T(\alpha_*,\alpha_2))$ – we have $k^\sigma(s) = [Q^\sigma(s)]_{\alpha_0\alpha_1}k_0 \in \mathcal{K}_{\alpha_1}$, recall (4.3).

For the first summand we have

$$\int_{\Gamma_0} G(\eta)\int_0^t Q(t-s)(A-A^\sigma)k_s^\sigma(\eta)ds\lambda(d\eta) =$$

$$= -\int_0^t \int_{\Gamma_0} G_{t-s}(\eta)\int_{\mathbb{R}^d}(1-\psi_\sigma(z))\sum_{\{x,y\}\subset\eta} c_1(x,y;z)dzk_s^\sigma(\eta)\lambda(d\eta)ds$$

Because $G \in \mathcal{B}_{bs}$, it lies in $\mathcal{G}_\theta$ with any θ, see (4.5), in particular with $\theta = \alpha_*$. Next, by (4.9) we have $G_{t-s} \in \mathcal{G}_{\alpha_2}$ for $t-s < T(\alpha_*,\alpha_2)$. This gives us by (3.12) and (4.10) estimates

$$||k_s^\sigma||_{\alpha_1} \le \frac{T(\alpha_1,\alpha_0)}{T(\alpha_1,\alpha_0)-s}||k_0||_{\alpha_0}, \quad s < T(\alpha_1,\alpha_0),$$

$$|G_{t-s}|_{\alpha_2} \le \frac{T(\alpha,\alpha_2)}{T(\alpha,\alpha_2)-(t-s)}|G_0|_\alpha, \quad t-s < T(\alpha,\alpha_2). \tag{4.14}$$

Denote

$$C(z,\eta)=\frac{1}{1+\frac{|\eta|(|\eta|-1)}{2}}\sum_{\{x,y\}\subset\eta}c_1(x,y;z)dz$$

$$g(z)=\operatorname*{ess\,sup}_{\eta\in\Gamma_0}C(z,\eta)$$

We have then

$$\int_{\mathbb{R}^d}g(z)dz\le c_1^{\max}.$$

Using above, we write

$$\left|\int_{\Gamma_0}G_{t-s}(\eta)\int_{\mathbb{R}^d}(1-\psi_\sigma(z))\sum_{\{x,y\}\subset\eta}c_1(x,y;z)dzk_s^\sigma(\eta)\lambda(d\eta)\right|\le$$

$$\le\int_{\mathbb{R}^d}(1-\psi_\sigma(z))g(z)dz\int_{\Gamma_0}\left(1+\frac{|\eta|(|\eta|-1)}{2}\right)|G_{t-s}(\eta)k_s^\sigma(\eta)|\lambda(d\eta).$$

Notice that

$$\int_{\Gamma_0}\left(1+\frac{|\eta|(|\eta|-1)}{2}\right)|G_{t-s}(\eta)k_s^\sigma(\eta)|\lambda(d\eta)\le$$

$$\int_{\Gamma_0}e^{\alpha_2|\eta|}|G_{t-s}(\eta)|\left(1+\frac{|\eta|(|\eta|-1)}{2}\right)e^{(\alpha_1-\alpha_2)|\eta|}||k_s^\sigma||_{\alpha_1}\lambda(d\eta)\le$$

$$\le\left(1+\frac{2}{e^2(\alpha_2-\alpha_1)^2}\right)||k_s^\sigma||_{\alpha_1}|G_{t-s}|_{\alpha_2}$$

$$\le\left(1+\frac{2}{e^2(\alpha_2-\alpha_1)^2}\right)D(t),$$

where

$$D(t)=\frac{T(\alpha_1,\alpha_0)}{T(\alpha_1,\alpha_0)-t}||k_0||_{\alpha_0}\frac{T(\alpha,\alpha_2)}{T(\alpha,\alpha_2)-t}|G_0|_\alpha \tag{4.15}$$

and the last estimate was performed by means of (4.14) for $s\le t<\min(T(\alpha_1,\alpha_0),T(\alpha,\alpha_2))$.

Finally by means of above estimates we obtain for $t<\min(T(\alpha_1,\alpha_0),T(\alpha,\alpha_2))$

$$\left|\int_0^t\int_{\Gamma_0}G_{t-s}(\eta)\int_{\mathbb{R}^d}(1-\psi_\sigma(z))\sum_{\{x,y\}\subset\eta}c_1(x,y;z)dzk_s^\sigma(\eta)\lambda(d\eta)ds\right|\le$$

$$\le\left(1+\frac{2}{e^2(\alpha_2-\alpha_1)^2}\right)tD(t)\int_{\mathbb{R}^d}(1-\psi_\sigma(z))g(z)dz\le tD(t)c_1^{\max}<\infty$$

and by means of Lebesgue's dominated convergence theorem

$$\lim_{\sigma\to0}\left|\int_0^t\int_{\Gamma_0}G_{t-s}(\eta)\int_{\mathbb{R}^d}(1-\psi_\sigma(z))\sum_{\{x,y\}\subset\eta}c_1(x,y;z)dzk_s^\sigma(\eta)\lambda(d\eta)ds\right|\le$$

$$\le\left(1+\frac{2}{e^2(\alpha_2-\alpha_1)^2}\right)tD(t)\int_{\mathbb{R}^d}\lim_{\sigma\to0}(1-\psi_\sigma(z))g(z)dz=0,$$

that is

$$\lim_{\sigma\to 0}\int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(A-A^\sigma)k_s^\sigma(\eta)\lambda(d\eta)ds = 0 \tag{4.16}$$

Now let us move on to the second summand of (4.3). Just like for the first one, we write

$$\int\limits_{\Gamma_0} G(\eta)\int_0^t Q(t-s)(B-B^\sigma)k_s^\sigma(\eta)ds\lambda(d\eta) = \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(B-B^\sigma)k_s^\sigma(\eta)\lambda(d\eta)ds =$$

$$= \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{11}^\Delta - L_{11}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds + \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{12}^\Delta - L_{12}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds$$

$$+ \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{13}^\Delta - L_{13}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds + \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{21}^\Delta - L_{21}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds$$

$$+ \int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{22}^\Delta - L_{22}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds \tag{4.17}$$

and we deal with the convergence of each summand separately. By the Minlos lemma in the form (2.21) and (2.22) we have

$$\left|\int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{11}^\Delta - L_{11}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)\right| \leq$$

$$\leq \frac{1}{2}\int\limits_{\Gamma_0} |G_{t-s}(\eta)| \int\limits_{(\mathbb{R}^d)^2} \sum_{z\in\eta}(1-\psi_\sigma(z))c_1(x,y;z)|k_s^\sigma(\eta\backslash z\cup\{x,y\})|dxdy\lambda(d\eta) =$$

$$= \int\limits_{\Gamma_0}\int\limits_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} |G_{t-s}(\eta\backslash\{x,y\}\cup z)|(1-\psi_\sigma(z))c_1(x,y;z)dz|k_s^\sigma(\eta)|\lambda(d\eta) \leq$$

$$\leq ||k_s^\sigma||_{\alpha_1}\int\limits_{\mathbb{R}^d}(1-\psi_\sigma(z))\int\limits_{\Gamma_0} \sum_{\{x,y\}\subset\eta} |G_{t-s}(\eta\backslash\{x,y\}\cup z)|c_1(x,y;z)e^{\alpha_1|\eta|}\lambda(d\eta)dz.$$

Therefore, we obtain

$$\left|\int_0^t \int\limits_{\Gamma_0} G_{t-s}(\eta)(L_{11}^\Delta - L_{11}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds\right| \leq \int\limits_{\mathbb{R}^d}(1-\psi_\sigma(z))g(z)dz$$

with

$$g(z) = \int\limits_{\Gamma_0}\int_0^t ||k_s^\sigma||_{\alpha_1} \sum_{\{x,y\}\subset\eta} |G_{t-s}(\eta\backslash\{x,y\}\cup z)|c_1(x,y;z)e^{\alpha_1|\eta|}\lambda(d\eta)ds$$

Notice that, again by (2.21), (2.22) and (4.14) for $s \le t < \min(T(\alpha_1, \alpha_0), T(\alpha, \alpha_2))$ we obtain

$$\int_{\mathbb{R}^d} g(z)dz = \int_0^t ||k_s^\sigma||_{\alpha_1} \int_{\mathbb{R}^d} \int_{\Gamma_0} \sum_{\{x,y\}\subset\eta} |G_{t-s}(\eta\backslash\{x,y\} \cup z)| c_1(x,y;z) e^{\alpha_1|\eta|}\lambda(d\eta)dzds =$$

$$= \frac{1}{2}e^{\alpha_1} \int_0^t ||k_s^\sigma||_{\alpha_1} \int_{\Gamma_0} e^{\alpha_2|\eta|} |G_{t-s}(\eta)| e^{(\alpha_1-\alpha_2)|\eta|} \sum_{z\in\eta} \int_{(\mathbb{R}^d)^2} c_1(x,y;z)dxdy\lambda(d\eta)ds \le$$

$$\le e^{\alpha_1}\langle c_1\rangle \frac{1}{2(\alpha_2-\alpha_1)e} \int_0^t ||k_s^\sigma||_{\alpha_1} \int_{\Gamma_0} e^{\alpha_2|\eta|}|G_{t-s}(\eta)|\lambda(d\eta)ds =$$

$$= e^{\alpha_1}\langle c_1\rangle \frac{1}{2(\alpha_2-\alpha_1)e} \int_0^t ||k_s^\sigma||_{\alpha_1} |G_{t-s}|_{\alpha_2} ds \le e^{\alpha_1}\langle c_1\rangle \frac{1}{2(\alpha_2-\alpha_1)e} tD(t) < \infty,$$

where $D(t)$ is as in (4.15). By the Lebesgue's theorem, we have

$$\lim_{\sigma\to 0} \int_{\mathbb{R}^d} (1-\psi_\sigma(z))g(z)dz = \int_{\mathbb{R}^d} \lim_{\sigma\to 0}(1-\psi_\sigma(z))g(z)dz = 0.$$

One can show by the direct repetition of above steps the required convergence of the next two summands of (4.17). Let us move on to the fourth one, which we will approach with the similar method. Again using Minlos lemma we obtain

$$\left| \int_0^t \int_{\Gamma_0} G_{t-s}(\eta)(L_{21}^\Delta - L_{21}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds \right| \le$$

$$\le \int_{\mathbb{R}^d} (1-\psi_\sigma(y)) \int_0^t \int_{\Gamma_0} \sum_{x\in\eta} c_2(x;y)|G_{t-s}(\eta\backslash x \cup y)|$$

$$\int_{\Gamma_0} |k_s^\sigma(\eta\cup\xi)| \prod_{u\in\xi}\left(1-e^{-\phi(y-u)}\right)\lambda(d\xi)\lambda(d\eta)dsdy.$$

Denote

$$g(y) = \int_0^t \int_{\Gamma_0} \sum_{x\in\eta} c_2(x;y)|G_{t-s}(\eta\backslash x\cup y)| \int_{\Gamma_0} |k_s^\sigma(\eta\cup\xi)| \prod_{u\in\xi}\left(1-e^{-\phi(y-u)}\right)\lambda(d\xi)\lambda(d\eta)ds.$$

We have by (3.10) and (2.21) for $s \le t < \min(T(\alpha_1,\alpha_0), T(\alpha,\alpha_2))$

$$\int_{\mathbb{R}^d} g(y)dy \le \exp(\langle\phi\rangle e^{\alpha_1}) \int_0^t ||k_s^\sigma||_{\alpha_1} \int_{\Gamma_0} e^{\alpha_1|\eta|}|G_{t-s}(\eta)| \sum_{y\in\eta} \int_{\mathbb{R}^d} c_2(x;y)dx\lambda(d\eta)ds \le$$

$$\le \exp(\langle\phi\rangle e^{\alpha_1})\langle c_2\rangle \frac{1}{(\alpha_2-\alpha_1)e} tD(t) < \infty$$

with $D(t)$ given by (4.15). Again, by dominated convergence theorem

$$\lim_{\sigma\to 0} \left| \int_0^t \int_{\Gamma_0} G_{t-s}(\eta)(L_{21}^\Delta - L_{21}^{\Delta,\sigma})k_s^\sigma(\eta)\lambda(d\eta)ds \right| = 0.$$

The last summand of (4.17) can be dealt by the same method, which finally yields

$$\lim_{\sigma\to 0}\int_0^t \int_{\Gamma_0} G_{t-s}(\eta)(B-B^\sigma)k_s^\sigma(\eta)\lambda(d\eta)ds = 0$$

and together with (4.16) proves (4.12) for $t < \min(T(\alpha_1,\alpha_0), T(\alpha_*,\alpha_2))$. Now take $\alpha_1 = \frac{\alpha_*+\alpha_0}{2}$ and $\alpha_2 = \frac{\alpha_*+\alpha_0}{2} + \varepsilon\beta(\alpha_*)$ with $\varepsilon > 0$ small enough – such that $\alpha_2 < \alpha_*$. Then $T(\alpha_1,\alpha_0) \geq \frac{1}{2}T(\alpha_*,\alpha_0)$ and $T(\alpha_*,\alpha_2)) = T(\alpha_*,\alpha_0) - \varepsilon$, which in the limit $\varepsilon \to 0$ gives us the result for $t < \frac{1}{2}T(\alpha_*,\alpha_0)$.

4.4. The local evolution. In this subsection we deal with the evolution described by $L^{\Delta,\sigma}$ but with the localized initial condition $k_0^{\Lambda,N}$, which is precised below by (4.19). The dynamics will be considered in spaces $\mathcal{K}_\theta$, recall Definition 2.23 and in $\mathcal{G}_\theta^{\text{fac}}$, introduced in Subsection 4.4.1, where the positivity property (4.1) is easy to show, as the evolution there is related to the evolution of densities of local states, see (4.26). In order to connect these two cases, we introduce $\mathcal{U}_\theta^\sigma$, a subset of both $\mathcal{K}_\theta$ and $\mathcal{G}_{\theta'}^{\text{fac}}$ (with appropriate θ', see Subsection 4.4.1, in particular (4.24)) such that $k_0^{\Lambda,N} \in \mathcal{U}_{\alpha_0}^\sigma$. By showing the existence and uniqueness of the solutions in each of the above spaces with the same initial condition, we deduce that they describe the same evolution and therefore obtain required positivity property for solution in $\mathcal{K}_\theta$, see Subsection 4.4.5. To finish the proof of Theorem 3.3 it suffices to show that our solutions with increasing Λ and N converge to k_t^σ, which is performed in Subsection 4.4.6.

Recall that initial condition k_0 is the correlation function of sub-Poissonian state μ_0. Consider its projection μ_0^Λ on Γ_Λ for bounded borel $\Lambda \subset \mathbb{R}^d$, see (2.12). It is absolutely continuous with respect to Lebesgue-Poisson measure λ. Denote by R_0^Λ its density. Then by (2.17) for $\eta \in \Gamma_\Lambda$ we have

$$k_0(\eta) = \int_{\Gamma_\Lambda} R_0^\Lambda(\eta\cup\xi)\lambda(d\xi), \quad \eta \in \Gamma_\Lambda. \tag{4.18}$$

Localized initial condition $k_0^{\Lambda,N}$ is defined as

$$k_0^{\Lambda,N}(\eta) = \int_{\Gamma_\Lambda} R_0^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi), \quad \eta \in \Gamma_0, \tag{4.19}$$

where

$$R_0^{\Lambda,N}(\eta) = \begin{cases} R_0^\Lambda(\eta) & \text{if } |\eta| \leq N, \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Note that $k_0^{\Lambda,N} \leq k_0$ (both being positive) so that $k_0^{\Lambda,N} \in \mathcal{K}_{\alpha_0}$. Therefore by repeating the procedure used in Theorem 3.2, $k_t^{\Lambda,N} = Q^\sigma_{\alpha_0\alpha_*}(t)k_0^{\Lambda,N} \in \mathcal{K}_{\alpha_*}$ with $\alpha_* > \alpha_0$ and $t < T(\alpha_*,\alpha_0)$ is the unique classical solution of the problem

$$\frac{d}{dt}k_t^{\Lambda,N} = L^{\Delta,\sigma}k_t^{\Lambda,N}, \qquad k_{t=0}^{\Lambda,N} = k_0^{\Lambda,N} \in \mathcal{K}_{\alpha_0}. \tag{4.21}$$

4.4.1. *Spaces $\mathcal{G}_\theta^{fac}$ and $\mathcal{U}_\theta^\sigma$.* Before proceeding with the proof, let us introduce two more spaces used therein and look closer at the dependencies mentioned at the beginning of Subsection 4.4. The space $\mathcal{G}_\theta^{\text{fac}}$ is L^1-type space similar to $\mathcal{G}_\theta$, recall 4.4, but with the norm given by

$$|G|_{\text{fac},\theta} = \int_{\Gamma_0} |G(\eta)|e^{\theta|\eta|}|\eta|!\lambda(d\eta).$$

Obviously, for $\theta' > \theta$ we have $\mathcal{G}^{\text{fac}}_{\theta'} \subset \mathcal{G}^{\text{fac}}_{\theta}$.

The space $\mathcal{U}^{\sigma}_{\theta}$ is L^{∞}-type space with the norm

$$||u||_{\sigma,\theta} = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \frac{|u(\eta)|e^{-\theta|\eta|}}{e(\psi_\sigma;\eta)},$$

where $e(\psi_\sigma;\eta) = \prod_{x\in\eta} e^{-\sigma|x|^2}$. We have an analogue of (2.24)

$$|u(\eta)| \le e^{\theta|\eta|} e(\psi_\sigma;\eta)||u||_{\sigma,\theta}. \tag{4.22}$$

Notice that by $\psi_\sigma \le 1$ we immediately obtain $\mathcal{U}^{\sigma}_{\theta} \subset \mathcal{K}_\theta$. Additionally we have for $u \in \mathcal{U}^{\sigma}_{\theta}$

$$|u|_{\text{fac},\theta'} \le \int_{\Gamma_0} e(\psi_\sigma;\eta) e^{(\theta+\theta')|\eta|}|\eta|!||u||_{\sigma,\theta}\lambda(d\eta) =$$

$$= \sum_{n=0}^{\infty} \left(e^{\theta+\theta'} \int_{\mathbb{R}^d} \psi_\sigma(x)dx \right)^n ||u||_{\sigma,\theta} = \sum_{n=0}^{\infty} \left(e^{\theta+\theta'} \left(\frac{\pi}{\sigma}\right)^{d/2} \right)^n ||u||_{\sigma,\theta}$$

and therefore

$$\mathcal{U}^{\sigma}_{\theta} \subset \mathcal{G}^{\text{fac}}_{\theta'}, \qquad \theta' < -\theta - \frac{d}{2}(\ln\pi - \ln\sigma). \tag{4.23}$$

Notice that $k_0^{\Lambda,N} \in \mathcal{U}^{\sigma}_{\alpha_0}$, and therefore it belongs also to $\mathcal{G}^{\text{fac}}_{\beta_0}$ with

$$\beta_0 < -\alpha_0 - \frac{d}{2}(\ln\pi - \ln\sigma). \tag{4.24}$$

Indeed, by (4.19) we get

$$\operatorname*{ess\,sup}_{\eta\in\Gamma_0} \int_{\Gamma_0} R_0^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi)\frac{e^{-\alpha_0|\eta|}}{e(\psi_\sigma;\eta)} \le e^{\sigma \sup_{y\in\Lambda}|y|^2 N}||k_{\mu_0}||_{\alpha_0} < \infty.$$

4.4.2. *Evolution in spaces $\mathcal{U}^{\sigma}_{\theta}$.* By (4.22) and (2.21) together with (2.22) we may estimate for $\theta' > \theta$

$$\begin{aligned}
||L_{11}^{\Delta,\sigma}u||_{\sigma,\theta'} &\le \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \frac{e^{-\theta'|\eta|}}{e(\psi_\sigma;\eta)} \cdot \frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{z\in\eta} \psi_\sigma(z)c_1(x,y;z)|u(\eta\backslash z\cup\{x,y\})|dxdy \le \\
&\le \operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-(\theta'-\theta)|\eta|} \cdot \frac{1}{2}e^{\theta}||u||_{\sigma,\theta} \int_{(\mathbb{R}^d)^2} \sum_{z\in\eta} c_1(x,y;z)\psi_\sigma(x)\psi_\sigma(y)dxdy \le \\
&\le \frac{e^{\theta}\langle c_1\rangle}{2e(\theta'-\theta)}||u||_{\sigma,\theta}
\end{aligned}$$

and therefore, when considering $L_{11}^{\Delta,\sigma} : \mathcal{U}^{\sigma}_{\theta} \to \mathcal{U}^{\sigma}_{\theta'}$ we have

$$||L_{11}^{\Delta,\sigma}||_{\theta\theta'} \le \frac{e^{\theta}\langle c_1\rangle}{2e(\theta'-\theta)},$$

which is the same as the corresponding part of (3.7). Analogously, one can obtain the same for the other components of $L^{\Delta,\sigma}$, which allows us to use the scheme of Theorem 3.2 again and obtain the existence and uniqueness of the classical solution of the problem

$$\frac{d}{dt}u_t = L^{\Delta,\sigma}u_t, \qquad u_{t=0} = u_0 \in \mathcal{U}^{\sigma}_{\alpha_0} \tag{4.25}$$

for $t \in [0, T(\alpha_*, \alpha_0))$ in $\mathcal{U}^{\sigma}_{\alpha_*}$ where $\alpha_* > \alpha_0$.

4.4.3. *Evolution of densities $R_t^\Lambda(\eta)$.* As mentioned before, recall (2.17), correlation functions in the case of the local system are connected with respective measure densities. We cannot claim that our initial state is finite, actually we want to deal with infinite states. In order to use the mentioned dependence, we introduce the local correlation functions,

$$q_t^{\Lambda,N}(\eta) = \int_{\Gamma_0} R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi), \quad t \geq 0, \tag{4.26}$$

where $R_t^{\Lambda,N}$ is the solution to the problem

$$\frac{d}{dt} R_t^{\Lambda,N} = L^{\dagger,\sigma} R_0^{\Lambda,N}, \qquad R_{t=0}^{\Lambda,N} = R_0^{\Lambda,N} \tag{4.27}$$

with $L^{\dagger,\sigma}$ satisfying duality condition

$$\int_{\Gamma_0} (L^\sigma F)(\eta) R(\eta)\lambda(d\eta) = \int_{\Gamma_0} F(\eta)(L^{\dagger,\sigma} R)(\eta)\lambda(d\eta).$$

By (4.2), (2.21) and (2.22), one obtains

$$(L^{\dagger,\sigma} R)(\eta) = \frac{1}{2} \sum_{z \in \eta} \int_{(\mathbb{R}^d)^2} \psi_\sigma(z) c_1(x, y; z) R\big(\eta \cup \{x, y\} \backslash z\big) dx dy +$$

$$+ \sum_{y \in \eta} \int_{\mathbb{R}^d} \psi_\sigma(y) c_2(x; y) \prod_{u \in \eta \backslash y} e^{-\phi(y-u)} R(\eta \cup x \backslash y) dx - E^\sigma(\eta) R(\eta)$$

with

$$E^\sigma(\eta) = \sum_{\{x,y\} \subset \eta} \int_{\mathbb{R}^d} \psi_\sigma(z) c_1(x, y; z) dz + \sum_{x \in \eta} \int_{\mathbb{R}^d} \psi_\sigma(y) \tilde{c}_2(x; y; \eta) dy.$$

Note that $R_0^{\Lambda,N} \in \mathcal{G}_\theta^{\text{fac}}$ for any $\theta \in \mathbb{R}$, as

$$|R_0^{\Lambda,N}|_{\text{fac},\theta} = \sum_{n=0}^{N} e^{\theta n} \int_{\Lambda^n} R_0^\Lambda(x_1, \ldots, x_n) dx_1 \ldots dx_n < \infty.$$

We show the existence and uniqueness of solution of (4.27) in $\mathcal{G}_\theta^{\text{fac}}$ using Thieme-Voigt perturbation theorem, see Theorem 2.7 in [36] or Proposition 3.2 in [20] for its usage in a similar context.

Denote $X = L^1(\Gamma_0, \lambda)$ and $\tilde{X} = \mathcal{G}_\theta^{\text{fac}} \subset X$. Introduce functionals

$$f(R) = \int_{\Gamma_0} R(\eta)\lambda(d\eta), \quad \tilde{f}(R) = \int_{\Gamma_0} |\eta|! e^{\beta|\eta|} R(\eta)\lambda(d\eta),$$

which have property $f(R) = ||R||_X$ for positive $R \in X$ and respectively $\tilde{f}(R) = ||R||_{\tilde{X}}$ for positive $R \in \tilde{X}$. Notice that $\tilde{X} \hookrightarrow X$, that is $\tilde{X}$ is continuously embedded in X. In order to prove that, we show that there exists $C > 0$ such that $||R||_{\tilde{X}} \geq C||R||_X$ for all $R \in \tilde{X}$. Indeed,

$$||R||_{\tilde{X}} = \int_{\Gamma_0} |R(\eta)| e^{\beta|\eta|} |\eta|! \lambda(d\eta) \geq \min_{n \in \mathbb{N}_0} \{e^{\beta n} n!\} ||R||_X.$$

The minimum above exists, as $e^{\theta n} n! \to \infty$. Denote

$$L^{\dagger,\sigma} = A_0^{\dagger,\sigma} + B^{\dagger,\sigma},$$

where $(A_0^{\dagger,\sigma}R)(\eta) = -E^\sigma(\eta)R(\eta)$ is multiplication operator and $-A_0^{\dagger,\sigma}$ is positive. The remaining part $B^{\dagger,\sigma}$ is also positive and will be treated as perturbation of $A_0^{\dagger,\sigma}$. Define the natural domain

$$\mathcal{D}(A_0^{\dagger,\sigma}) = \{R \in X : \int_{\Gamma_0} |(A_0^{\dagger,\sigma}R)(\eta)|\lambda(d\eta) < \infty\}$$

and note that $B^{\dagger,\sigma} : \mathcal{D}(A_0^{\dagger,\sigma}) \to X$ as by Minlos lemma

$$\int_{\Gamma_0} |(B^{\dagger,\sigma}R)(\eta)|\lambda(d\eta) \le \int_{\Gamma_0} E^\sigma(\eta)|R(\eta)|\lambda(d\eta) < \infty$$

if $R \in \mathcal{D}(A_0^{\dagger,\sigma})$. Therefore we can endow both $A_0^{\dagger,\sigma}$ and $B^{\dagger,\sigma}$ (and also $L^{\dagger,\sigma}$) with the same domain $\mathcal{D}(A_0^{\dagger,\sigma})$ acting into X.

For the space $\tilde{X}$ we define

$$\tilde{\mathcal{D}}(A_0^{\dagger,\sigma}) = \{R \in \mathcal{D}(A_0^{\dagger,\sigma}) \cap \tilde{X} : A_0^{\dagger,\sigma}R \in \tilde{X}\}$$

with which also $B^{\dagger,\sigma} : \tilde{\mathcal{D}}(A_0^{\dagger,\sigma}) \to \tilde{X}$.

Notice that both semigroups $(S_0^\sigma(t)R)(\eta) = e^{-E^\sigma(\eta)t}R(\eta)$ generated by $(A_0^{\dagger,\sigma}, \mathcal{D}(A_0^{\dagger,\sigma}))$ and one generated by $(A_0^{\dagger,\sigma}, \tilde{\mathcal{D}}(A_0^{\dagger,\sigma}))$ are C_0-semigroups by the means of inequality $1 - e^{-x} \le x$.

In order to show inequality $f\big((A_0^{\dagger,\sigma} + B^{\dagger,\sigma})R\big) \le 0$ observe that

$$f\big((A_0^{\dagger,\sigma} + B^{\dagger,\sigma})R\big) = \int_{\Gamma_0} 1(\eta)(L^{\dagger,\sigma}R)(\eta)\lambda(d\eta) = \int_{\Gamma_0} (L^\sigma 1)(\eta)R(\eta)\lambda(d\eta) = 0.$$

as $(L^\sigma 1(\cdot))(\eta) = 0$.

To fulfill all of the required conditions, we need to show existence of positive constants c and ε such that for positive $R \in \tilde{\mathcal{D}}(A_0^{\dagger,\sigma})$ the following inequality holds:

$$\tilde{f}((A_0^{\dagger,\sigma} + B^{\dagger,\sigma})R) \le c||R||_{\tilde{X}} - \varepsilon||A_0^{\dagger,\sigma}R||_X. \tag{4.28}$$

First, notice that

$$\tilde{f}\left((A_0^{\dagger,\sigma} + B^{\dagger,\sigma})R\right) = \int_{\Gamma_0} (L^\sigma F_\theta)(\eta)R(\eta)$$

with $F_\theta(\eta) = |\eta|!e^{\theta|\eta|}$. We also have

$$(L^\sigma F_\theta)(\eta) = E_1^\sigma(\eta)\big((|\eta|-1)!e^{\theta(|\eta|-1)} - |\eta|!e^{\theta|\eta|}\big)$$

as F_θ depends only on the number of elements in configuration being its argument.

In order to show inequality (4.28), we prove for positive R

$$\tilde{f}((A_0^{\dagger,\sigma} + B^{\dagger,\sigma})R) - c||R||_{\tilde{X}} + \varepsilon||A_0^{\dagger,\sigma}R||_X =$$

$$= \int_{\Gamma_0} \Big[E_1^\sigma(\eta)\left((|\eta|-1)!e^{\theta(|\eta|-1)} - |\eta|!e^{\theta|\eta|}\right) - ce^{\theta|\eta|}|\eta|! + \varepsilon E^\sigma(\eta)\Big]R(\eta)\lambda(d\eta) \le 0$$

for which it is enough to find positive c and ε such that

$$c \ge h_{\varepsilon,\theta}(\eta) \tag{4.29}$$

where

$$h_{\varepsilon,\theta}(\eta) = \frac{e^{-\theta|\eta|}}{|\eta|!}\Big[E_1^\sigma(\eta)\left((|\eta|-1)!e^{\theta(|\eta|-1)} - |\eta|!e^{\theta|\eta|}\right) + \varepsilon E^\sigma(\eta)\Big].$$

In order to do that, notice

$$E^{\sigma}(\eta) = E_1^{\sigma}(\eta) + E_2^{\sigma}(\eta) \le |\eta|^2 c_1^{\max} + |\eta|\langle c_2\rangle,$$

choose arbitrary $\varepsilon > 0$ and consider two cases. First, assume that $|\eta| < e^{-\theta}$. Then

$$(|\eta|-1)!e^{\theta(|\eta|-1)} - |\eta|!e^{\theta|\eta|} \ge 0$$

and therefore

$$h_{\varepsilon,\theta}(\eta) \le m_1$$

where

$$m_1 = \max_{n\in\mathbb{N}, n<e^{-\theta}} \frac{e^{-\theta n}}{n!}\left[n^2 c_1^{\max}\left((n-1)!e^{\theta(n-1)} - n!e^{\theta n}\right) + \varepsilon\left(n^2 c_1^{\max} + n\langle c_2\rangle\right)\right].$$

In the second case, when $|\theta| \ge e^{-\beta}$ we have

$$h_{\varepsilon,\theta}(\eta) \le m_2$$

with

$$m_2 = \max_{n\in\mathbb{N}} \frac{\varepsilon e^{-\theta n}}{n!}\left(n^2 c_1^{\max} + n\langle c_2\rangle\right)$$

which exists as $m_2 \to 0$ as $n \to 0$.
Therefore we choose $c = \max(m_1, m_2)$ proving (4.29) so that (4.28) holds.

We have shown that all of the conditions of the Thieme-Voigt theorem hold, so that the closure of $(A_0^{\dagger,\sigma} + B^{\dagger,\sigma}, \mathcal{D}(A_0^{\dagger,\sigma}))$ in X is a generator of substochastic (in our case even stochastic) semigroup $S^{\dagger,\sigma}$ in X that leaves $\tilde{X}$ invariant. Therefore equation (4.27) posess the unique solution in $\tilde{X} = \mathcal{G}_{\theta}^{\text{fac}}$ for any $\theta \in \mathbb{R}$ and all $t > 0$.

4.4.4. *Evolution in spaces* $\mathcal{G}_{\theta}^{fac}$. Notice that the definition of localized initial condition $k_0^{\Lambda,N}$ given by (4.19) coincides with the definition (4.26) of local correlation function $q_t^{\Lambda,N}$ with $t = 0$. Consider equation

$$\frac{d}{dt}q_t^{\Lambda,N} = L^{\Delta,\sigma}q_t^{\Lambda,N}, \qquad q_0^{\Lambda,N} = k_0^{\Lambda,N}. \tag{4.30}$$

It posess the unique classical solution in space $\mathcal{G}_{\beta_0}^{\text{fac}}$ for all $t > 0$ with β_0 satisfying (4.24). Indeed, recall that $k_0^{\Lambda,N}$ lies in $\mathcal{G}_{\beta_0}^{\text{fac}}$. Next, we show that $q_t^{\Lambda,N}$ given by (4.26) is a solution to our problem (4.30). First, $R_t^{\Lambda,N} \in \mathcal{G}_{\beta_0}^{\text{fac}}$ implies $q_t^{\Lambda,N} \in \mathcal{G}_{\beta_0}^{\text{fac}}$ as by Minlos lemma (2.20)

$$|q_t^{\Lambda,N}|_{\text{fac},\beta} = \int_{\Gamma_0} R_t^{\Lambda,N}(\eta)\sum_{\xi\subset\eta} e^{\beta|\xi|}|\xi|!\lambda(d\eta) = \tag{4.31}$$

$$= \int_{\Gamma_0} R_t^{\Lambda,N}(\eta)\sum_{k=0}^{|\eta|}\frac{|\eta|!}{(|\eta|-k)!}e^{\beta k}\lambda(d\eta) \le e^{e^{-\beta}}|R_t^{\Lambda,N}|_{\text{fac},\beta}.$$

Next, observe that for arbitrary $G \in \mathcal{B}_{bs}$, again using Minlos lemma, we have

$$\int_{\Gamma_0} \frac{d}{dt} q_t^{\Lambda,N}(\eta) G(\eta) \lambda(d\eta) =$$

$$= \int_{\Gamma_0}\int_{\Gamma_0} L^{\dagger,\sigma} R_t^{\Lambda,N}(\eta \cup \xi) G(\eta) \lambda(d\xi)\lambda(d\eta) = \int_{\Gamma_0} L^{\dagger,\sigma} R_t^{\Lambda,N}(\eta) \sum_{\xi\subset\eta} G(\xi)\lambda(d\eta) =$$

$$= \int_{\Gamma_0} (L^\sigma K G)(\eta) R_t^{\Lambda,N}(\eta)\lambda(d\eta) = \int_{\Gamma_0} (K\hat{L}^\sigma G)(\eta) R_t^{\Lambda,N}(\eta)\lambda(d\eta) =$$

$$= \int_{\Gamma_0}\int_{\Gamma_0} (\hat{L}^\sigma G)(\eta) R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi)\lambda(d\eta) = \int_{\Gamma_0} (\hat{L}^\sigma G)(\eta) q_t^{\Lambda,N}(\eta)\lambda(d\eta) =$$

$$= \int_{\Gamma_0} G(\eta)(L^{\Delta,\sigma} q_t^{\Lambda,N})(\eta)\lambda(d\eta),$$

where $\hat{L}^\sigma = K^{-1}L^\sigma K$ (recall Definition 2.16) satisfies

$$\int_{\Gamma_0} \hat{L}^\sigma G(\eta) k(\eta)\lambda(d\eta) = \int_{\Gamma_0} G(\eta) L^{\Delta,\sigma} k(\eta)\lambda(d\eta)$$

and is given by (cf. (4.7) and [30]) $\hat{L}^\sigma = \hat{L}_1^\sigma + \hat{L}_2^\sigma$ with

$$\hat{L}_1^\sigma G(\eta) = \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \psi_\sigma(z) c_1(x,y;z)\Big(G(\eta\backslash\{x,y\}\cup z) - G(\eta\backslash y) - G(\eta\backslash x) - G(\eta)\Big)dz,$$

$$\hat{L}_2^\sigma G(\eta) = \int_{\mathbb{R}^d} \sum_{x\in\eta} \psi_\sigma(y) c_2(x;y) \sum_{\xi\subset\eta\backslash x} \big[G(\xi\cup y) - G(\xi\cup x)\big] \sum_{\zeta\subset\xi} \prod_{u\in\eta\backslash\xi\backslash x\cup\zeta} \Big(e^{-\phi(y-u)} - 1\Big)\, dy.$$

Because $\mathcal{B}_{bs}$ is a measure defining class on $(\Gamma_0, \mathcal{B}(\Gamma_0))$, it implies that $q_t^{\Lambda,N}$ is indeed a solution to 4.30.

To prove that this solution is unique, follow the procedure used in Theorem 3.2. Note that for any $\beta < \beta_0$ by (2.21), (2.22) and (3.1) we have for $L^{\Delta,\sigma} : \mathcal{G}_{\beta_0}^{\text{fac}} \to \mathcal{G}_\beta^{\text{fac}}$

$$||B^{\Delta,\sigma}||_{\beta_0\beta} \le \frac{\frac{3}{2}c_1^{\max}e^{-\beta} + 2e^{e^{-\beta}}\langle c_2\rangle}{(\beta_0-\beta)e}$$

$$||A^{\Delta,\sigma}||_{\beta_0\beta} \le \frac{2c_1^{\max}}{(\beta_0-\beta)^2e^2}$$

where $L^{\Delta,\sigma} = A^{\Delta,\sigma} + B^{\Delta,\sigma}$ and

$$A^{\Delta,\sigma}(\eta) = -\int_{\mathbb{R}^d} \psi_\sigma(z) \sum_{\{x,y\}\subset\eta} c_1(x,y;z)dz\; k(\eta).$$

By repeating steps taken in the proof of Theorem 3.2, one obtains uniqueness of the solution for $t < T_2(\beta,\beta_0)$ with

$$T_2(\beta,\beta_0) = \frac{\beta_0-\beta}{\frac{3}{2}c_1^{\max}e^{-\beta} + 2\exp(e^{-\beta})\langle c_2\rangle}.$$

But by (4.31) we know that $q_t^{\Lambda,N}$ actually lies in $\mathcal{G}_{\beta_0}^{\text{fac}}$, so we may repeat our procedure taking in equation (4.30) as initial condition k_T with $T = \frac{1}{2}T_2(\beta,\beta_0)$ and obtaining uniqueness of our solution for longer interval. By this method we may prolong uniqueness for all $t > 0$.

4.4.5. *Positivity property.* We have shown that equations (4.21), (4.25) and (4.30) with the same initial condition $k_0^{\Lambda,N}$ have unique classical solutions for $t < T(\alpha_*, \alpha_0)$ in spaces, respectively, $\mathcal{K}_{\alpha_*}$, $\mathcal{U}^{\sigma}_{\alpha_*}$ and $\mathcal{G}^{\text{fac}}_{\beta_0}$ with any β_0 satisfying (4.24). Taking β_0 such that (4.23) holds with $\theta' = \beta_0$ and $\theta = \alpha_*$, for which obviously (4.24) also holds, we have $\mathcal{U}^{\sigma}_{\alpha_*} \subset \mathcal{K}_{\alpha_*}$ and $\mathcal{U}^{\sigma}_{\alpha_*} \subset \mathcal{G}^{\text{fac}}_{\beta_0}$. Therefore $k_t^{\Lambda,N} = u_t^{\Lambda,N}$ and $q_t^{\Lambda,N} = u_t^{\Lambda,N}$, which leads to

$$k_t^{\Lambda,N} = q_t^{\Lambda,N}.$$

By this equality, (4.26) and (2.20) we may write for $G \in \mathcal{B}^*_{bs}$

$$\int_{\Gamma_0} G(\eta) k_t^{\Lambda,N}(\eta)\lambda(d\eta) = \int_{\Gamma_0} R_t^{\Lambda,N}(\eta) \sum_{\xi\subset\eta} G(\xi)\lambda(d\eta) \geq 0,$$

which is desired positivity property, recall (4.1), for $k_t^{\Lambda,N}$.

4.4.6. *Convergence with respect to* Λ *and* N*.* To finish the proof of Theorem 3.3 it suffices to show that for increasing sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ of bounded, measurable subsets of $\mathbb{R}^d$ which is cofinal (that is $\forall_{x\in\mathbb{R}^d}\exists_{n\in\mathbb{N}} : x \in \Lambda_n$) and $G \in \mathcal{B}_{bs}$ for $t < T(\alpha_*, \alpha_0)$ the following holds

$$\lim_{n\to\infty}\lim_{N\to\infty}\int_{\Gamma_0} G(\eta) k_t^{\Lambda_n,N}(\eta)\lambda(d\eta) = \int_{\Gamma_0} G(\eta) k_t^{\sigma}(\eta)\lambda(d\eta). \tag{4.32}$$

We prove it in the similar manner, as it is done in the proof of (3.54) in appendix of [7]. By (4.6) we have

$$\lim_{n\to\infty}\lim_{N\to\infty}\int_{\Gamma_0} G(\eta) k_t^{\Lambda_n,N}(\eta)\lambda(d\eta) = \lim_{n\to\infty}\lim_{N\to\infty}\int_{\Gamma_0} H_{\alpha_*\alpha_0}(t) G(\eta) k_0^{\Lambda_n,N}(\eta)\lambda(d\eta)$$

and

$$\int_{\Gamma_0} G(\eta) k_t^{\sigma}(\eta)\lambda(d\eta) = \int_{\Gamma_0} H_{\alpha_*\alpha_0}(t) G(\eta) k_0(\eta)\lambda(d\eta).$$

Next, denoting $G_t = H_{\alpha_*\alpha_0}(t)G$,

$$\int_{\Gamma_0} G_t(\eta)\left(k_0(\eta) - k_0^{\Lambda_n,N}(\eta)\right)\lambda(d\eta) = J_n^{(1)} + J_{n,N}^{(2)}$$

with

$$J_n^{(1)} = \int_{\Gamma_0} G_t(\eta) k_0(\eta)\left(1 - I_{\Gamma_{\Lambda_n}}(\eta)\right)\lambda(d\eta),$$

$$J_{n,N}^{(2)} = \int_{\Gamma_0} G_t(\eta)\left(k_0(\eta) I_{\Gamma_{\Lambda_n}}(\eta) - k_0^{\Lambda_n,N}(\eta)\right)\lambda(d\eta).$$

Take an arbitrary $\varepsilon > 0$.
We have

$$|J_n^{(1)}| \leq \sum_{k=1}^{\infty}\frac{1}{k!}\int_{(\mathbb{R}^d)^k} |G_t^{(k)}(x_1,\ldots,x_k)| k_0^{(k)}(x_1,\ldots,x_k)\sum_{l=1}^{k} I_{\Lambda_N^c}(x_l) dx_1\ldots dx_k.$$

Because $k_0 \in \mathcal{K}_{\alpha_0}$ and $G_t^{(k)}$ i $k_0^{(k)}$ are symmetric for all k, we may write

$$|J_n^{(1)}| \leq ||k_0||_{\alpha_0}\sum_{k=1}^{\infty}\frac{k}{k!}e^{\alpha_0 k}\int_{\Lambda_n^c}\int_{(\mathbb{R}^d)^{k-1}} |G_t^{(k)}(x_1,\ldots,x_k)| dx_1\ldots dx_k.$$

Recall that $G_0 \in \mathcal{B}_{bs}$ implies $G_0 \in \mathcal{G}_\theta$ for any $\theta \in \mathbb{R}$. In principle, we can take $\theta = \beta_0 > \alpha_*$. Then for $\beta_* \in (\alpha_0, \alpha_*)$ for $t < T(\beta_0, \beta_*)$ we have $G_t \in \mathcal{G}_{\beta_*}$, recall (4.9). With such β_* we estimate

$$|J_n^{(1)}| \leq \frac{||k_0||_{\alpha_0}}{(\beta_* - \alpha_0)e} \sum_{k=1}^{\infty} \frac{1}{k!} e^{\beta_* k} \int\limits_{\Lambda_n^c} \int\limits_{(\mathbb{R}^d)^{k-1}} |G_t^{(k)}(x_1, \ldots, x_k)| dx_1 \ldots dx_k.$$

Taking into account that by $G_t \in \mathcal{G}_{\beta_*}$ we have

$$\sum_{k=1}^{\infty} \frac{1}{k!} e^{\beta_* k} \int\limits_{(\mathbb{R}^d)^k} |G_t^{(k)}(x_1, \ldots, x_k)| dx_1 \ldots dx_k = |G_t|_{\beta_*} < \infty,$$

we can pick an integer M such that

$$\frac{||k_0||_{\alpha_0}}{(\beta_* - \alpha_0)e} \sum_{k=M+1}^{\infty} \frac{1}{k!} e^{\beta_* k} \int\limits_{(\mathbb{R}^d)^k} |G_t^{(k)}(x_1, \ldots, x_k)| dx_1 \ldots dx_k < \frac{\varepsilon}{4},$$

as it is a tail of convergent series. Next, as $G_t^{(k)} \in L^1\left((\mathbb{R}^d)^k\right)$ and Λ_n is cofinal, there exists n_1 such that for $n > n_1$

$$\frac{||k_0||_{\alpha_0}}{(\alpha' - \alpha_0)e} \sum_{k=1}^{M} \frac{1}{k!} e^{\alpha' k} \int\limits_{\Lambda_n^c} \int\limits_{(\mathbb{R}^d)^{k-1}} |G_t^{(k)}(x_1, \ldots, x_k)| dx_1 \ldots dx_k < \frac{\varepsilon}{4}.$$

Therefore

$$|J_n^{(1)}| < \frac{\varepsilon}{2}.$$

By (4.18), (4.19), (4.20) and (2.20), denoting

$$I_N(\eta) = \begin{cases} 1 & \text{if } |\eta| \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} J_{n,N}^{(2)} &= \int\limits_{\Gamma_0} \Big[G_t(\eta) \int\limits_{\Gamma_{\Lambda_n}} R_0^{\Lambda_n}(\eta \cup \xi) I_{\Gamma_{\Lambda_n}}(\eta) \left(1 - I_N(\eta \cup \xi)\right) \lambda(d\xi) \Big] \lambda(d\eta) = \\ &= \int\limits_{\Gamma_0} G_t(\eta) \int\limits_{\Gamma_0} R_0^{\Lambda_n}(\eta \cup \xi) I_{\Gamma_{\Lambda_n}}(\eta \cup \xi) \left(1 - I_N(\eta \cup \xi)\right) \lambda(d\xi) \lambda(d\eta) = \\ &= \int\limits_{\Gamma_{\Lambda_n}} \sum_{\xi \subset \eta} G_t(\xi) R_0^{\Lambda_n}(\eta) \left(1 - I_N(\eta)\right) \lambda(d\eta) = \\ &= \sum_{m=N+1}^{\infty} \frac{1}{m!} \int\limits_{(\Lambda_n)^m} R_0^{\Lambda_n}(\{x_1, \ldots, x_m\}) \sum_{k=0}^{m} \sum_{\{i_1, \ldots i_k\} \subset \{1, \ldots m\}} G_t^{(k)}(x_{i_1}, \ldots, x_{i_k}) dx_1 \ldots dx_m. \end{aligned}$$

Note that equation (4.18) may be rewritten for $\eta = \{y_1, \ldots, y_s\} \in \Gamma_\Lambda$ in the form

$$k_0(\{y_1, \ldots, y_s\}) = R_0^\Lambda(\{y_1, \ldots, y_s\}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int\limits_{\Lambda^k} R_0^\Lambda(y_1, \ldots, y_s, x_1, \ldots, x_k) dx_1 \ldots dx_k,$$

which implies

$$R_0^\Lambda(\{y_1, \ldots, y_s\}) \leq k_0(\{y_1, \ldots, y_s\}) \leq e^{\alpha_0 s} ||k_0||_{\alpha_0}$$

as $k_0 \in \mathcal{K}_{\alpha_0}$. Then, denoting by $l(\Lambda_n)$ the Lebesgue's measure of set Λ_n, we have

$$|J^{(2)}_{n,N}| \leq ||k_0||_{\alpha_0} \sum_{m=N+1}^{\infty} \frac{1}{m!} e^{\alpha_0 m} \int\limits_{(\Lambda_n)^m} \sum_{k=0}^{m} \sum_{\{i_1,\dots i_k\}\subset\{1,\dots m\}} G^{(k)}_t(x_{i_1},\dots,x_{i_k}) dx_1 \dots dx_m \leq$$

$$\leq ||k_0||_{\alpha_0} \sum_{m=N+1}^{\infty} \frac{1}{m!} e^{\alpha_0 m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} ||G^{(k)}_t||_{L^1\left((\mathbb{R}^d)^k\right)} [l(\Lambda_n)]^{m-k}.$$

For sufficiently big N we may estimate

$$|J^{(2)}_{n,N}| < \frac{\varepsilon}{2}$$

as above is a tail of convergent series

$$||k_0||_{\alpha_0} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{e^{\alpha_0 k}}{k!} ||G^{(k)}_t||_{L^1\left((\mathbb{R}^d)^k\right)} \frac{e^{\alpha_0 (m-k)}}{(m-k)!} [l(\Lambda_n)]^{m-k} =$$

$$= ||k_0||_{\alpha_0} \sum_{k=0}^{\infty} \frac{e^{\alpha_0 k}}{k!} ||G^{(k)}_t||_{L^1\left((\mathbb{R}^d)^k\right)} \sum_{m=0}^{\infty} \frac{e^{\alpha_0 m}}{m!} [l(\Lambda_n)]^{m} =$$

$$= ||k_0||_{\alpha_0} |G_t|_{\alpha_0} \exp\left(e^{\alpha_0} l(\Lambda_n)\right) < \infty.$$

Finally, because $\beta_0 > \alpha_*$ and $\beta_* \in (\alpha_0, \alpha_*)$ are arbitrarily chosen (so we can pick β_* as close to α_0 as we want and the same with β_0 and α_*), we obtain (4.32) for $t < T(\alpha_*, \alpha_0)$.

5. Mesoscopic dynamics

In this section we consider the case of the system with initial state μ_0 being a Poisson measure with density $\rho_0 \in L^\infty(\mathbb{R}^d)$. It means that its correlation function is of the product form

$$r_0(\eta) = \prod_{x\in\eta} \rho_0(x).$$

In such case, instead of the precise dynamics of the system given with the use of operator L, recall (3.1), or L^Δ, recall (4.2), one can consider an approximated scheme in which the Poissonity of the state is preserved. We will call it the mesoscopic dynamics, as its description is less precise than in the case of the original microscopic case (higher-level correlations are neglected) but it is still not a macroscopic level at which the detailed spatial structure of the system would be ignored. The idea here is to obtain a description of the system by the means of densities of the corresponding Poisson measures.

A slightly more general case can also be treated by the approach presented within this section, where the initial state μ_0 is not necessarily Poissonian, but sub-Poissonian with its correlation function $k_0 \in \mathcal{K}_{\alpha_0}$. We then assume being given a scale of functions $q_{0,\varepsilon} \in \mathcal{K}_{\alpha_0}$ with $\varepsilon \in [0,1]$ such that

$$q_{0,1} = k_0, \qquad q_{0,0}(\eta) = \prod_{x\in\eta} \rho_0(x)$$

and the dependence on ε is continuous, i.e.

$$||q_{0,\varepsilon} - q_{0,\varepsilon'}||_{\alpha_0} \to 0, \qquad \varepsilon' \to \varepsilon,$$

where $||\cdot||_{\alpha_0}$ is the norm in $\mathcal{K}_{\alpha_0}$ space, recall Definition 2.23.

Remark 5.1. Note that the mentioned scale of functions always exists, one can take any Poisson measure with its correlation function

$$r_0 = \prod_{x \in \cdot} \rho_0(x) \in \mathcal{K}_{\alpha_0}$$

and consider

$$q_{0,\varepsilon} = \varepsilon k_0 + (1-\varepsilon) r_0.$$

The mesoscopic dynamics is given with the use of a Vlasov operator V, see (5.3) below, which preserves the product form of the argument. It is obtained by a scaling procedure described within this section, cf. e.g. Section 1.3 of [14] or Section 4 in [5], see also [2], where multi-scale models are discussed. The performed scaling is closely related to so-called moment closure, see e.g. [27]. Instead of the dynamics of the functions from $\mathcal{K}_\vartheta$ spaces, one can consider evolution of L^∞-type factors ρ_t, that correspond to the densities of Poisson states. This evolution will be given by a kinetic equation (5.5).

The mesoscopic description of the system, while being only an approximation of the actual evolution, is worth investigating. One of its advantages is that it allows one to employ numerical methods for finding approximated solution which are not applicable at the microscopic level. An example of such approach is given in Section 6.

5.1. Vlasov Scaling. For the scale parameter $\varepsilon \in (0,1]$ we define operator L_ε^Δ as L^Δ with c_1 substituted by εc_1 and ϕ by $\varepsilon\phi$. That is, we can express it in the form $L_\varepsilon^\Delta = L_{\varepsilon,11}^\Delta + L_{\varepsilon,12}^\Delta + L_{\varepsilon,13}^\Delta + L_{\varepsilon,14}^\Delta + L_{\varepsilon,21}^\Delta + L_{\varepsilon,22}^\Delta$ with

$$\begin{aligned}
L_{\varepsilon,11}^\Delta k(\eta) &= \frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{z\in\eta} \varepsilon c_1(x,y;z) k(\eta\backslash z \cup \{x,y\}) dxdy, \\
L_{\varepsilon,12}^\Delta k(\eta) &= -\frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{x\in\eta} \varepsilon c_1(x,y;z) k(\eta \cup y) dydz, \\
L_{\varepsilon,13}^\Delta k(\eta) &= -\frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{y\in\eta} \varepsilon c_1(x,y;z) k(\eta \cup x) dxdz, \\
L_{\varepsilon,14}^\Delta k(\eta) &= -\int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \varepsilon c_1(x,y;z) dz \; k(\eta)
\end{aligned}$$

and

$$\begin{aligned}
L_{\varepsilon,21}^\Delta k(\eta) &= \int_{\mathbb{R}^d} \sum_{y\in\eta} c_2(x;y) \prod_{u\in\eta\backslash y} e^{-\varepsilon\phi(y-u)} \; (Q_y^\varepsilon k)(\eta\backslash y \cup x) dx, \\
L_{\varepsilon,22}^\Delta k(\eta) &= -\int_{\mathbb{R}^d} \sum_{x\in\eta} c_2(x;y) \prod_{u\in\eta\backslash x} e^{-\varepsilon\phi(y-u)} \; (Q_y^\varepsilon k)(\eta) dy,
\end{aligned}$$

where

$$(Q_y^\varepsilon k)(\eta) = \int_{\Gamma_0} k(\eta \cup \xi) \prod_{u\in\xi} (e^{-\varepsilon\phi(y-u)} - 1) \lambda(d\xi),$$

cf. (3.5).

The performed alteration can be interpreted as weakening the interactions between particles in the system. Then the renormed operator that corresponds to the ε-rescaled evolution L_ε^{ren} given by formula

$$L_\varepsilon^{ren} k(\eta) = \varepsilon^{|\eta|} L_\varepsilon^{\Delta}(\varepsilon^{-|\eta|} k(\eta)) \tag{5.1}$$

consists of the following six summands

$$L_{\varepsilon,11}^{ren} = L_{11}^{\Delta},\ L_{\varepsilon,12}^{ren} = L_{12}^{\Delta},\ L_{\varepsilon,13}^{ren} = L_{13}^{\Delta},\ L_{\varepsilon,14}^{ren} k(\eta) = \varepsilon L_{14}^{\Delta},$$

$$L_{\varepsilon,21}^{ren} k(\eta) = \int_{\mathbb{R}^d} \sum_{y\in\eta} c_2(x;y) \prod_{u\in\eta\setminus y} e^{-\varepsilon\phi(y-u)}\ (Q_y^{\varepsilon,ren} k)(\eta\setminus y \cup x) dx,$$

$$L_{\varepsilon,22}^{ren} k(\eta) = -\int_{\mathbb{R}^d} \sum_{x\in\eta} c_2(x;y) \prod_{u\in\eta\setminus x} e^{-\varepsilon\phi(y-u)}\ (Q_y^{\varepsilon,ren} k)(\eta) dy,$$

where

$$(Q_y^{\varepsilon,ren} k)(\eta) = \int_{\Gamma_0} k(\eta\cup\xi) \prod_{u\in\xi} \frac{1}{\varepsilon}(e^{-\varepsilon\phi(y-u)} - 1)\lambda(d\xi)$$

This altered action can be interpreted as follows. First, the density of the system is increased (with $\varepsilon^{-|\eta|}$), then the weakened action of L^{Δ} is performed (with L_ε^{Δ}) and finally the system's density is decreased back with $\varepsilon^{|\eta|}$.

As $\varepsilon \leq 1$, L_ε^{ren} undergoes the same norm estimation (3.7) as L^{Δ}. Therefore, by direct repetition of arguments used in Theorem 3.2, we can show the existence and uniqueness of the solution $q_{t,\varepsilon} = S_{\alpha_0\alpha_*}^{\varepsilon}(t) q_{0,\varepsilon} \in \mathcal{K}_{\alpha_*}$ of equation

$$\begin{cases} \frac{d}{dt} q_{t,\varepsilon} = L_\varepsilon^{ren} q_{t,\varepsilon} \\ q_{t,\varepsilon|t=0} = q_{0,\varepsilon} \in \mathcal{K}_{\alpha_0} \end{cases} \tag{5.2}$$

In the scaling limit $\varepsilon \to 0$, the corresponding Vlasov operator $V = \lim\limits_{\varepsilon\to 0} L_\varepsilon^{ren}$ can be expressed as

$$V = L_{11}^{\Delta} + L_{12}^{\Delta} + L_{13}^{\Delta} + V_{21} + V_{22} \tag{5.3}$$

with

$$V_{21} k(\eta) = \int_{\mathbb{R}^d} \sum_{y\in\eta} c_2(x;y) \int_{\Gamma_0} k(\eta\setminus y \cup \xi \cup x) \prod_{u\in\xi} (-\phi(y-u))\lambda(d\xi) dx$$

and

$$V_{22} k(\eta) = -\int_{\mathbb{R}^d} \sum_{x\in\eta} c_2(x;y) \int_{\Gamma_0} k(\eta\cup\xi) \prod_{u\in\xi} (-\phi(y-u))\lambda(d\xi) dy$$

Treating V as an operator acting between $\mathcal{K}_\theta$ and $\mathcal{K}_{\theta'}$ with $\theta' > \theta$ one can estimate its operator norm

$$||V||_{\theta\theta'} \leq \frac{2\langle c_2\rangle \exp(\langle\phi\rangle e^{\theta}) + \frac{3}{2} e^{\theta}\langle c_1\rangle}{(\theta'-\theta)e}.$$

Indeed, the second summand of nominator comes from the similar estimation for L^{Δ} (Lemma 3.1) and the first summand comes from the following calculations.

$$\begin{aligned} e^{-\theta'|\eta|} |V_{21} k(\eta)| &\leq e^{-(\theta'-\theta)|\eta|} ||k||_\theta \sum_{y\in\eta} \int_{\mathbb{R}^d} \int_{\Gamma_0} c_2(x;y) \prod_{u\in\xi} e^{\theta}(\phi(y-u))\lambda(d\xi) dx \\ &\leq \frac{\langle c_2\rangle \exp(\langle\phi\rangle e^{\theta})}{(\theta'-\theta)e}, \end{aligned}$$

as $xe^{-(\theta'-\theta)x} \le \frac{1}{(\theta'-\theta)e}$ for all $x>0$ and

$$\int_{\Gamma_0} \prod_{u\in\xi} e^{\theta}\phi(y-u)\lambda(d\xi) = \exp(\langle\phi\rangle e^{\theta})$$

The estimation for $||V_{22}||_{\theta\theta'}$ is analogical.

Therefore, using a method as in Theorem 3.2, one can show the existence and uniqueness of the solution to the problem

$$\begin{cases} \frac{d}{dt} r_t = V r_t \\ r_{t|t=0} = r_0 \in \mathcal{K}_{\alpha_0} \end{cases} \tag{5.4}$$

with $r_t = S^0_{\alpha_0\alpha_*}(t) r_0 \in \mathcal{K}_{\alpha_*}$ for $t < T(\alpha_*, \alpha_0)$.

For $r_t(\eta) = \prod\limits_{x\in\eta} \rho_t(x)$ we can write, e.g. for V_{11},

$$V_{11} r_t(\eta) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \sum_{z\in\eta} c_1(x,y;z) r_t(\eta\backslash z \cup \{x,y\}) dxdy$$

$$= \frac{1}{2} \sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} c_1(z,y;x) \prod_{u\in\eta\backslash x} \rho_t(u)\rho_t(z)\rho_t(y) dzdy = \sum_{x\in\eta} \prod_{u\in\eta\backslash x} v_{11}(\rho_t, x),$$

with

$$v_{11}(\rho_t, x) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(z,y;x)\rho_t(y)\rho_t(z) dydz.$$

In a similar way, one can rewrite the other summands of V as

$$V_{ij} r_t(\eta) = \sum_{x\in\eta} \prod_{u\in\eta\backslash x} v_{ij}(\rho_t, x),$$

where

$$v_{12}(\rho_t, x) = -\frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(x,y;z)\rho_t(x)\rho_t(y) dydz$$

$$v_{13}(\rho_t, x) = -\frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(y,x;z)\rho_t(x)\rho_t(y) dydz$$

$$v_{21}(\rho_t, x) = \int_{\mathbb{R}^d} c_2(y;x) \exp\Big(-\int_{\mathbb{R}^d} \phi(x-u)\rho_t(u)du\Big)\rho_t(y) dy$$

$$v_{22}(\rho_t, x) = -\int_{\mathbb{R}^d} c_2(x;y) \exp\Big(-\int_{\mathbb{R}^d} \phi(y-u)\rho_t(u)du\Big)\rho_t(x) dy$$

It allows us to consider the following problem – kinetic equation – for ρ_t instead of (5.4) for r_t:

$$\begin{cases} \frac{d}{dt} \rho_t(x) = v(\rho_t, x) \\ \rho_{t|t=0}(x) = \rho_0(x), \qquad \rho_0 \in L^\infty(\mathbb{R}^d), \end{cases} \tag{5.5}$$

where $v = v_{11} + v_{12} + v_{13} + v_{21} + v_{22}$. One can show the existence and uniqueness of the classical solution for a finite time horizon to this problem. It is done in a more general case in Theorem 5.9.

While equation (5.5) is complicated in general, it drastically simplifies in the spatially homogeneous case, i.e. when $\rho_0(x) = \rho_0 \in \mathbb{R}$ for all $x \in \mathbb{R}^d$. In this case, having in mind (3.1) we can rewrite (5.5) as

$$\frac{d\rho_t}{dt} = -\langle c_1 \rangle \rho_t^2.$$

This can be easily solved leading to

$$\rho_t = \frac{\rho_0}{1 + \langle c_1 \rangle \rho_0 t}.$$

In the preceding part, we analysed the problem for $q_{t,\varepsilon}$ (which is equal to k_t for $\varepsilon = 1$) with the scale parameter $\varepsilon \in (0,1]$ and the case in limit $\varepsilon \to 0$ for r_t. We discussed the existence and uniqueness of the corresponding problems (5.2) and (5.4), but we treated each ε separately. Let us look closer at the dependence on ε - it appears that the scaling procedure described above is continuous w.r.t ε. This result is described more precisely in Theorem 5.4. Let us start with two technical lemmas.

LEMMA 5.2. *Suppose that $\theta < \theta'$ and $k \in \mathcal{K}_\theta$. Then $(L_\varepsilon^{ren} - V)k \in \mathcal{K}_{\theta'}$ and*

$$||L_\varepsilon^{ren} - V||_{\theta\theta'} \to 0, \qquad \varepsilon \to 0.$$

PROOF. $L_\varepsilon^{ren} - V$ acting on $k \in \mathcal{K}_\theta$ has the following form:

$$(L_\varepsilon^{ren} - V)k(\eta) = -\varepsilon \int_{\mathbb{R}^d} \sum_{\{x,y\} \subset \eta} c_1(x, y; z) dz\ k(\eta)$$

$$+ \int_{\mathbb{R}^d} \int_{\Gamma_0} \sum_{y \in \eta} k(\eta \backslash y \cup \xi \cup y) c_2(x; y) \Big[\prod_{u \in \xi} \frac{1}{\varepsilon} (e^{-\varepsilon \phi(y-u)} - 1) \prod_{u \in \eta \backslash y} e^{-\varepsilon \phi(y-u)}$$

$$- \prod_{u \in \xi} (-\phi(y-u)) \Big] \lambda(d\xi) dx$$

$$- \int_{\mathbb{R}^d} \int_{\Gamma_0} k(\eta \cup \xi) \sum_{x \in \eta} c_2(x; y) \Big[\prod_{u \in \xi} \frac{1}{\varepsilon} (e^{-\varepsilon \phi(y-u)} - 1) \prod_{u \in \eta \backslash x} e^{-\varepsilon \phi(y-u)}$$

$$- \prod_{u \in \xi} (-\phi(y-u)) \Big] \lambda(d\xi) dy.$$

Denote the above three summands as $(L_\varepsilon^{ren} - V)_i$ with $i = 1, 2, 3$. Then for $k \in \mathcal{K}_\theta$ and $\theta' > \theta$ we have

$$|(L_\varepsilon^{ren} - V)_1 k(\eta)| \le \varepsilon c_1^{\max} ||k||_\theta e^{\theta|\eta|} \frac{|\eta|^2}{2} \le \varepsilon c_1^{\max} ||k||_\theta \frac{2}{(\theta' - \theta)^2 e^2} e^{\theta'|\eta|}$$

Next,

$$|(L_\varepsilon^{ren} - V)_2 k(\eta)| \le \int_{\mathbb{R}^d} \int_{\Gamma_0} \sum_{y \in \eta} ||k||_\theta e^{\theta(|\eta| + |\xi|)} c_2(x; y) \Big| \prod_{u \in \xi} \frac{1}{\varepsilon} (e^{-\varepsilon \phi(y-u)} - 1) \prod_{u \in \eta \backslash y} e^{-\varepsilon \phi(y-u)}$$

$$- \prod_{u \in \xi} (-\phi(y-u)) \Big| \lambda(d\xi) dx.$$

Observe that

$$\Big| \prod_{u \in \xi} \frac{1}{\varepsilon} \Big(e^{-\varepsilon \phi(y-u)} - 1 \Big) \prod_{u \in \eta \backslash y} e^{-\varepsilon \phi(y-u)} - \prod_{u \in \xi} (-\phi(y-u)) \Big|$$

$$\le \Big| \prod_{u \in \xi} \frac{1}{\varepsilon} \Big(1 - e^{-\varepsilon \phi(y-u)} \Big) - \prod_{u \in \xi} \phi(y-u) \Big| + \Big(1 - \prod_{u \in \eta \backslash y} e^{-\varepsilon \phi(y-u)} \Big) \prod_{u \in \xi} \frac{1}{\varepsilon} \Big(1 - e^{-\varepsilon \phi(y-u)} \Big)$$

By the means of inequality $\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \le \sum_{i=1}^{n} \frac{b_i - a_i}{b_i} \prod_{j=1}^{n} b_j$ for $b_i \ge a_i > 0$, cf. [7] p. 30 or [5] p. 27, we may further estimate the first summand of the above with

$$\sum_{u\in\xi} \Big[\phi(y-u) - \frac{1}{\varepsilon}\Big(1 - e^{-\varepsilon\phi(y-u)}\Big)\Big] \prod_{v\in\xi\setminus u} \phi(y-v)$$

and the second one with

$$\sum_{u\in\eta\setminus y} \Big(1 - e^{-\varepsilon\phi(y-u)}\Big) \prod_{u\in\xi} \frac{1}{\varepsilon}\Big(1 - e^{-\varepsilon\phi(y-u)}\Big).$$

As for the first one, notice that

$$\phi(y-u) - \frac{1}{\varepsilon}\Big(1 - e^{-\varepsilon\phi(y-u)}\Big) = \varepsilon\phi^2(y-u)\frac{\varepsilon\phi(y-u) - 1 + e^{-\varepsilon\phi(y-u)}}{\varepsilon^2\phi^2(y-u)} \le \frac{\varepsilon}{2}\phi^2(y-u),$$

as $\frac{e^{-x}-1+x}{x^2} \le \frac{1}{2}$ for $x > 0$. Therefore

$$\Big|\prod_{u\in\xi} \frac{1}{\varepsilon}\Big(1 - e^{-\varepsilon\phi(y-u)}\Big) - \prod_{u\in\xi} \phi(y-u)\Big| \le \frac{\varepsilon}{2} \sum_{u\in\xi} \phi^2(y-u) \prod_{v\in\xi\setminus u} \phi(y-v).$$

In the second summand, we use $1 - e^{-x} \le x$, which gives us estimation

$$\Big(1 - \prod_{u\in\eta\setminus y} e^{-\varepsilon\phi(y-u)}\Big) \prod_{u\in\xi} \frac{1}{\varepsilon}\Big(1 - e^{-\varepsilon\phi(y-u)}\Big) \le \varepsilon \sum_{u\in\eta\setminus y} \phi(y-u) \prod_{v\in\xi} \phi(y-v).$$

Coming back to $|(L_\varepsilon^{ren} - V)_2 k(\eta)|$, we have, recall assumptions (3.1)

$$\int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta \; e^{\theta(|\eta|+|\xi|)} c_2(x;y) \frac{\varepsilon}{2} \sum_{u\in\xi} \phi^2(y-u) \prod_{v\in\xi\setminus u} \phi(y-v)\lambda(d\xi)dx$$
$$= \varepsilon \; \frac{\langle c_2\rangle}{2} e^{\theta(|\eta|+1)} ||k||_\theta \sum_{y\in\eta} \int_{\Gamma_0} \sum_{u\in\xi} \phi^2(y-u) \prod_{v\in\xi\setminus u} e^\theta \phi(y-v)\lambda(d\xi)$$

Next, observe that by Lemma 2.21 and again by (3.1)

$$\int_{\Gamma_0} \sum_{u\in\xi} \phi^2(y-u) \prod_{v\in\xi\setminus u} e^\theta\phi(y-v)\lambda(d\xi) = \int_{\Gamma_0}\int_{\mathbb{R}^d} \phi^2(y-u) \prod_{v\in\xi} e^\theta\phi(y-v)du\lambda(d\xi)$$
$$\le |\phi|\langle\phi\rangle \int_{\Gamma_0} \prod_{v\in\xi} e^\theta\phi(y-v)\lambda(d\xi) = |\phi|\langle\phi\rangle \exp\big(\langle\phi\rangle e^\theta\big)$$

which allows us to estimate the first part of $|(L_\varepsilon^{ren} - V)_2 k(\eta)|$ by

$$\varepsilon\frac{\langle c_2\rangle}{2} |\eta| e^{\theta(|\eta|+1)} |\phi|\langle\phi\rangle \exp\big(\langle\phi\rangle e^\theta\big) ||k||_\theta$$

and for $\theta' > \theta$ by

$$\varepsilon\frac{\langle c_2\rangle}{2} \frac{e^\theta}{e(\theta'-\theta)} |\phi|\langle\phi\rangle \exp\big(\langle\phi\rangle e^\theta\big) e^{\theta'|\eta|} ||k||_\theta.$$

The second part of $|(L_\varepsilon^{ren} - V)_2 k(\eta)|$ we can estimate by

$$\varepsilon \int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta e^{\theta|\eta|} c_2(x;y) \sum_{u\in\eta\setminus y} \phi(y-u) \prod_{v\in\xi} e^\theta \phi(y-v)\lambda(d\xi)dx$$

and this by

$$\varepsilon\langle c_2\rangle |\phi| \exp\big(e^\theta\langle\phi\rangle\big) |\eta|^2 e^{\theta|\eta|} ||k||_\theta$$

which, for $\theta' > \theta$ is less than or equal to

$$\varepsilon\langle c_2\rangle|\phi|\exp\left(e^{\theta}\langle\phi\rangle\right)\frac{4}{(\theta'-\theta)^2e^2}e^{\theta'|\eta|}||k||_{\theta}$$

Summing up, we obtained for $\theta' > \theta$ and $k \in \mathcal{K}_\theta$

$$|(L_{\varepsilon}^{ren}-V)_2k(\eta)| \le \varepsilon\langle c_2\rangle|\phi|\exp\left(e^{\theta}\langle\phi\rangle\right)\Big[\frac{e^{\theta}\langle\phi\rangle}{2e(\theta'-\theta)}+\frac{4}{(\theta'-\theta)^2e^2}\Big]e^{\theta'|\eta|}||k||_{\theta}.$$

By similar calculations one can obtain exactly the same estimate for $|(L_{\varepsilon}^{ren}-V)_3k(\eta)|$, which together with the estimate obtained for $|(L_{\varepsilon}^{ren}-V)_1k(\eta)|$ and the one just shown, gives us finally

$$|(L_{\varepsilon}^{ren}-V)k(\eta)| \le \varepsilon\Big[\frac{2c_1^{\max}}{(\theta'-\theta)^2e^2}+\langle c_2\rangle|\phi|\exp\left(e^{\theta}\langle\phi\rangle\right)\Big(\frac{e^{\theta}\langle\phi\rangle}{e(\theta'-\theta)}+\frac{8}{(\theta'-\theta)^2e^2}\Big)\Big]e^{\theta'|\eta|}||k||_{\theta}.$$

From the above, the statement of the lemma directly follows. □

LEMMA 5.3. *Suppose that $\theta < \theta'$ and $k \in \mathcal{K}_\theta$. Then $(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})k \in \mathcal{K}_{\theta'}$ and*

$$||L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren}||_{\theta\theta'} \to 0, \qquad \varepsilon \to 0.$$

PROOF. By (5.1) we have

$$\begin{aligned}
&L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren} = (\varepsilon-\varepsilon')L_{14}^{\Delta}\\
&+\int_{\mathbb{R}^d}\sum_{y\in\eta}c_2(x;y)\Big[\prod_{u\in\eta\setminus y}e^{-\varepsilon\phi(y-u)}\ (Q_y^{\varepsilon,ren}k)(\eta\setminus y\cup x)-\prod_{u\in\eta\setminus y}e^{-\varepsilon'\phi(y-u)}\ (Q_y^{\varepsilon',ren}k)(\eta\setminus y\cup x)\Big]dx\\
&-\int_{\mathbb{R}^d}\sum_{x\in\eta}c_2(x;y)\Big[\prod_{u\in\eta\setminus x}e^{-\varepsilon\phi(y-u)}\ (Q_y^{\varepsilon,ren}k)(\eta)-\prod_{u\in\eta\setminus x}e^{-\varepsilon'\phi(y-u)}\ (Q_y^{\varepsilon',ren}k)(\eta)\Big]dy.
\end{aligned}$$

Denote the above three summands of $(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})$ as $(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})_1$, $(L_{\varepsilon}^{ren}-L_{\varepsilon'})_2$ and $(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})_3$ respectively. For the first one we have

$$||(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})_1||_{\theta\theta'} \le |\varepsilon'-\varepsilon|\ ||L_{14}^{\Delta}||_{\theta\theta'}.$$

For the next one, assuming that $\varepsilon' > \varepsilon$, we may write

$$\begin{aligned}
|(L_{\varepsilon}^{ren}-L_{\varepsilon'}^{ren})_2k(\eta)| \le &\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta}||k||_{\theta}e^{\theta'|\eta|}c_2(x;y)e^{\theta|\xi|}\Big[\prod_{u\in\xi}f_{y-u}(\varepsilon)\prod_{v\in\eta\setminus y}e^{-\varepsilon\phi(y-v)}\\
&-\prod_{u\in\xi}f_{y-u}(\varepsilon')\prod_{v\in\eta\setminus y}e^{-\varepsilon'\phi(y-v)}\Big]\lambda(d\xi)dx,
\end{aligned}$$

where

$$f_x(\varepsilon) = \frac{1}{\varepsilon}\Big(1-e^{-\varepsilon\phi(x)}\Big).$$

Notice that both ϕ and ε are non-negative, so that

$$f_x(\varepsilon) \le \phi(x). \tag{5.6}$$

Moreover, for $\varepsilon' > \varepsilon$, by the means of inequality $(\varepsilon')^n-\varepsilon^n \le (n+1)(\varepsilon'-\varepsilon)$ (both ε and ε' are from $(0,1]$) we have

$$\begin{aligned}
f_x(\varepsilon)-f_x(\varepsilon') &\le \sum_{n=1}^{\infty}\frac{\phi^{n+1}(x)}{(n+1)!}\Big[(\varepsilon')^n-\varepsilon^n\Big]\\
&\le (\varepsilon'-\varepsilon)\phi(x)\Big[\frac{\phi(x)}{2}+\sum_{n=2}^{\infty}\frac{\phi^n(x)}{n!}\Big] \le (\varepsilon'-\varepsilon)\phi(x)e^{\phi(x)}.
\end{aligned} \tag{5.7}$$

By inequality $\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \le \sum_{i=1}^{n} \frac{b_i - a_i}{b_i} \prod_{j=1}^{n} b_j$ for $b_i \ge a_i > 0$ we may estimate further $|(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren})_2 k(\eta)|$ by

$$\int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta e^{\theta|\eta|} c_2(x;y) e^{\theta|\xi|} \sum_{u\in\xi} \Big[f_{y-u}(\varepsilon)$$
$$- f_{y-u}(\varepsilon') \prod_{v\in\eta\setminus y} e^{-(\varepsilon'-\varepsilon)\phi(y-v)}\Big] \prod_{w\in\xi\setminus u} f_{y-w}(\varepsilon)\lambda(d\xi)dx$$
$$= \int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta e^{\theta|\eta|} c_2(x;y) e^{\theta|\xi|} \sum_{u\in\xi} \Big[f_{y-u}(\varepsilon) - f_{y-u}(\varepsilon')$$
$$+ f_{y-u}(\varepsilon')\Big(1 - \prod_{v\in\eta\setminus y} e^{-(\varepsilon'-\varepsilon)\phi(y-v)}\Big)\Big] \prod_{w\in\xi\setminus u} f_{y-w}(\varepsilon)\lambda(d\xi)dx$$
$$\le \int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta e^{\theta|\eta|} c_2(x;y) e^{\theta|\xi|} \sum_{u\in\xi} \Big[f_{y-u}(\varepsilon) - f_{y-u}(\varepsilon')$$
$$+ (\varepsilon'-\varepsilon) f_{y-u}(\varepsilon') \sum_{v\in\eta\setminus y} f_{y-v}(\varepsilon'-\varepsilon)\Big] \prod_{w\in\xi\setminus u} f_{y-w}(\varepsilon)\lambda(d\xi)dx.$$

Now, using (5.6) and (5.7) we obtain

$$|(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren})_2 k(\eta)| \le \int_{\mathbb{R}^d}\int_{\Gamma_0} \sum_{y\in\eta} ||k||_\theta e^{\theta|\eta|} c_2(x;y) e^{\theta|\xi|} \sum_{u\in\xi} \Big[(\varepsilon'-\varepsilon)\phi(y-u) e^{\phi(y-u)}$$
$$+ (\varepsilon'-\varepsilon)\phi(y-u) \sum_{v\in\eta\setminus y} \phi(y-v)\Big] \prod_{w\in\xi\setminus u} f_{y-w}(\varepsilon)\lambda(d\xi)dx.$$

By Lemma 2.21 and model assumptions (3.1) we can estimate furtlher by

$$(\varepsilon'-\varepsilon)||k||_\theta e^\theta \langle c_2\rangle e^{\theta|\eta|}\Big(e^{|\phi|} + (|\eta|-1)|\phi|\Big) \int_{\Gamma_0} \sum_{y\in\eta} \int_{\mathbb{R}^d} \phi(y-u) \prod_{w\in\xi} e^\theta \phi(y-w) du\lambda(d\xi).$$

Finally, by (3.8) and (3.9) we obtain

$$|(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren})_2 k(\eta)| \le (\varepsilon'-\varepsilon)||k||_\theta e^\theta \langle c_2\rangle e^{\theta|\eta|}\Big(|\eta| e^{|\phi|} + |\eta|^2|\phi|\Big)\langle\phi\rangle \exp\left(e^\theta\langle\phi\rangle\right)$$
$$\le (\varepsilon'-\varepsilon)||k||_\theta e^{\theta'|\eta|} e^\theta \langle c_2\rangle\langle\phi\rangle \exp\left(e^\theta\langle\phi\rangle\right)\Big(\frac{e^{|\phi|}}{(\theta'-\theta)e} + \frac{4|\phi|}{(\theta'-\theta)^2 e^2}\Big)$$

so that

$$||(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren})_2||_{\theta\theta'} \le (\varepsilon'-\varepsilon) e^\theta \langle c_2\rangle\langle\phi\rangle \exp\left(e^\theta\langle\phi\rangle\right)\Big(\frac{e^{|\phi|}}{(\theta'-\theta)e} + \frac{4|\phi|}{(\theta'-\theta)^2 e^2}\Big)$$

Analogical calculations lead to the same norm estimate for the third part of $L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren}$, i.e.

$$||(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren})_3||_{\theta\theta'} \le (\varepsilon'-\varepsilon) e^\theta \langle c_2\rangle\langle\phi\rangle \exp\left(e^\theta\langle\phi\rangle\right)\Big(\frac{e^{|\phi|}}{(\theta'-\theta)e} + \frac{4|\phi|}{(\theta'-\theta)^2 e^2}\Big).$$

Therefore

$$||L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren}||_{\theta\theta'} \to 0, \qquad \varepsilon' \to \varepsilon.$$

□

Theorem 5.4. *For* $\alpha_0 < \alpha_*$ *and* $t < T(\alpha_*, \frac{\alpha_0+\alpha_*}{2})$ *the scaling is continuous, i.e.*

$$||q_{t,\varepsilon} - q_{t,\varepsilon'}||_{\alpha_*} \to 0, \qquad \varepsilon' \to \varepsilon.$$

and

$$||q_{t,\varepsilon} - r_t||_{\alpha_*} \to 0, \qquad \varepsilon \to 0,$$

where r_t *is solution to equation* (5.4) *and* $q_{t,\varepsilon}, q_{t,\varepsilon'}$ *to equations* (5.2).

Proof. For $\varepsilon, \varepsilon' \in (0, 1]$ we have

$$q_{t,\varepsilon} - q_{t,\varepsilon'} = S_t^{\varepsilon} q_{0,\varepsilon} - S_t^{\varepsilon'} q_{0,\varepsilon'} = S_t^{\varepsilon}\left(q_{0,\varepsilon} - q_{0,\varepsilon'}\right) + \left(S_t^{\varepsilon} - S_t^{\varepsilon'}\right) q_{0,\varepsilon'}.$$

Estimating the norm of the above in $\mathcal{K}_{\alpha_*}$, the first summand either zeroes out if the initial state is Poisson measure, so that $q_{0,\varepsilon} = q_0$ for all $\varepsilon \in [0, 1]$, or

$$||S_t^{\varepsilon}\left(q_{0,\varepsilon} - q_{0,\varepsilon'}\right)||_{\alpha_*} \to 0, \qquad \varepsilon' \to \varepsilon.$$

It comes from the fact that $||S_t^{\varepsilon}||_{\alpha_0\alpha_*} \le \frac{T(\alpha_*,\alpha_0)}{T(\alpha_*,\alpha_0)-t}$ and $||q_{0,\varepsilon} - q_{0,\varepsilon'}||_{\alpha_0} \to 0$ as $\varepsilon' \to \varepsilon$.

The second summand can be rewritten as

$$\int_0^t S_s^{\varepsilon}\Big(L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren}\Big) S_{t-s}^{\varepsilon'} q_{0,\varepsilon'} ds$$

so that

$$\begin{aligned}\left|\left|\left(S_t^{\varepsilon} - S_t^{\varepsilon'}\right) q_{0,\varepsilon'}\right|\right|_{\alpha_*} &\le \int_0^t \frac{T(\alpha_*, \alpha_2)}{T(\alpha_*, \alpha_2) - s} \cdot \frac{T(\alpha_1, \alpha_0)}{T(\alpha_1, \alpha_0) - (t-s)} ds\ ||L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren}||_{\alpha_1\alpha_2}\ ||q_{0,\varepsilon'}||_{\alpha_0} \\ &\le t \frac{T(\alpha_*, \alpha_2)}{T(\alpha_*, \alpha_2) - t} \cdot \frac{T(\alpha_1, \alpha_0)}{T(\alpha_1, \alpha_0) - t}\ ||L_{\varepsilon}^{ren} - L_{\varepsilon'}^{ren}||_{\alpha_1\alpha_2}\ ||q_{0,\varepsilon'}||_{\alpha_0}\end{aligned}$$

for $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_*$.

By Lemma 5.3 it means that

$$\left|\left|\left(S_t^{\varepsilon} - S_t^{\varepsilon'}\right)\right|\right|_{\alpha_0\alpha_*} \to 0, \qquad \varepsilon' \to \varepsilon.$$

for $t < \min(T(\alpha_*, \alpha_2), T(\alpha_1, \alpha_0))$. In particular, we can choose α_1 and α_2 in such a way, that the above convergence holds true for $t < T(\alpha_*, \frac{\alpha_0+\alpha_*}{2})$.

Therefore for $t < T(\alpha_*, \frac{\alpha_0+\alpha_*}{2})$ we have

$$||q_{t,\varepsilon} - q_{t,\varepsilon'}||_{\alpha_*} \to 0, \qquad \varepsilon \to 0,$$

i.e. the first statement of the theorem holds true. Now, let us investigate the case $\varepsilon \to 0$.

$$q_{t,\varepsilon} - r_t = (q_{t,\varepsilon} - S_t^{\varepsilon} r_0) + (S_t^{\varepsilon} r_0 - S_t^0 r_0) = S_t^{\varepsilon}(q_{0,\varepsilon} - r_0) + \int_0^t S_s^{\varepsilon}(L_{\varepsilon}^{ren} - V) r_{t-s} ds,$$

so that by the means of Lemma 5.2 below and $||q_{0,\varepsilon} - r_0||_{\alpha_0} \to 0$ as $\varepsilon \to 0$ (recall the assumptions given at the beginning of this section), we obtain

$$||q_{t,\varepsilon} - r_t||_{\alpha_*} \to 0, \qquad \varepsilon \to 0$$

for $t < T(\alpha_*, \frac{\alpha_0+\alpha_*}{2})$. □

5.2. Repulsive coalescence. The analog of kinetic equation (5.5) can be obtained using the same scaling technique for a more general model, where coalescence is also repulsive. As the microscopic theory for this model is not developed, the continuity of the scale is not discussed. In Theorem 5.9, the main result of this section, we show that obtained kinetic equation (5.14) has the unique classical solution for some finite time horizon. The results presented in this section were published in [30]. Within this more general model, the operator L, cf. (3.1), has the form

$$LF(\gamma) = \sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d} \tilde{c}_1(x,y;z;\gamma)\Big(F\big(\gamma\backslash\{x,y\}\cup z\big) - F(\gamma)\Big)dz$$
$$+\sum_{x\in\gamma} \int_{\mathbb{R}^d} \tilde{c}_2(x;y;\gamma)\Big(F\big(\gamma\backslash x\cup y\big) - F(\gamma)\Big)dy,$$

where $\tilde{c}_2$ is as previously (with ϕ denoted by ϕ_2 now)

$$\tilde{c}_2(x;y;\gamma) = c_2(x;y)\prod_{u\in\gamma\backslash x} e^{-\phi_2(y-u)}$$

and $\tilde{c}_1$ is defined similarly as

$$\tilde{c}_1(x,y;z;\gamma) = c_1(x,y;z)\prod_{u\in\gamma\backslash\{x,y\}} e^{-\phi_2(z-u)},$$

where c_1, c_2, ϕ_1, ϕ_2 are non-negative real functions that satisfy, analogously as in the original model,

$$c_1(x,y;z) = c_1(y,x;z),$$
$$\int_{(\mathbb{R}^d)^2} c_1(x_1,x_2;x_3)dx_i dx_j = \langle c_1\rangle \ < \infty,\ i,j = 1,2,3,\ i \neq j,$$
$$\int_{\mathbb{R}^d} c_2(x;y)dx = \int_{\mathbb{R}^d} c_2(x;y)dy = \langle c_2\rangle \ < \infty,$$
$$\int_{\mathbb{R}^d} \phi_1(x)dx = \langle \phi_1\rangle \ < \infty, \quad \int_{\mathbb{R}^d} \phi_2(x)dx = \langle \phi_2\rangle \ < \infty.$$

Let

$$\tilde{c}_1(x,y;z;\gamma) = (KC^1_{x,y;z})(\gamma\backslash\{x,y\}),$$
$$\tilde{c}_2(x;y;\gamma) = (KC^2_{x;y})(\gamma\backslash x) \tag{5.8}$$

for some $C^1_{x,y;z}$ and $C^2_{x;y}$.

Suppose that $F = KG$, where $G : \Gamma_0 \to \mathbb{R}$. Then, by writing $K\hat{L}G = LF$, we define

$$\hat{L} = K^{-1}LK. \tag{5.9}$$

By the properties of the K-transform we derive an explicit formula for $\hat{L}$.

Lemma 5.5. *$\hat{L}$ defined as above has the following form*

$$\hat{L}G(\eta) \ = \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\eta\backslash\{x,y\})dz$$
$$+ \int_{\mathbb{R}^d} \sum_{x\in\eta} \Big[C^2_{x;y} \star H^2_{x;y}\Big](\eta\backslash x)dy,$$

where

$$H^1_{x,y;z}(\eta) = G(\eta\cup z) - G(\eta\cup x) - G(\eta\cup y) - G(\eta\cup\{x,y\}),$$
$$H^2_{x;y}(\eta) = G(\eta\cup y) - G(\eta\cup x). \tag{5.10}$$

Proof. First let us rewrite the operator L in a more convenient form. Using (5.8) and taking into account that any configuration treated as a subset of R^d is Lebesgue measure-zero, cf. (2.15), we have

$$L_1F(\gamma) = \sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d\setminus\gamma} \Big(KC^1_{x,y;z}(\cdot)\Big[KG\big(\,\cdot\cup z\big) - KG\big(\,\cdot\cup\{x,y\}\big)\Big]\Big)(\gamma\setminus\{x,y\})dz.$$

Observe that for any $\xi\in\Gamma,\ x,y,z\notin\xi$ we have

$$KG(\xi\cup z) = \sum_{\eta\subset\subset\xi\cup z} G(\eta) = \sum_{\eta\subset\subset\xi}\Big[G(\eta)+G(\eta\cup z)\Big] = K\Big[G(\cdot)+G(\cdot\cup z)\Big](\xi)$$

and analogously

$$KG(\xi\cup\{x,y\}) = K\Big[G(\cdot)+G(\cdot\cup x)+G(\cdot\cup y)+G(\cdot\cup\{x,y\})\Big](\xi).$$

Using linearity of the K-transform and above observations, we obtain

$$L_1F(\gamma) = \sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d} \Big(KC^1_{x,y;z}(\cdot)K\Big[G(\cdot\cup z) - G(\cdot\cup x) - G(\cdot\cup y) - G(\cdot\cup\{x,y\})\Big](\cdot)\Big)(\gamma\setminus\{x,y\})dz.$$

Considering the second part of the operator, we have

$$\begin{aligned} L_2F(\gamma) &= \sum_{x\in\gamma}\int_{\mathbb{R}^d} KC^2_{x;y}(\gamma\setminus x)\Big[KG(\gamma\setminus x\cup y) - KG(\gamma)\Big]dy \\ &= \sum_{x\in\gamma}\int_{\mathbb{R}^d} \Big(KC^2_{x;y}(\cdot)K\Big[G(\cdot\cup y) - G(\cdot\cup x)\Big](\cdot)\Big)(\gamma\setminus x)dy. \end{aligned}$$

Using notion (5.10) and property (2.6) of the product of K-transforms, we derive

$$L_1F(\gamma) = \sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d} K\Big[C^1_{x,y;z}\star H^1_{x,y;z}\Big](\gamma\setminus\{x,y\})dz,$$

$$L_2F(\gamma) = \sum_{x\in\gamma}\int_{\mathbb{R}^d} K\Big[C^2_{x;y}\star H^2_{x;y}\Big](\gamma\setminus x)dy.$$

Therefore

$$LF(\gamma) = \sum_{\{x,y\}\subset\gamma} \int_{\mathbb{R}^d} K\Big[C^1_{x,y;z}\star H^1_{x,y;z}\Big](\gamma\setminus\{x,y\})dz + \sum_{x\in\gamma}\int_{\mathbb{R}^d} K\Big[C^2_{x;y}\star H^2_{x;y}\Big](\gamma\setminus x)dy.$$

Recalling the definition (5.9) of the operator $\hat{L}$ and denoting

$$\hat{L}_1G(\eta) = K^{-1}L_1F(\eta), \quad \hat{L}_2G(\eta) = K^{-1}L_2F(\eta)$$

we obtain

$$\hat{L}_1 G(\eta) = \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{\{x,y\}\subset\xi} \int_{\mathbb{R}^d} K\Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\xi\setminus\{x,y\})dz$$

$$= \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \sum_{\xi\subset\eta\setminus\{x,y\}} (-1)^{|\eta\setminus\{x,y\}\setminus\xi|} K\Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\xi)dz$$

$$= \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} K^{-1}K\Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\eta\setminus\{x,y\})dz$$

$$= \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\eta\setminus\{x,y\})dz$$

and analogously

$$\hat{L}_2 G(\eta) = \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} \sum_{x\in\xi} \int_{\mathbb{R}^d} K\Big[C^2_{x;y} \star H^2_{x;y}\Big](\xi\setminus x)dy$$

$$= \int_{\mathbb{R}^d} \sum_{x\in\eta} \sum_{\xi\subset\eta\setminus x} (-1)^{|\eta\setminus x\setminus\xi|} K\Big[C^2_{x;y} \star H^2_{x;y}\Big](\xi)dy$$

$$= \int_{\mathbb{R}^d} \sum_{x\in\eta} K^{-1}K\Big[C^2_{x;y} \star H^2_{x;y}\Big](\eta\setminus x)dy$$

$$= \int_{\mathbb{R}^d} \sum_{x\in\eta} \Big[C^2_{x;y} \star H^2_{x;y}\Big](\eta\setminus x)dy.$$

Therefore

$$\hat{L}G(\eta) = \int_{\mathbb{R}^d} \sum_{\{x,y\}\subset\eta} \Big[C^1_{x,y;z} \star H^1_{x,y;z}\Big](\eta\setminus\{x,y\})dz + \int_{\mathbb{R}^d} \sum_{x\in\eta} \Big[C^2_{x;y} \star H^2_{x;y}\Big](\eta\setminus x)dy.$$

□

The next step is to pass with the action of the operator $\hat{L}$ to the correlation functions, i.e. to obtain L^Δ, cf. (3.3) and (3.4). The latter can be derived from the equation $\int_{\Gamma_0} \big(\hat{L}G\big)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} G\big(L^\Delta k\big)(\eta)\lambda(d\eta)$.

Lemma 5.6. *Operator L^Δ for the repulsive coalescence model is of the form*

$$L^\Delta = L_1^\Delta + L_2^\Delta,$$

where

$$L_1^\Delta k(\eta) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{z\in\eta} c_1(x,y;z) k(\eta\setminus z \cup \xi \cup \{x,y\}) e(t_z^{(1)} - 1, \xi) e(t_z^{(1)}, \eta\setminus z)\lambda(d\xi)dxdy$$

$$- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{x\in\eta} c_1(x,y;z) k(\eta \cup \xi \cup y) e(t_z^{(1)} - 1, \xi) e(t_z^{(1)}, \eta\setminus x)\lambda(d\xi)dydz$$

$$- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{y\in\eta} c_1(x,y;z) k(\eta \cup \xi \cup x) e(t_z^{(1)} - 1, \xi) e(t_z^{(1)}, \eta\setminus y)\lambda(d\xi)dxdz$$

$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{\{x,y\}\subset\eta} c_1(x,y;z)k(\eta\cup\xi)e(t_z^{(1)}-1,\xi)e(t_z^{(1)},\eta\backslash\{x,y\})\lambda(d\xi)dz$$

and

$$\begin{aligned}L_2^{\Delta}k(\eta) &= \int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta} k(\eta\backslash y\cup\xi\cup x)c_2(x;y)e(t_y^{(2)}-1,\xi)e(t_y^{(2)}-1,\eta\backslash y)\lambda(d\xi)dx\\ &\quad-\int_{\mathbb{R}^d}\int_{\Gamma_0} k(\eta\cup\xi)\sum_{x\in\eta} c_2(x;y)\prod_{u\in\xi} e(t_y^{(2)}-1,\xi)e(t_y^{(2)}-1,\eta\backslash x)\lambda(d\xi)dy.\end{aligned}$$

with

$$t_z^{(1)}(u)=e^{-\phi_1(z-u)},\ t_y^{(2)}(u)=e^{-\phi_2(y-u)},$$

see (2.5) *for definition of* $e(f,\eta)$.

PROOF. Using Lemmas 2.22 and 5.5 we get

$$\begin{aligned}\int_{\Gamma_0}(\hat{L}_1G)(\eta)k(\eta)\lambda(d\eta) &= \int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{\{x,y\}\subset\eta}\Big[C^1_{x,y;z}\star H^1_{x,y;z}\Big](\eta\backslash\{x,y\})k(\eta)dz\lambda(d\eta)\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\Big[C^1_{x,y;z}\star H^1_{x,y;z}\Big](\eta)k(\eta\cup\{x,y\})\lambda(d\eta)dxdydz.\end{aligned}$$

Recalling the definition (2.7) of the convolution $\star$ and using Lemma 2.20 twice, we obtain

$$\begin{aligned}&\int_{\Gamma_0}(\hat{L}_1G)(\eta)k(\eta)\lambda(d\eta)\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\sum_{\xi\subset\eta}C^1_{x,y;z}(\xi)\sum_{\zeta\subset\xi}H^1_{x,y;z}(\eta\backslash\xi\cup\zeta)k(\eta\cup\{x,y\})\lambda(d\eta)dxdydz\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\int_{\Gamma_0}C^1_{x,y;z}(\xi)\sum_{\zeta\subset\xi}H^1_{x,y;z}(\eta\cup\zeta)k(\eta\cup\xi\cup\{x,y\})\lambda(d\eta)\lambda(d\xi)dxdydz\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}C^1_{x,y;z}(\xi\cup\zeta)H^1_{x,y;z}(\eta\cup\zeta)k(\eta\cup\xi\cup\zeta\cup\{x,y\})\lambda(d\eta)\lambda(d\xi)\lambda(d\zeta)dxdydz.\end{aligned}$$

Using again Lemma 2.20, but in the opposite direction, we have

$$\begin{aligned}&\int_{\Gamma_0}(\hat{L}_1G)(\eta)k(\eta)\lambda(d\eta)\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta\subset\eta}C^1_{x,y;z}(\xi\cup\zeta)H^1_{x,y;z}(\eta)k(\eta\cup\xi\cup\{x,y\})\lambda(d\eta)\lambda(d\xi)dxdydz\\ &= \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}H^1_{x,y;z}(\eta)\Big[\int_{\Gamma_0}k(\eta\cup\xi\cup\{x,y\})\sum_{\zeta\subset\eta}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)\Big]\lambda(d\eta)dxdydz.\end{aligned}$$

Let us rewrite above using the notion (5.10) of $H^1_{x,y;z}(\eta)$.

$$\int_{\Gamma_0}(\hat{L}_1G)(\eta)k(\eta)\lambda(d\eta) = \frac{1}{2}\int_{(\mathbb{R}^d)^3}\int_{\Gamma_0}\Big[G(\eta\cup z) - G(\eta\cup x) - G(\eta\cup y) - G(\eta\cup\{x,y\})\Big]$$

$$\Big[\int_{\Gamma_0} k(\eta\cup\xi\cup\{x,y\})\sum_{\zeta\subset\eta}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)\Big]\lambda(d\eta)dxdydz$$

Using Lemas 2.21 and 2.22 we can rewrite RHS of the above as

$$\int_{\Gamma_0}G(\eta)\Big[\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{z\in\eta}k(\eta\setminus z\cup\xi\cup\{x,y\})\sum_{\zeta\subset\eta\setminus z}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dxdy$$

$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{x\in\eta}k(\eta\cup\xi\cup y)\sum_{\zeta\subset\eta\setminus x}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dydz$$

$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dxdz$$

$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{\{x,y\}\subset\eta}k(\eta\cup\xi)\sum_{\zeta\subset\eta\setminus\{x,y\}}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dz\Big]\lambda(d\eta).$$

Employing the same technique to the second part of the operator $\hat{L}$, we derive

$$\int_{\Gamma_0}(\hat{L}_2G)(\eta)k(\eta)\lambda(d\eta)$$

$$= \int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta\subset\eta}C^2_{x;y}(\xi\cup\zeta)H^2_{x;y}(\eta)k(\eta\cup\xi\cup x)\lambda(d\eta)\lambda(d\xi)dxdy$$

$$= \int_{\Gamma_0}G(\eta)\Big[\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\setminus y\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}C^2_{x;y}(\xi\cup\zeta)\lambda(d\xi)dx$$

$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}k(\eta\cup\xi)\sum_{x\in\eta}\sum_{\zeta\subset\eta\setminus x}C^2_{x;y}(\xi\cup\zeta)\lambda(d\xi)dy\Big]\lambda(d\eta)$$

Therefore, we obtain

$$L^{\Delta}k(\eta) = \frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{z\in\eta}k(\eta\setminus z\cup\xi\cup\{x,y\})\sum_{\zeta\subset\eta\setminus z}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dxdy$$

$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{x\in\eta}k(\eta\cup\xi\cup y)\sum_{\zeta\subset\eta\setminus x}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dydz$$

$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dxdz$$

$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{\{x,y\}\subset\eta}k(\eta\cup\xi)\sum_{\zeta\subset\eta\setminus\{x,y\}}C^1_{x,y;z}(\xi\cup\zeta)\lambda(d\xi)dz$$

$$+\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\setminus y\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}C^2_{x;y}(\xi\cup\zeta)\lambda(d\xi)dx$$

$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{x\in\eta}k(\eta\cup\xi)\sum_{\zeta\subset\eta\setminus x}C^2_{x;y}(\xi\cup\zeta)\lambda(d\xi)dy. \tag{5.11}$$

Note that so far we have not used any assumption about coefficients $\tilde{c}_1$ and $\tilde{c}_2$ apart from that they can be written as results of action of the K-transform on corresponding functions $C^1_{x,y;z}$ and $C^2_{x;y}$. Let us calculate explicit forms of these functions. Recall that

$$\tilde{c}_1(x,y;z;\gamma)=c_1(x,y;z)\prod_{u\in\gamma\setminus\{x,y\}}e^{-\phi_1(z-u)},$$
$$\tilde{c}_2(x;y;\gamma)=c_2(x;y)\prod_{u\in\gamma\setminus x}e^{-\phi_2(y-u)}.$$

We have

$$KC^1_{x,y;z}=c_1(x,y;z)e(t^{(1)}_z,\cdot),$$

that is

$$C^1_{x,y;z}=K^{-1}c_1(x,y;z)e(1+t^{(1)}_z-1,\cdot)=c_1(x,y;z)K^{-1}\sum_{\xi\subset\cdot}e(t^{(1)}_z-1,\xi)$$
$$=c_1(x,y;z)K^{-1}Ke(t^{(1)}_z-1,\cdot)=c_1(x,y;z)e(t^{(1)}_z-1,\cdot).$$

Therefore

$$C^1_{x,y;z}(\eta)=c_1(x,y;z)e(t^{(1)}_z-1,\eta). \tag{5.12}$$

Analogously we can derive

$$C^2_{x;y}(\eta)=c_2(x;y)e(t^{(2)}_y-1,\eta). \tag{5.13}$$

We use the above to rewrite the operator L^Δ. For convenience let us denote as previously for the original model: part corresponding to the coalescence, that is the first four terms of (5.11), as L^Δ_1 and the part corresponding to the jumps, that is the last two terms of (5.11), as L^Δ_2. Substituting (5.12) we derive

$$L^\Delta_1k(\eta)=\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{z\in\eta}k(\eta\setminus z\cup\xi\cup\{x,y\})\sum_{\zeta\subset\eta\setminus z}c_1(x,y;z)$$
$$e(t^{(1)}_z-1,\xi\cup\zeta)\lambda(d\xi)dxdy$$
$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{x\in\eta}k(\eta\cup\xi\cup y)\sum_{\zeta\subset\eta\setminus x}c_1(x,y;z)e(t^{(1)}_z-1,\xi\cup\zeta)\lambda(d\xi)dydz$$
$$-\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}c_1(x,y;z)e(t^{(1)}_z-1,\xi\cup\zeta)\lambda(d\xi)dxdz$$
$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{\{x,y\}\subset\eta}k(\eta\cup\xi)\sum_{\zeta\subset\eta\setminus\{x,y\}}c_1(x,y;z)e(t^{(1)}_z-1,\xi\cup\zeta)\lambda(d\xi)dz$$

and analogously using (5.13) we obtain

$$L^\Delta_2k(\eta)=\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta}k(\eta\setminus y\cup\xi\cup x)\sum_{\zeta\subset\eta\setminus y}c_2(x;y)e(t^{(2)}_y-1,\xi\cup\zeta)\lambda(d\xi)dx$$
$$-\int_{\mathbb{R}^d}\int_{\Gamma_0}k(\eta\cup\xi)\sum_{x\in\eta}\sum_{\zeta\subset\eta\setminus x}c_2(x;y)e(t^{(2)}_y-1,\xi\cup\zeta)\lambda(d\xi)dy.$$

Consider the first component L_{11}^{Δ} of L_1^{Δ}, i.e.

$$L_{11}^{\Delta}k(\eta) = \frac{1}{2} \int\limits_{(\mathbb{R}^d)^2} \int\limits_{\Gamma_0} \sum_{z\in\eta} k(\eta\backslash z \cup \xi \cup \{x,y\})$$
$$\sum_{\zeta\subset\eta\backslash z} c_1(x,y;z) e(t_z^{(1)} - 1, \xi\cup\zeta)\lambda(d\xi)dxdy.$$

For a given η let us introduce $C(\eta) = \{\xi \in \Gamma_0 : \xi\cap\eta \neq \emptyset\}$. Then, because any configuration treated as a measurable subset of $\mathbb{R}^d$ is of Lebesgue measure 0 and the empty configuration does not belong to $C(\eta)$ for any $\eta \in \Gamma_0$, we have $\lambda(C(\eta)) = 0$ for every $\eta \in \Gamma_0$. Indeed, using the Definition 2.18 of the Lebesgue-Poisson integral, we obtain

$$\lambda(C(\eta)) = \int\limits_{\Gamma_0} I_{C(\eta)}(\xi)\lambda(d\xi) = I_{C(\eta)}^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^d)^n} I_{C(\eta)}^{(n)}(x_1, ..., x_n)dx_1...dx_n.$$

First, notice that $I_{C(\eta)}^{(0)} = 0$, as empty configuration is disjoint with any configuration. Then, because

$$I_{C(\eta)}^{(n)}(x_1, ..., x_n) \le I_{C(\eta)}^{(1)}(x_1) + I_{C(\eta)}^{(1)}(x_2) + ... + I_{C(\eta)}^{(1)}(x_n),$$

we have for every $n \in \mathbb{N}$

$$\int\limits_{(\mathbb{R}^d)^n} I_{C(\eta)}^{(n)}(x_1, ..., x_n)dx_1...dx_n \le n \int\limits_{(\mathbb{R}^d)^{n-1}} \Big[\int\limits_{\mathbb{R}^d} I_{C(\eta)}^{(1)}(x)dx\Big]dx_1...dx_{n-1}.$$

Taking into account that

$$\int\limits_{\mathbb{R}^d} I_{C(\eta)}^{(1)}(x)dx = \int\limits_{\mathbb{R}^d} I_\eta(x)dx = l(\eta) = 0,$$

where l denotes the Lebesgue measure, one can clearly see that $\lambda(C(\eta)) = 0$.

Therefore, when integrating over $\Gamma_0\backslash C(\eta)$ instead of Γ_0, the result is the same. However, all subconfigurations ζ of η are disjoint with any $\xi \in \Gamma_0\backslash C(\eta)$, which allows us to separate the product taken over $\xi\cup\zeta$ into one taken over ξ and another taken over ζ. Thus we can write

$$L_{11}^{\Delta}k(\eta) = \frac{1}{2} \int\limits_{(\mathbb{R}^d)^2} \int\limits_{\Gamma_0} \sum_{z\in\eta} c_1(x,y;z) k(\eta\backslash z \cup \xi \cup \{x,y\})$$
$$e(t_z^{(1)} - 1, \xi) \sum_{\zeta\subset\eta\backslash z} e(t_z^{(1)} - 1, \zeta)\lambda(d\xi)dxdy.$$

Recalling the Definition 2.16 of the K-transform and its property (2.8) we have

$$\sum_{\zeta\subset\eta\backslash z} e(t_z^{(1)} - 1, \zeta) = K\big(e(t_z^{(1)} - 1, \cdot)\big)(\eta\backslash z) = e(t_z^{(1)}, \eta\backslash z).$$

Therefore, we can rewrite the action of L_{11}^{Δ} in the form

$$L_{11}^{\Delta}k(\eta) = \frac{1}{2} \int\limits_{(\mathbb{R}^d)^2} \int\limits_{\Gamma_0} \sum_{z\in\eta} c_1(x,y;z) k(\eta\backslash z \cup \xi \cup \{x,y\})$$
$$e(t_z^{(1)} - 1, \xi) e(t_z^{(1)}, \eta\backslash z)\lambda(d\xi)dxdy.$$

Applying the same method for the rest of the L_1^Δ and for the L_2^Δ, we obtain the result. $\square$

Using the same scaling technique, as described for the original model (L_ε^Δ obtained by substitutions $c_1 \to \varepsilon c_1$, $\phi_1 \to \varepsilon\phi_1$ and $\phi_2 \to \varepsilon\phi_2$), we obtain $V = \lim_{\varepsilon\to 0} L_\varepsilon^{ren}$ of the form

$$
\begin{aligned}
Vk(\eta) = {} & \frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{z\in\eta} c_1(x,y;z)k(\eta\backslash z\cup\xi\cup\{x,y\})\prod_{u\in\xi}(-\phi_1(z-u))\lambda(d\xi)dxdy\\
& -\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{x\in\eta} c_1(x,y;z)k(\eta\cup\xi\cup y)\prod_{u\in\xi}(-\phi_1(z-u))\lambda(d\xi)dydz\\
& -\frac{1}{2}\int_{(\mathbb{R}^d)^2}\int_{\Gamma_0}\sum_{y\in\eta} c_1(x,y;z)k(\eta\cup\xi\cup x)\prod_{u\in\xi}(-\phi_1(z-u))\lambda(d\xi)dxdz\\
& +\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y\in\eta} k(\eta\backslash y\cup\xi\cup x)c_2(x;y)\prod_{u\in\xi}(-\phi_2(y-u))\lambda(d\xi)dx\\
& -\int_{\mathbb{R}^d}\int_{\Gamma_0} k(\eta\cup\xi)\sum_{x\in\eta} c_2(x;y)\prod_{u\in\xi}(-\phi_2(y-u))\lambda(d\xi)dy.
\end{aligned}
$$

The corresponding kinetic equation can be written as

$$
\frac{d}{dt}\rho_t(x) = R_1(\rho_t,x) + R_2(\rho_t,x), \quad \rho_{t=0}(x) = \rho_0(x) \in L^\infty(\mathbb{R}^d) \tag{5.14}
$$

where

$$
\begin{aligned}
R_1(\rho_t,x) &= -\frac{1}{2}\rho_t(x)\int_{(\mathbb{R}^d)^2}\Big(c_1(x,y;z)+c_1(y,x;z)\Big)\rho_t(y)dydz - h(\rho_t,x)\int_{\mathbb{R}^d} c_2(x;y)dy\\
&= -\rho_t(x)h(\rho_t,x)
\end{aligned}
$$

with

$$
h(\rho_t,x) = \frac{1}{2}\int_{(\mathbb{R}^d)^2}\Big(c_1(x,y;z)+c_1(y,x;z)\Big)\rho_t(y)dydz \ + \langle c_2\rangle
$$

and

$$
\begin{aligned}
R_2(\rho_t,x) = {} & \frac{1}{2}\int_{(\mathbb{R}^d)^2} c_1(y,z;x)\exp\Big(-\int_{\mathbb{R}^d}\phi_1(x-u)\rho_t(u)du\Big)\rho_t(y)\rho_t(z)dydz\\
& +\frac{1}{2}\int_{(\mathbb{R}^d)^2}\Big(c_1(x,y;z)+c_1(y,x;z)\Big)\\
& \qquad \Big[1-\exp\Big(-\int_{\mathbb{R}^d}\phi_1(z-u)\rho_t(u)du\Big)\Big]\rho_t(x)\rho_t(y)dydz\\
& +\int_{\mathbb{R}^d} c_2(y;x)\exp\Big(-\int_{\mathbb{R}^d}\phi_2(x-u)\rho_t(u)du\Big)\rho_t(y)dy\\
& +\int_{\mathbb{R}^d} c_2(x;y)\Big[1-\exp\Big(-\int_{\mathbb{R}^d}\phi_2(y-u)\rho_t(u)du\Big)\Big]\rho_t(x)dy.
\end{aligned}
$$

Kinetic equation (5.14) can be written in the equivalent form of the integral equation

$$\rho_t(x) = \rho_0(x) \exp\Big(-\int_0^t h(\rho_s, x)ds\Big) + \int_0^t R_2(\rho_s, x) \exp\Big(-\int_s^t h(\rho_\sigma, x)d\sigma\Big)ds. \quad (5.15)$$

Now we will move to the main result of this section, the existence and uniqueness of the local classical solution to equation (5.14). Before that, let us introduce some additional notions and prove two technical lemmas: Lemma 5.7 and Lemma 5.8.

Consider $X_T = C([0,T] \to L^\infty(\mathbb{R}^d)),\ T > 0$ with the norm

$$||\rho||_{T,\gamma} = \sup_{t\in[0,T]} e^{-\gamma\langle c_2\rangle t}||\rho_t||_{L^\infty}.$$

Denote

$$B_{T,\gamma}(r) = \{\rho \in X_T :\ ||\rho||_{T,\gamma} \le r,\ \rho_t \ge 0\ \forall t \in [0,T]\},$$
$$B_{T,\gamma}(r, \rho_0) = \{\psi \in B_{T,\gamma}(r) : \psi_0 = \rho_0\},$$

where $\rho_0 \in L^\infty(\mathbb{R}^d), \rho_0 \ge 0,\ r \ge ||\rho_0||_{L^\infty}$ and $T, \gamma > 0$.

Lemma 5.7. *Given $r > 0$, there exist $\gamma, \tilde{T} > 0$ such that F defined by the RHS of (5.15) with the domain $B_{T^*,\gamma}(r) \subset X_{T^*}$ acts again to the $B_{T^*,\gamma}(r)$ for any $T^* \in [0, \tilde{T}]$.*

Proof of Lemma 5.7. Take arbitrary $T, \gamma > 0$ and $\rho \in B_{T,\gamma}(r)$. Note that

$$h(\rho_t, x) \ge\ \langle c_2\rangle,$$
$$R_2(\rho_t, x) \le \frac{3}{2}||\rho_t||^2_{L^\infty}\langle c_1\rangle + 2||\rho_t||_{L^\infty}\langle c_2\rangle,$$
$$\rho_t(x) \le ||\rho_t||_{L^\infty} \le e^{\gamma\langle c_2\rangle t}||\rho||_{T,\gamma}. \quad (5.16)$$

It is obvious that F preserves positiveness of ρ. Furthermore, using above estimates and the definition of $B_{T,\gamma}(r)$, we derive

$$\Big(F(\rho)\Big)_t(x) = \rho_0(x) \exp\Big(-\int_0^t h(\rho_s, x)ds\Big) + \int_0^t R_2(\rho_s, x) \exp\Big(-\int_s^t h(\rho_\sigma, x)d\sigma\Big)ds$$
$$\le ||\rho_0||_{L^\infty} e^{-t\langle c_2\rangle} + \int_0^t R_2(\rho_s, x)e^{(s-t)\langle c_2\rangle}ds$$
$$\le e^{-t\langle c_2\rangle}\Big[||\rho||_{T,\gamma} + \int_0^t \Big(\frac{3}{2}\langle c_1\rangle e^{(2\gamma+1)\langle c_2\rangle s}||\rho||^2_{T,\gamma} + 2\langle c_2\rangle e^{(\gamma+1)\langle c_2\rangle s}||\rho||_{T,\gamma}\Big)ds\Big].$$

Therefore we obtain

$$\Big|\Big|\Big(F(\rho)\Big)_t\Big|\Big|_{L^\infty} \le e^{-t\langle c_2\rangle} r\Big[1 + \frac{3\langle c_1\rangle r}{2(2\gamma+1)\langle c_2\rangle}\Big(e^{(2\gamma+1)\langle c_2\rangle t} - 1\Big) + \frac{2}{\gamma+1}\Big(e^{(\gamma+1)\langle c_2\rangle t} - 1\Big)\Big].$$

Thus

$$\Big|\Big|\Big(F(\rho)\Big)\Big|\Big|_{T,\gamma} \le r \sup_{t\in[0,T]} f(t),$$

where

$$f(t) = e^{-(\gamma+1)\langle c_2\rangle t}\Big[1 + \frac{3\langle c_1\rangle r}{2(2\gamma+1)\langle c_2\rangle}\Big(e^{(2\gamma+1)\langle c_2\rangle t} - 1\Big) + \frac{2}{\gamma+1}\Big(e^{(\gamma+1)\langle c_2\rangle t} - 1\Big)\Big].$$

Note that $f(0) = 1$. Additionally

$$f'(t) = -(\gamma+1)\langle c_2\rangle e^{-(\gamma+1)\langle c_2\rangle t}\Big[1 + \frac{3\langle c_1\rangle r}{2(2\gamma+1)\langle c_2\rangle}\Big(e^{(2\gamma+1)\langle c_2\rangle t} - 1\Big)$$
$$+\frac{2}{(\gamma+1)}\Big(e^{(\gamma+1)\langle c_2\rangle t} - 1\Big)\Big] + e^{-(\gamma+1)\langle c_2\rangle t}\Big[\frac{3\langle c_1\rangle r}{2}e^{(2\gamma+1)\langle c_2\rangle t} + 2\langle c_2\rangle e^{(\gamma+1)\langle c_2\rangle t}\Big]$$

and hence

$$f'(0) = -(\gamma+1)\langle c_2\rangle + \Big(\frac{3}{2}\langle c_1\rangle r + 2\langle c_2\rangle\Big).$$

Choosing $\gamma > 1 + \frac{3\langle c_1\rangle r}{2\langle c_2\rangle}$ we have $f'(0) < 0$, which guarantees existence of $\tilde{T}$ such that $\sup_{t\in[0,\tilde{T}]} f(t) = 1$. Taking $T = T^*$ for $T^* \in [0, \tilde{T}]$ yields

$$||F(\rho)||_{T^*,\gamma} \leq r.$$

Therefore $F(\rho) \in B_{T^*,\gamma}(r)$ for $\rho \in B_{T^*,\gamma}(r)$. □

LEMMA 5.8. *Let $\rho_0 \in L^\infty(\mathbb{R}^d), \rho_0 \geq 0$ and $r \geq ||\rho_0||_{L^\infty}$. Let $\tilde{T}, \gamma$ satisfy lemma 5.7 for this r. We can choose $T^* \in [0, \tilde{T}]$ in such a way that for any ρ, ψ in $B_{T^*,\gamma}(r, \rho_0)$ the inequality $||F(\rho) - F(\psi)||_{T^*,\gamma} \leq C||\rho - \psi||_{T^*,\gamma}$ holds for some constant $C < 1$.*

PROOF. We have

$$\Big(F(\rho) - F(\psi)\Big)_t(x) = \rho_0(x)\exp\Big(-\int_0^t h(\rho_s, x)ds\Big)$$
$$+\int_0^t R_2(\rho_s, x)\exp\Big(-\int_s^t h(\rho_\sigma, x)d\sigma\Big)ds - \rho_0(x)\exp\Big(-\int_0^t h(\psi_s, x)ds\Big)$$
$$-\int_0^t R_2(\psi_s, x)\exp\Big(-\int_s^t h(\psi_\sigma, x)d\sigma\Big)ds = D_1 + \int_0^t D_2 ds, \tag{5.17}$$

where

$$D_1 = \rho_0(x)\Big[\exp\Big(-\int_0^t h(\rho_s, x)ds\Big) - \exp\Big(-\int_0^t h(\psi_s, x)ds\Big)\Big]$$

and

$$D_2 = \int_0^t \Big[R_2(\rho_s, x)\exp\Big(-\int_s^t h(\rho_\sigma, x)d\sigma\Big)$$
$$- R_2(\psi_s, x)\exp\Big(-\int_s^t h(\psi_\sigma, x)d\sigma\Big)\Big]ds.$$

Take an arbitrary $T^* \in [0, \tilde{T}]$. We have

$$|D_1| \leq ||\rho_0||_{L^\infty}\langle c_1\rangle\int_0^t ||\rho_s - \psi_s||_{L^\infty}ds \leq r\langle c_1\rangle te^{\gamma\langle c_2\rangle t}||\rho - \psi||_{T^*,\gamma}. \tag{5.18}$$

In order to estimate $|D_2|$, consider two cases. First, suppose that

$$\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma x)\Big) d\sigma \geq 0.$$

Then

$$|D_2| \leq \Big| R_2(\rho_s, x) \exp\Big[-\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma, x)\Big) d\sigma\Big]$$

$$-R_2(\psi_s, x) \exp\Big[-\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma, x)\Big) d\sigma\Big]\Big|$$

$$+\Big| R_2(\psi_s, x) \exp\Big[-\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma, x)\Big) d\sigma\Big] - R_2(\psi_s, x)\Big|$$

$$\leq \Big| R_2(\rho_s, x) - R_2(\psi_s, x)\Big| + R_2(\psi_s, x)\Big\{1 - \exp\Big[-\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma, x)\Big) d\sigma\Big]\Big\}.$$

In the other case, when $\int_s^t \Big(h(\rho_\sigma, x) - h(\psi_\sigma, x)\Big) d\sigma < 0$, we have analogously

$$|D_2| \leq \Big| R_2(\rho_s, x) - R_2(\psi_s, x)\Big| + R_2(\rho_s, x)\Big\{1 - \exp\Big[-\int_s^t \Big(h(\psi_\sigma, x) - h(\rho_\sigma, x)\Big) d\sigma\Big]\Big\}.$$

Note that both $R_2(\rho_s, x)$ and $R_2(\psi_s, x)$, as both belong to $B_{T^*,\gamma}(r)$, satisfy the same estimate (cf (5.2))

$$R_2(\rho_s, x), R_2(\psi_s, x) \leq \frac{3}{2}\langle c_1\rangle e^{2\gamma\langle c_2\rangle s} r^2 + 2\langle c_2\rangle e^{\gamma\langle c_2\rangle s} r,$$

which allows us to write

$$|D_2| \leq \Big| R_2(\rho_s, x) - R_2(\psi_s, x)\Big| + \Big(\frac{3}{2}\langle c_1\rangle e^{2\gamma\langle c_2\rangle s} r^2 + 2\langle c_2\rangle e^{\gamma\langle c_2\rangle s} r\Big)$$

$$\Big\{1 - \exp\Big[-\int_s^t \Big| h(\psi_\sigma, x) - h(\rho_\sigma, x)\Big| d\sigma\Big]\Big\}. \tag{5.19}$$

We have

$$1 - \exp\Big[-\int_s^t \Big| h(\psi_\sigma, x) - h(\rho_\sigma, x)\Big| d\sigma\Big] \leq \int_s^t \Big| h(\psi_\sigma, x) - h(\rho_\sigma, x)\Big| d\sigma$$

$$= \frac{1}{2}\int_s^t \Big| \int_{(\mathbb{R}^d)^2} \Big(c_1(x, y; z) + c_1(y, x; z)\Big)\Big(\rho_\sigma(y) - \psi_\sigma(y)\Big) dy dz\Big| d\sigma$$

$$\leq \int_s^t \langle c_1\rangle ||\rho_\sigma - \psi_\sigma||_{L^\infty} d\sigma \leq \int_s^t \langle c_1\rangle e^{\gamma\langle c_2\rangle\sigma} ||\rho - \psi||_{T^*,\gamma} d\sigma$$

$$\leq \langle c_1\rangle e^{\gamma\langle c_2\rangle t}(t - s)||\rho - \psi||_{T^*,\gamma},$$

which yields

$$1-\exp\Big[-\int\limits_s^t \Big|h(\psi_\sigma,x)-h(\rho_\sigma,x)\Big|d\sigma\Big] \leq \ \langle c_1\rangle t e^{\gamma\langle c_2\rangle t}||\rho-\psi||_{T^*,\gamma}. \tag{5.20}$$

Let us estimate

$$\begin{aligned}
\Big|R_2(\rho_s,x)-R_2(\psi_s,x)\Big| \leq \frac{1}{2}\int\limits_{(\mathbb{R}^d)^2} c_1(y,z;x)\ \Big|&\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_1(x-u)\rho_s(u)du\Big)\rho_s(y)\rho_s(z)\\
&-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_1(x-u)\psi_s(u)du\Big)\psi_s(y)\psi_s(z)\Big|dydz\\
+\frac{1}{2}\int\limits_{(\mathbb{R}^d)^2}\Big(c_1(x,y;z)+c_1(y,x;z)\Big)\ \Big|&\Big[1-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_1(z-u)\rho_s(u)du\Big)\Big]\\
\rho_s(x)\rho_s(y)-&\Big[1-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_1(z-u)\psi_s(u)du\Big)\Big]\psi_s(x)\psi_s(y)\Big|dydz\\
+\int\limits_{\mathbb{R}^d}c_2(y;x)\ \Big|&\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_2(x-u)\rho_s(u)du\Big)\rho_s(y)\\
&-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_2(x-u)\psi_s(u)du\Big)\psi_s(y)\Big|dy\\
+\int\limits_{\mathbb{R}^d}c_2(x;y)\ \Big|&\Big[1-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_2(y-u)\rho_s(u)du\Big)\Big]\rho_s(x)\\
&-\Big[1-\exp\Big(-\int\limits_{\mathbb{R}^d}\phi_2(y-u)\psi_s(u)du\Big)\Big]\psi_s(x)\Big|dy.
\end{aligned}$$

Denote by I_i the i-th component of the RHS of the above inequality for $i=1,2,3,4$. Then estimating analogously as above we derive

$$I_3, I_4 \leq \ \langle c_2\rangle\Big(e^{2\gamma\langle c_2\rangle s}\langle\phi_2\rangle r+e^{\gamma\langle c_2\rangle s}\Big)\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma}.$$

Moreover, noting that

$$\begin{aligned}
&\Big|\rho_s(y)\rho_s(z)-\psi_s(y)\psi_s(z)\Big|\\
&\leq \frac{1}{2}\Big(\rho_s(z)+\psi_s(z)\Big)\Big|\rho_s(y)-\psi_s(y)\Big|+\frac{1}{2}\Big(\rho_s(y)+\psi_s(y)\Big)\Big|\rho_s(z)-\psi_s(z)\Big|,
\end{aligned}$$

we obtain

$$\begin{aligned}
I_1 &\leq \frac{1}{2}\langle c_1\rangle\Big(2e^{2\gamma\langle c_2\rangle s}r+e^{3\gamma\langle c_2\rangle s}\langle\phi_1\rangle r^2\Big)\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma},\\
I_2 &\leq \ \langle c_1\rangle\Big(2e^{2\gamma\langle c_2\rangle s}r+e^{3\gamma\langle c_2\rangle s}\langle\phi_1\rangle r^2\Big)\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Big|R_2(\rho_s,x)-R_2(\psi_s,x)\Big| \leq \Big[&\frac{3}{2}\langle c_1\rangle e^{\gamma\langle c_2\rangle s}\Big(2e^{\gamma\langle c_2\rangle s}r+e^{2\gamma\langle c_2\rangle s}\langle\phi_1\rangle r^2\Big)\\
&+2\langle c_2\rangle e^{\gamma\langle c_2\rangle s}\Big(e^{\gamma\langle c_2\rangle s}\langle\phi_2\rangle r+1\Big)\Big]\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma}.
\end{aligned} \tag{5.21}$$

Substituting (5.20) and (5.2) into (5.2) and using it together with (5.18), we obtain (cf 5.2)

$$\left|\Big(F(\rho)-F(\psi)\Big)_t(x)\right| \leq e^{\gamma\langle c_2\rangle t} f(t)\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma},$$

where

$$f(t) = t\,\Big[\frac{3}{2}r^2\langle c_1\rangle e^{2\gamma\langle c_2\rangle t}\Big(\langle c_1\rangle t+\langle\phi_1\rangle\Big)$$
$$+re^{\gamma\langle c_2\rangle t}\Big(2\langle c_1\rangle\langle c_2\rangle t+3\langle c_1\rangle+2\langle c_2\rangle\langle\phi_2\rangle\Big)+2\langle c_2\rangle\Big].$$

Therefore

$$\Big|\Big|F(\rho)-F(\psi)\Big|\Big|_{T^*,\gamma} \leq \sup_{t\in[0,T^*]} f(t)\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma}.$$

Note that $f(t)$ is continuous, increasing function of t and $f(0)=0$. Thus, there exists $T^{**}>0$ such that $f(T^{**})<1$ and $f(t)\in[0,f(T^{**})]$ for $t\in[0,T^{**}]$. Choosing $T^*=\min(T^{**},\tilde{T})$, we obtain

$$\Big|\Big|F(\rho)-F(\psi)\Big|\Big|_{T^*,\gamma} \leq C\Big|\Big|\rho-\psi\Big|\Big|_{T^*,\gamma}$$

with $C=f(T^*)\leq f(T^{**})<1$. □

THEOREM 5.9. *Problem (5.14) with the initial condition $\rho_0\in L^\infty(\mathbb{R}^d)$), $\rho_0\geq 0$ has the unique local classical solution.*

PROOF. Choose $r>||\rho_0||_{L^\infty}$ and take corresponding $\gamma,\tilde{T}$ from lemma 5.7. Take T^* as in lemma 5.8. Define the sequence of Picard iterations $(\rho^{(n)})_{n\in\mathbb{N}_0}$ in the following way

$$\begin{aligned}\rho^{(0)}_t &= \rho_0\ \forall t\in[0,T^*],\\ \rho^{(n)} &= F(\rho^{(n-1)}),\ n\in\mathbb{N}.\end{aligned} \tag{5.22}$$

Obviously, $\rho^{(0)}\in B_{T^*,\gamma}(r)$. Therefore, by lemma 5.7, $\rho^{(n)}\in B_{T^*,\gamma}(r)$ for all $n\in\mathbb{N}$ and from lemma 5.8 we obtain

$$||\rho^{(n+k)}-\rho^{(n)}||_{T^*,\gamma}\leq||\rho^{(1)}-\rho^{(0)}||_{T^*,\gamma}\sum_{i=1}^{k}C^{n+i-1}\leq||\rho^{(1)}-\rho^{(0)}||_{T^*,\gamma}\frac{C^n}{1-C},$$

where $C<1$ is a positive constant. Therefore $\big(\rho^{(n)}\big)_{n\in\mathbb{N}_0}$ defined by (5.2) is a Cauchy sequence. As $B_{T^*,\gamma}(r)$ is a closed subset of a Banach space, there exists

$$\lim_{n\to\infty}\rho^{(n)}=\rho\in B_{T^*,\gamma}(r).$$

Clearly $F(\rho)=\rho$ and therefore ρ_t satisfies the integral equation (5.15) for $t\in[0,T^*]$. Thus it is a local classical solution of (5.14).

Now suppose there is another local classical solution of this equation, say ψ. Then $\psi_0=\rho_0$ and for r,γ,T^* as above, there exists $T\leq T^*$ such that $\psi\in B_{T,\gamma}(r)$. However, from lemma 5.8 we have

$$||\rho-\psi||_{T,\gamma}=||F(\rho)-F(\psi)||_{T,\gamma}\leq C||\rho-\psi||_{T,\gamma}$$

for $C<1$, which means that

$$||\rho-\psi||_{T,\gamma}=0$$

and thus ρ is the unique local classical solution. □

6. Numerical simulations

This section is devoted to the numerical simulations of the studied system by finding approximated solution to kinetic equation (5.5). Here, we consider only one-dimensional system, i.e. $d = 1$. Most of the presented results were published in [22] and [29]. Contents of this section are based mainly on the latter article, with some editorial changes, additional examples coming from the other article and finally some unpublished results.

First, we make additional assumptions regarding intensity coalescence c_1 and jump kernel c_2. Namely, let

$$c_1(x, y, z) = b(x - y)\delta((x + y)/2 - z) \tag{6.1}$$

and

$$c_2(x, y) = a(x - y),$$

where a and b are non-negative, even functions and δ stands for the Dirac δ-function. It means that both jumps and coalescence intensities depend on the distance between involved particles and the target point. Moreover, the resulting point of the coalescence of two particles in x and y is at the middle $z = (x + y)/2$ of their locations.

The δ-function was used to simplify the calculations as it allows us to lower dimensionality of the integration. Indeed, integrating over z to eliminate δ-function, the kinetic equation (5.5) transforms to

$$\begin{aligned}\frac{\partial \rho_t(x)}{\partial t} = &-\int_{-\infty}^{\infty} a(x - y)\exp\left(-\int_{-\infty}^{\infty} \varphi(y - u)\rho_t(u)du\right)\rho_t(x)dy\\ &+\int_{-\infty}^{\infty} a(x - y)\exp\left(-\int_{-\infty}^{\infty} \varphi(x - u)\rho_t(u)du\right)\rho_t(y)dy\\ &-2\int_{-\infty}^{\infty} \left(b(x - y)\rho_t(x) - b(2(x - y))\rho_t(2x - y)\right)\rho_t(y)dy,\end{aligned} \tag{6.2}$$

where the assumed properties of the kernels have been taken into account. Imposing an initial condition $\rho_0(x), x \in \mathbb{R}$, (6.2) leads to a complicated partial integro-differential equation. Because of the presence of spatial integrals and nonlinearity, we doubt it can be solved analytically in general. That is why we develop a numerical approach for solving this equation.

6.1. Numerical algorithm. In order to solve numerically the kinetic equation, it is necessary, first of all, to perform its discretization in coordinate space. Let x_i be the grid points uniformly distributed over $\mathbb{R}$ with mesh h. Then the discrete analog of (6.2) is

$$\begin{aligned}\frac{dn_i}{dt} =&h\sum_j\left(a_{i-j}\exp\Big[-h\sum_k \varphi_{i-k}n_k\Big]n_j - a_{i-j}\exp\Big[-h\sum_k \varphi_{j-k}n_k\Big]n_i\right)\\ &-2h\sum_j b_{i-j}n_in_j + 2h\sum_j b_{2i-2j}n_jn_{2i-j},\end{aligned} \tag{6.3}$$

where $n_i = n(x_i, t)$ and $n(x_i, 0) = \rho_0(x_i)$ with $a_{i-j} = a(x_i - x_j) = a((i - j)h)$, $b_{i-j} = b(x_i - x_j) = b((i - j)h)$, $\varphi_{i-k} = \varphi(x_i - x_k) = \varphi((i - k)h)$, and the infinite sums over j and k correspond to the spatial integrals. It is obvious that in the limit $h \to 0$, the discretized kinetic equation (6.3) coincides with its original continuous version (6.2). Replacing $i - j$ by j, taking into account that the summation is carried out over the infinite number of terms, and introducing the auxiliary quantities

$$\lambda_i = \exp\Big[- h \sum_k \varphi_{i-k} n_k \Big], \tag{6.4}$$

one can rewrite (6.3) as

$$\frac{dn_i}{dt} = h \sum_j \Big(a_j (\lambda_i n_{i-j} - \lambda_{i-j} n_i) - 2b_j n_i n_{i-j} + 2b_{2j} n_{i-j} n_{i+j} \Big) \tag{6.5}$$

where $a_j = a(jh)$, $b_j = b(jh)$, and $b_{2j} = b(2jh)$.

In computer simulations we cannot operate with infinite-size samples leading to the infinite summation over j in (6.3), (6.4), and (6.5). Because of this, we consider a finite number N of grid points x_i uniformly distributed over the area $[-L/2, L/2]$ with spacing $h = L/N$, where $i = 1, 2, \ldots, N$. The finite length L should be sufficiently big and the number N of grid points must be large enough to minimize the discretization noise. Then h will be sufficiently small to provide a high accuracy of the spatial integration. The finite-size effects can be reduced by applying the corresponding boundary conditions (BC) when mapping infinite range $(-\infty, \infty)$ by finite area $[-L/2, L/2]$. In view of the aforesaid, (6.5) represents a coupled system of N autonomous equations, where $i = 1, 2, \ldots, N$ and summation over j is performed according to BC.

We consider three types of BC: Dirichlet (DBC), periodic (PBC), and asymptotic (ABC) boundary conditions. The choice depends on initial function $\rho_0(x)$ and expected properties of solution $\rho_t(x)$. For example, if $\rho_0(x)$ takes nonzero values only within a narrow interval $[-l/2, l/2]$ with $l \ll L$, we can apply the DBC by letting $n_j = 0$ for all $|x_j| > L/2$. This means that during the finite simulation time $0 \le t \le T$, the non-zero values of $n_i(t)$ do not approach the boundaries $x_B = \pm(L/2 - \max\sigma)$, where $\max\sigma$ is the maximal radius of the kernels (see Section 6.2). In numerical calculations this can be expressed by the condition $n(x_B, t) < \varepsilon\ \max_x n(x,t)$, where $0 < \varepsilon \ll 1$ is the relative tolerable level (a negligibly small quantity slightly exceeding machine zero). When the propagation front becomes too close to the boundaries, i.e., $n(x_B, t) > \varepsilon\ \max_x n(x,t)$, we should enlarge L (e.g. gradually doubling it) until to satisfy the required first condition, use DBC again, and continue the simulation for $t > T$. We also consider a case (see Section 6.2.3) in which members of infinite configuration are initially absent in one half-space. This requires a modified BC that combines DBC and ABC with an addition of adjustable system-size approach.

If ρ_0 and, thus, $n(\cdot, t)$ are periodic functions, it is necessary to apply PBC. According to PBC, the summation in (6.5) for each $i = 1, 2, \ldots, N$ is performed not only over all $j = 1, 2, \ldots, N$ but also over all infinite number of images j' of j. The images are obtained by repeating the basic interval $[-L/2, L/2]$ by the infinite number of times to the left and to the right of it, so that $x_{j'} = x_j \pm KL$, where $K = 1, 2, \ldots$ and $n_{j'} = n_j$. This reproduces the periodicity $n(x \pm KL, 0) = n(x, 0)$, where $x \in [-L/2, L/2]$. The solution $n(x \pm KL, t) = n(x, t)$ will also be periodic for any time $t > 0$ with the same (finite) period L. In such a way the infinite system can be handled by a finite-size sample. Because the kernel values a_j and b_j decrease to zero with increasing the inter-particle distance, the summation over j in (6.5) can be actually truncated to a finite number of terms. The truncation radiuses $R_{a,b}$ are chosen to satisfy the conditions $a(|x|) \approx 0$ and $b(|x|) \approx 0$ for $|x| > R_a$ and $|x| > R_b$, respectively.

In the spatially homogeneous case when $n(x, t) = n(t)$, we should apply ABC, i.e. $n_{j'} = n(t)$ for all $x_{j'} < -L/2$ and $n_{j'} = n(t)$ for all $x_{j'} > L/2$. For this case, PBC and ABC lead to the same results. The ABC can also be used for spatially inhomogeneous solutions $n(x, t)$ which are flat for $x < -L/2$ and $x > L/2$ at a given

t where they take non-zero constant values. Then $n_{j'} = n(-L/2, t)$ for all $x_{j'} < -L/2$ while $n_{j'} = n(L/2, t)$ for all $x_{j'} > L/2$. If in the course of time the flatness is violated at a current L, the basic length should be enlarged using the automatically adjustable system-size approach mentioned above.

The uniform knot distribution over $[-L/2, L/2]$ can be chosen in the form $x_i = -L/2 + (i - 1/2)h$, where $i = 1, 2, \ldots, N$ with even N. This provides the symmetricity of knot positions with respect to $x = 0$. Then according to PBC the calculations of $n_{i\pm j}$ should be performed as

$$n_{i-j} = \begin{cases} n_{i-j+N}, & i-j<1 \\ n_{i-j}, & 1 \le i-j \le N \end{cases}, \quad n_{i+j} = \begin{cases} n_{i+j-N} & i+j>N \\ n_{i+j} & 1 \le i-j \le N \end{cases}. \tag{6.6}$$

The application of DBC result in

$$n_{i-j} = \begin{cases} 0, & i-j<1 \\ n_{i-j}, & 1 \le i-j \le N \end{cases}, \quad n_{i+j} = \begin{cases} 0 & i+j>N \\ n_{i+j} & 1 \le i-j \le N \end{cases}. \tag{6.7}$$

and and ABC in

$$n_{i-j} = \begin{cases} n_1, & i-j<1 \\ n_{i-j}, & 1 \le i-j \le N \end{cases}, \quad n_{i+j} = \begin{cases} n_N & i+j>N \\ n_{i+j} & 1 \le i-j \le N \end{cases}. \tag{6.8}$$

respectively.

In order to solve the problem (6.5), we use Runge-Kutta scheme of the fourth order (RK4) to calculate the values of $n_i(x, t)$ for increasing t.

6.2. Results of the performed simulations. In order to apply the algorithm described in the preceeding section and numerically solve equation (5.5), we need to precise initial population density and exact form of parameter functions involved. The jump $a(x)$, coalescence $b(x)$, and repulsion $\varphi(x)$ kernels were modelled by Gaussian

$$G_{\mu,\sigma}(x) = \frac{\mu}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{6.9}$$

or rectangle

$$C_{\mu,\sigma}(x) = \begin{cases} \frac{\mu}{2\sigma}, & |x| \le \sigma \\ 0, & |x| > \sigma \end{cases} \tag{6.10}$$

functions, where $\mu = \mu_a$, μ_b, μ_φ are the intensities and $\sigma = \sigma_a$, σ_b, σ_φ the ranges of the corresponding interactions. A symmetrical pair of shifted Gaussian or rectangle functions was involved as well,

$$F_{\mu,\sigma,s}(x) = \frac{1}{2}\Big(F_{\mu,\sigma}(x-s) + F_{\mu,\sigma}(x+s)\Big), \tag{6.11}$$

where F stands for G or C and s is the shifting interval.

In order to consistently analyze the influence of parameter functions on the dynamics, the following four situations were considered: (i) pure free jumps; (ii) repulsive jumps; (iii) pure coalescence; and (iv) repulsive jumps with coalescence. The initial conditions were chosen in the form of Gaussians (6.9), rectangles (6.10), trigonometric or step functions.

6.2.1. *Rectangle initial density.* The first example relates to the initial condition in the form of periodic rectangle function. The infinite system is reproduced by repeating the single rectangle segment $C_{1,1}(x)$ at $x \in [-L/2, L/2]$ with a period of L and applying PBC (6.6). The jump $a(x) = G_{\mu_a,\sigma_a}(x)$, repulsion $\varphi(x) = G_{\mu_\varphi,\sigma_\varphi}(x)$, and coalescence $b(x) = G_{\mu_b,\sigma_b}(x)$ kernels are modelled by the Gaussians.

First, let us consider the case of free jumps (i.e. $b \equiv 0$, $\varphi \equiv 0$) with jump kernel $a = G_{1,1}$. The discontinuity of initial density quickly dissipates, transforming the initial density into a function resembling periodic Gaussian. With increasing time, the

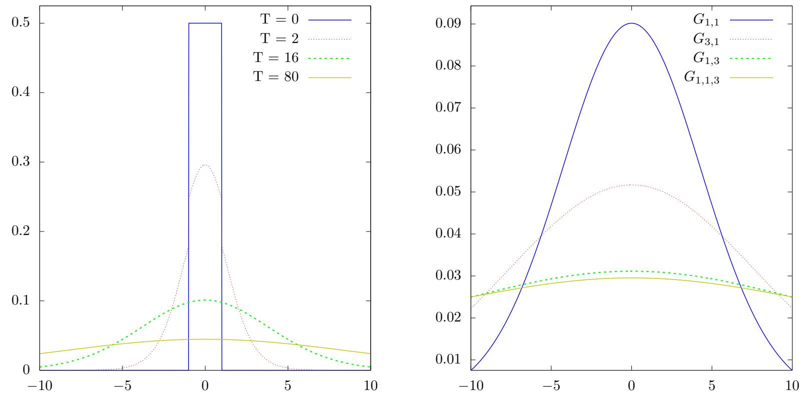

FIGURE 1. Free jumps without coalescence. Approximated density ρ_T on $[-10, 10]$ for periodic $C_{1,1}$-type initial condition with period $L = 40$. On the left: evolution in time with $G_{1,1}$ jump kernel. On the right: comparison at $T = 20$ with different choices of jump kernel. [22]

system tends to homogeneity, see left plot in Figure 1. Increase of the strength (μ_a) or range (σ_a) parameter, as well as the shift of the jump kernel result in acceleration of this process, see right plot in Figure 1.

Next, we consider addition of repulsion and/or coalescence. We consider corresponding intensities to be either zero or $\mu_a = 1$, $\mu_\varphi = 20$, and $\mu_b = 1$. Three sets of kernel ranges were taken into account, $\sigma_a = \sigma_\varphi = \sigma_b = 1$, $\sigma_a < \sigma_\varphi < \sigma_b$ with $\sigma_a = 0.5$, $\sigma_\varphi = 1$, $\sigma_b = 2$ and vice versa, $\sigma_a > \sigma_\varphi > \sigma_b$ with $\sigma_a = 2$, $\sigma_\varphi = 1$, $\sigma_b = 0.5$. The corresponding time evolution of spatial structure $n(x, t)$ is presented in Fig. 2 for the cases of pure free jumps, repulsive jumps, pure coalescence, and repulsive jumps with coalescence with $\sigma_a = \sigma_\varphi = \sigma_b = 1$, parts (a), (b), (c), and (d), respectively, as well, with $\sigma_a < \sigma_\varphi < \sigma_b$ and $\sigma_a > \sigma_\varphi > \sigma_b$ for coalescing repulsive jumps, parts (e) and (f).

In Fig. 2(a) we see the dynamics similar to one observed previously in Figure 1. In the limit $t \to \infty$ we expect absolutely flat density $\lim_{t\to\infty} \rho_t(x) = \mu_a/L$. In (b) we see that addition of relatively strong repulsion to jumps results in the slowing down of the process of homogenization. Additionally, we observe the emergence of two additional local maxima in the ranges $x \approx \pm 2$ due to the repulsion between particles. We believe that all the maxima disappear at $t \to \infty$ with the same asymptotic behaviour $\lim_{t\to\infty} \rho_t(x) = 1/L$ as for the free jumps. In contrast, for free coalescence, see part (c), we expect decay of $\rho_t(x)$ to zero at $t \to \infty$. Moreover, here the particles remain to be located exclusively within the initial intervals $[-1, 1] \pm kL, k \in \mathbb{Z}$ and they are absent outside of it at any t.

When the repulsive jumps are carried out in the presence of coalescence at equal interaction ranges $\sigma_a = \sigma_\varphi = \sigma_b = 1$, see part (d) of Figure 2, the pattern is somewhat similar to that of part (b). However, the three-maximum structure dissipates now much faster. For short-ranged jumps, where $\sigma_a = 0.5$, $\sigma_\varphi = 1, \sigma_b = 2$, the central peaks at $x = 0$ become sharper, while the secondary side maxima at $x \approx \pm 2$ do not seem to appear, see part (e) and compare it with (d). In the case of short-ranged coalescence when $\sigma_a = 2, \sigma_\varphi = 1, \sigma_b = 0.5$, the central peaks transform into a more complicated structure with one central minimum at $x = 0$ and two side maxima at $x \approx \pm 0.5$, see part (f). The secondary maxima at $x \approx \pm 2$ become more visible with respect to those for equal-range interactions, cf. part (d). Thus, the influence of jumps on the dynamics increases not only with increasing their intensity but range as well.

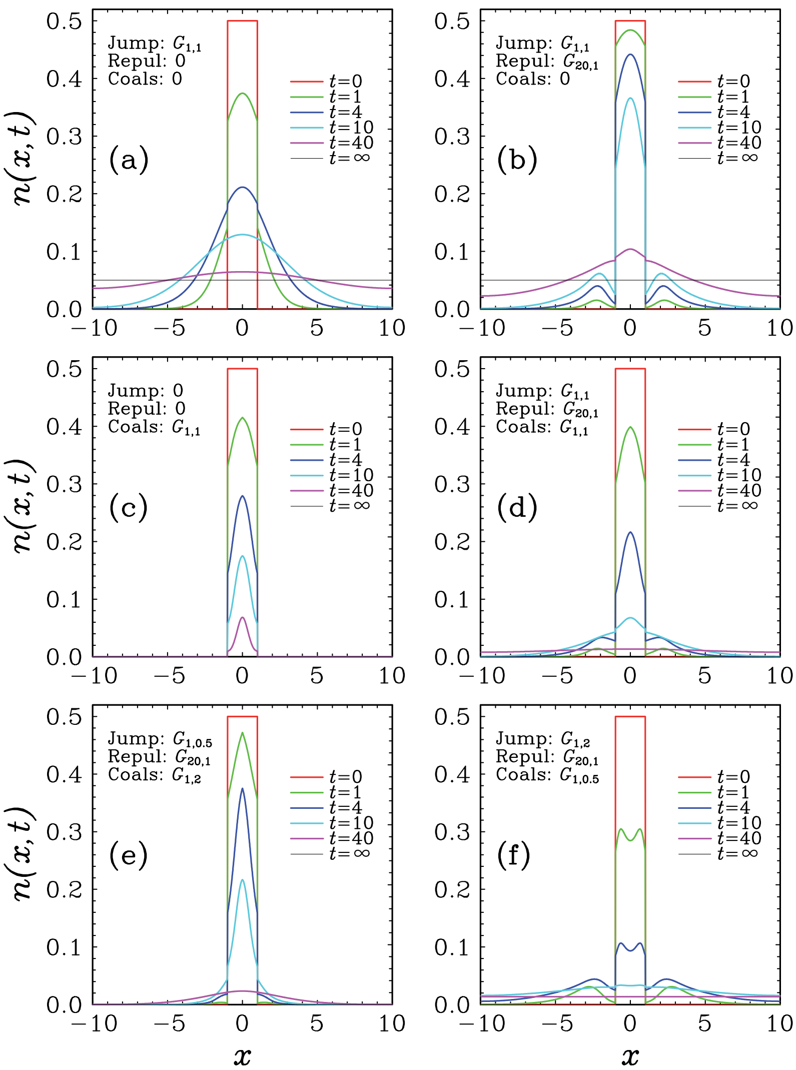

FIGURE 2. Time evolution of population density at several moments of time t for periodic $C_{1,1}$-type initial condition (with period $L = 20$) in the cases: (a) pure free jumps; (b) repulsive jumps; and (c) pure coalescence; as well as repulsive jumps with coalescence for (d) equal interaction ranges; (e) short-ranged jumps; and (f) short-ranged coalescence. [29]

The same concerns the coalescence. Note also that the density profiles in Figures 2 (a)–(f) are symmetric, i.e., $\rho_t(-x) = \rho_t(x)$, like the initial condition, $\rho_0(-x) = \rho_0(x)$. This follows from the symmetry of the kinetic equation.

As another example, consider asymmetric initial condition in the form of $\mathcal{N}_0$ shifted single rectangle functions $C_{v_k,\sigma_k}(x + s_k)$ with intensities v_k and ranges σ_k, namely,

$$\rho_0(x) = \sum_{k=1}^{\mathcal{N}_0} C_{v_k,\sigma_k}(x + s_k) \tag{6.12}$$

where $s_k = -L/2 + (k - 1/2)L/\mathcal{N}_0$ are shifting parameters. Repeating (6.12) with period L, we should apply PBC to deal with the infinite system. We used a particular case of (6.12) with $\mathcal{N}_0 = 3$ and $L = 20$ as well as three different amplitudes $v_{1,2,3}$

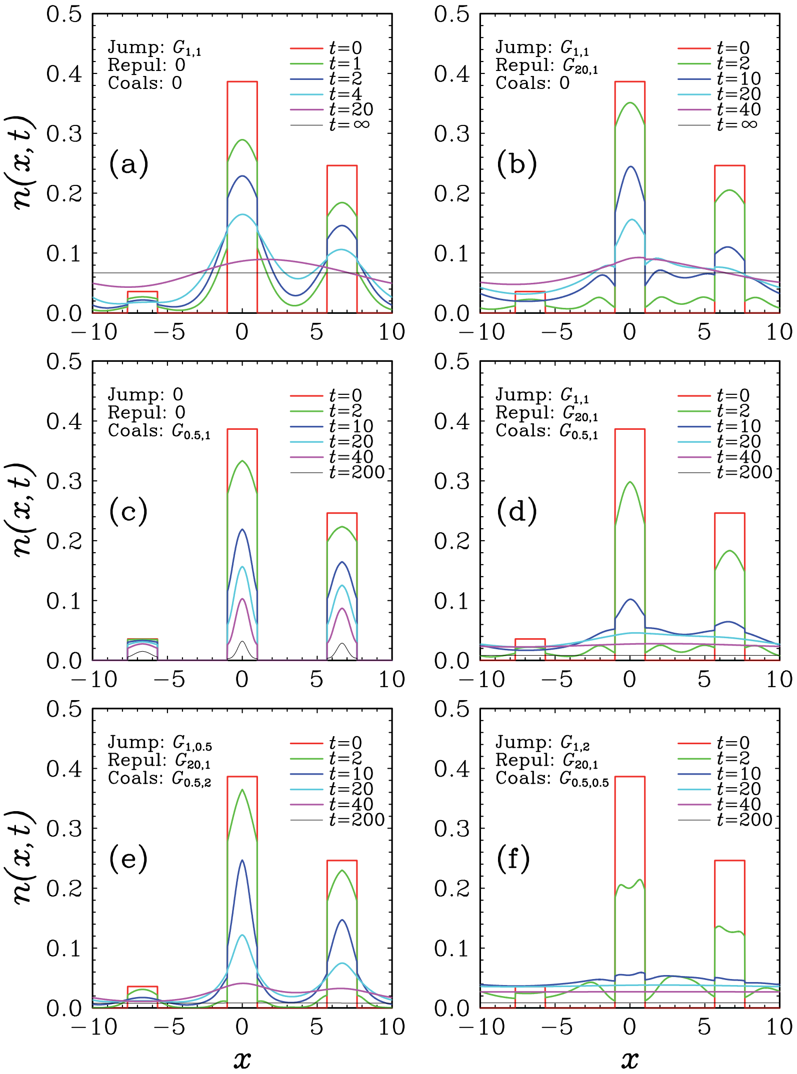

FIGURE 3. Time evolution of density for asymmetric initial condition in the form of three rectangle functions with different amplitudes. [29]

generated at random in the interval $(0, 1)$. The corresponding result is shown in Figure 3.

Looking at Figure 3 and comparing it with Figure 2 we see that the evolution of density for short time interval can be approximately presented as a sum of independent separate solutions obtained for single rectangle initial densities $C_{v_k,\sigma_k}(x + s_k)$. With time increment, it ceases to be the case. Again, in the absence of coalescence, see parts (a) and (b), the density seems to tend with $t \to \infty$ to the nonzero constant $(v_1 + v_2 + v_3)/L$. For $\mu_b > 0$, the density seems to take its zeroth asymptotics at $t \to \infty$, see parts (c)–(f), except special cases, see below. For populations with periodic initial densities $\rho_0(x \pm kL) = \rho_0(x)$ with $x \in [-L/2, L/2]$ and $k \in \mathbb{Z}$, the solution ρ_t will also be periodic for any time $t > 0$ with the same period L. We can observe this in particular for $-L/2$ and $L/2$. Investigations show that the increase of the strength μ and range σ of the jump and coalescence kernels accelerates the speed of density evolution.

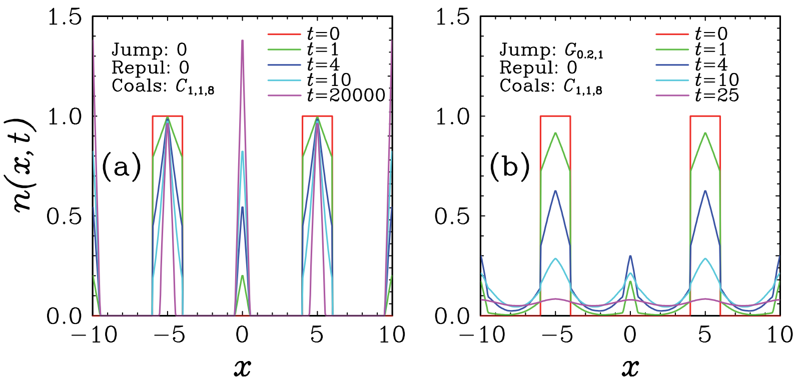

FIGURE 4. Evolution of initial density in the form of two rectangle functions $C_{1,1,5,2}$ in the cases: (a) pure coalescence and (b) coalescence with free jumps. The interactions are modelled by shifted pair rectangle coalescence $C_{1,1,8}$ and Gaussian jump $G_{0.2,1}$ kernels. [29]

As visualized in Figures 2 and 3, the presence of coalescence seems to lead to the zeroth asymptotic provided the kernels are single rectangle functions with positive values around zero (the same concerns simple Gaussians). However, the coalescence kernel can be chosen in the form $b = C_{\mu,\sigma,s}$ of a pair of shifted rectangle functions, recall (6.11), with appropriate shifting parameter s to avoid the zeroth density limit. The initial inhomogeneous density should also be chosen correspondingly. For instance, we can consider ρ_0 to be $C_{1,1,5,2}$-type periodic function with period $L = 20$, recall (6.12). In the absence of jumps and with coalescence intensity $b = C_{1,1,8}$ the evolution of density is depicted in Figure 4(a). We see that the initial rectangles soon transform into triangle-shaped peaks centered at $x = \pm(5 + kL)$. Additionally, new peaks appear exactly in the middle of them at $x = 0 \pm kL$. At long times the modification of the density profile slows down to a level suggesting that the system approaches a non-trivial stationary state.

Allowing particles to jump changes the situation radically, as is demonstrated in Figure 4(b) for Gaussian jump kernel $G_{0.2,1}(x)$. Even for relatively small jump intensity $\mu_a = 0.2$ and range $\sigma_a = 1$, the density quickly decreases to zero everywhere, after initial period of time, when new peaks are formed. It is interesting to remark that the monotonic decrease of main maxima in $x \pm 5$ is accompanied by nonmonotonic change of the magnitude of the newly formed peaks in $x = 0$ (and $\pm kL$). This magnitude first increases (cf. densities at $t = 1$ and $t = 4$) and then decreases.

6.2.2. *Trigonometric initial density.* Next example is the initial density in the form of a trigonometric function

$$T_{n_0,\mu_0,k}(x) = n_0\Big(1 + \mu_0 \cos(2\pi k x/L)\Big),$$

where $0 < \mu_0 \leq 1$ is the coefficient of the modulation and $k \geq 1$ defines the period L/k. Then PBC should be used to reproduce the infinite system. The obtained solutions for $\rho_0(x) = T_{1,1,3}(x)$ with $n_0 = 1$, $\mu_0 = 1$, $k = 3$, and $L = 20$ are plotted in Figure 5 when jump, repulsion, and coalescence kernels are Gaussians $a(x) = G_{1,1}(x)$, $\varphi(x) = G_{8,1}(x)$, and $b(x) = G_{0.5,1}(x)$, respectively.

From Figure 5(a) we see that pure free jumps do not change the form of density profile which remains to be of the trigonometric shape with the same periodicity. In particular, the density continues to oscillate around the same level $n_0(1+\mu_0)/2 = 1$ for any t. However, the amplitude of these oscillations decreases with increasing t, leading

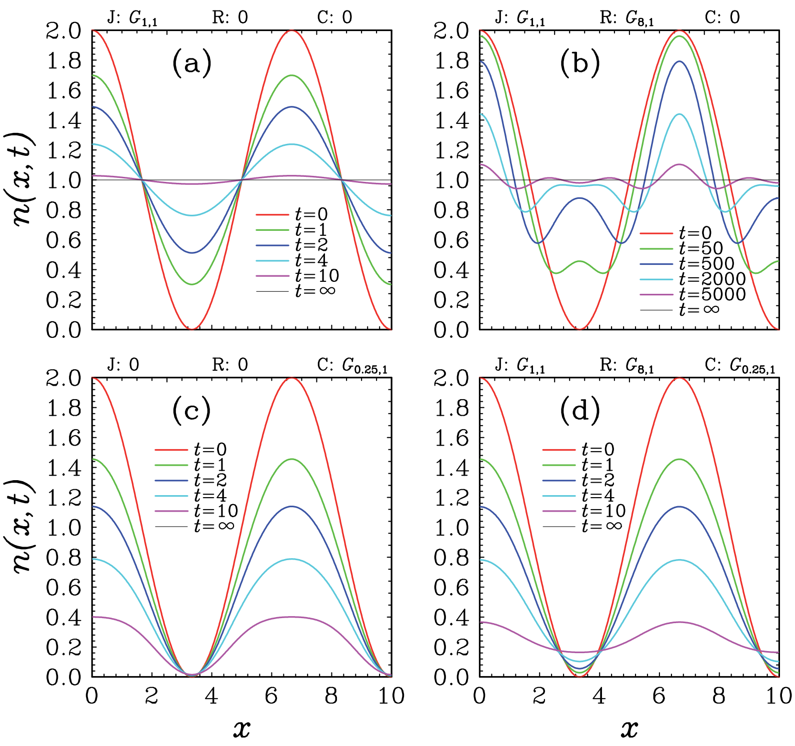

FIGURE 5. Evolution of density with trigonometric initial condition $T_{1,1,3}(x)$. The jump (J), repulsion (R), and coalescence (C) kernels are Gaussians $G_{1,1}(x)$, $G_{8,1}(x)$, and $G_{0.25,1}(x)$, as indicated above plots. [29]

to homogeneity in long term. The addition of repulsion alters the simple trigonometric form. Additional local maxima and minima emerge, see Figure 5(b). Moreover, the homogeneity is being achieved here much slower than in the case of free jumps (compare density at $t = 5000$ in Figure 5(b) versus for $t = 10$ in part (a)). This is caused by the strong intensity of repulsion potential $\mu_\varphi = 8$.

For pure coalescence in Figure 5(c), the density decreases to zero at $t \to \infty$. The inclusion of jumps, see Figure 5(d), changes the behaviour of solution. It approaches spatial homogeneity faster, as near minima density initially grows, contrary to the case of pure coalescence, where these minima remain to be zero for all t.

An interesting case is a system with strongly repulsive jumps where kernels are shifted (recall (6.11)). When repulsion is strong enough, with appropriate shift it seems to lead to appearance of non-trivial stationary states. It is presented in Figure 6, where jump kernel was taken as $G_{1,1,2}$ and repulsion intensity in the form of $G_{8,1,2.5}$. When the shift of kernels get reduced, the repulsion is no longer able to counteract the process of homogenization. It makes it considerably slower though. In the Figure 7 similar forms of kernels as previously were utilized, but with reduced shift parameters. For the jump kernel this parameter was changed from 2 to 0 and for repulsion intensity from 2.5 to 1.5. When comparing Figure 6 with Figure 7, one can see that the alteration of shift parameters changes drastically the behaviour of the system. In the first case, where the range of repulsion is longer, initial peaks becomes higher and thinner and the regions of small density between them get wider. When the repulsion range is shorter, the amplitude of the initial trigonometric function gets smaller with time and the system clearly tends to the spatial homogeneity.

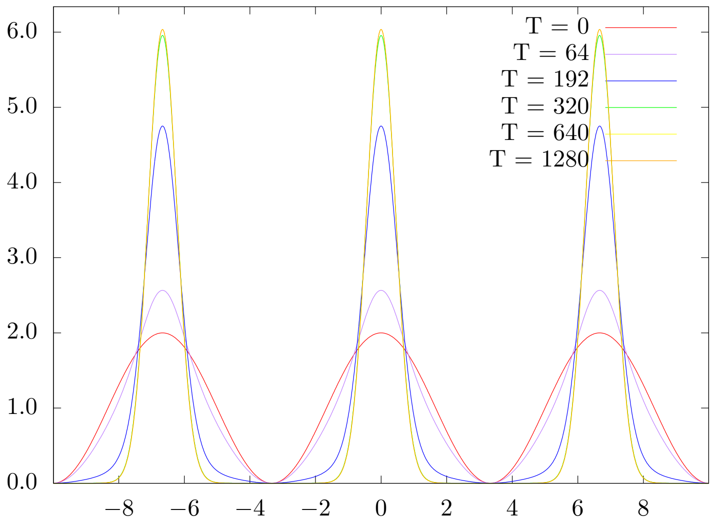

FIGURE 6. Density dynamics with initial trigonometric condition $T_{1,1,3}(x)$ in the presence of repulsive jumps with strong shifted repulsion and no coalescence. Jump and repulsion intensities in the form of shifted double Gaussians $G_{1,1,2}$ and $G_{8,1,2.5}$ respectively.

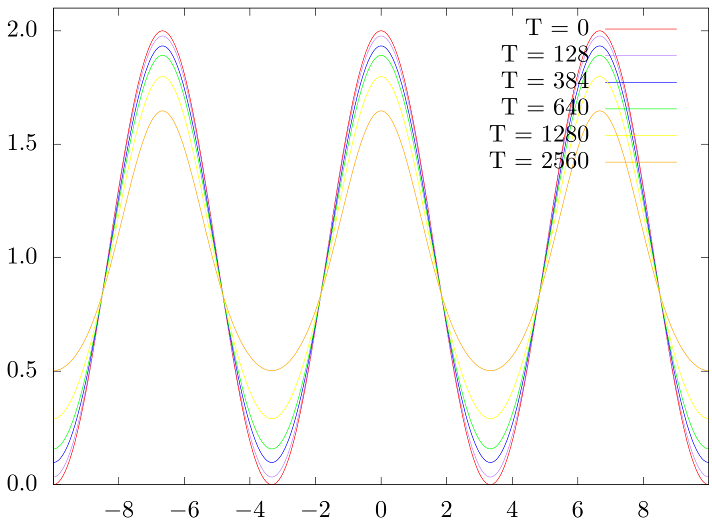

FIGURE 7. Density dynamics with initial trigonometric condition $T_{1,1,3}(x)$ in the presence of repulsive jumps with shifted repulsion and no coalescence. Jump and repulsion intensities in the form of double Gaussians $G_{1,1}$ and $G_{8,1,1.5}$ respectively.

6.2.3. *Step initial density.* Another interesting case to study is the initial condition in the form of the Heaviside step function

$$H_{n_0}(x) = \begin{cases} n_0, & x \le 0 \\ 0, & x > 0 \end{cases} .$$

Here, the density is not periodic, so a different approach should be used. The size of the initial interval $[-L/2, L/2]$ on which the simulations are performed is gradually increased with increasing t, according to the automatically adjusted system-size approach. Additionally, a modified BC should be applied by combination of DBC and ABC. The DBC (recall (6.7)) is used from the right, where $\lim_{x\to\infty} \rho_t(x) = 0$ for all t. From the left, we must employ ABC (recall (6.8)). When nonzero values approach the right boundary, the system size L is enlarged and the simulations are continued. From the left, we measure the difference between the actual values of $n(x,t)$ near boundary and their homogeneous counterpart $n^h_{RK}(t)$. When this difference exceeds the predefined level $\varepsilon \max_x n(x,t)$, the system is enlarged. $n^h_{RK}(t)$ is obtained by solving numerically the kinetic equation for the spatially homogeneous initial condition $\rho_0 \equiv n_0$ in parallel to our spatially inhomogeneous case.

Time evolution of density for step initial condition $\rho_0(x) = H_1(x)$ is presented in Figure 8. Gaussian jump G_{μ_a,σ_a}, repulsion $G_{\mu_\varphi,\sigma_\varphi}$, and coalescence G_{μ_b,σ_b} kernels were employed. Respective intensities were set to $\mu_a = 1$, $\mu_\varphi = 8$, and $\mu_b = 0.1$. All range parameters were set to equal value $\sigma = 1$ in parts (a)–(d). In part (e) coalescence range was increased and jump range descreased, namely $\sigma_a = 0.5$, $\sigma_\varphi = 1$, and $\sigma_b = 2$ and in part (f) vice-versa with $\sigma_a = 2$, $\sigma_\varphi = 1$, and $\sigma_b = 0.5$.

As can be seen in Figure 8(a) for pure free jumps, the discontinuous step function $\rho_0(x) = H_1(x)$ with the flow of time transforms into a continuous S-shaped curve. The density for negative x decrease and for positive increase. After obtaining continuous shape, it remains unchanged in $x = 0$ and symmetric with respect to point $(0, 0.5)$, the value 0.5 being the arithmetic mean of two initial values – 1 to the left and 0 to the right. The slope of these curves becomes smaller with increasing time and seem to tend to the mid-value everywhere.

In part (b) of Figure 8, the repulsion effect is added. Here, density at $x = 0$ also remains constant after initial discontinuity disappears, but it is lower than in the case of free jumps. Additionally, the shape is more complicated, including the appearance of local maximum for positive x, which becomes more and more flat with the flow of time and eventually vanishes. In the long term, the density seems to approach homogeneity as in (a), but this process is slower.

For pure coalescence we can observe in Figure 8(c) that the initial step function H_1 changes to a more complicated shape with a small peak to the left of the initial discontinuity at $x = 0$. The density for positive x remains unchanged at zero level and for negative arguments it tends to zero as well. Moreover, the initial discontinuity does not vanish even for relatively long times.

The inclusion of repulsive jumps, see Figure 8(d–f), reduces the peak appearing to the left from $x = 0$, making the density profile more homogeneous around initial discontinuity, when compared to (c). When jumps are stronger (longer range) compared to coalescence, this effect is more visible (see Figure 8(f) and (d) vs (e)). Additionally, in contrast to the case presented in (c), the density obtains positive values also for $x > 0$ due to the jumps. Repulsion causes another peak to the right of the initial discontinuity point to appear, similarly as in case (b). Stronger jumps make it more visible, compare Figure 8(f) to (d) and (e), where this peak is barely visible. Ultimately, the system approaches zero everywhere like in (c).

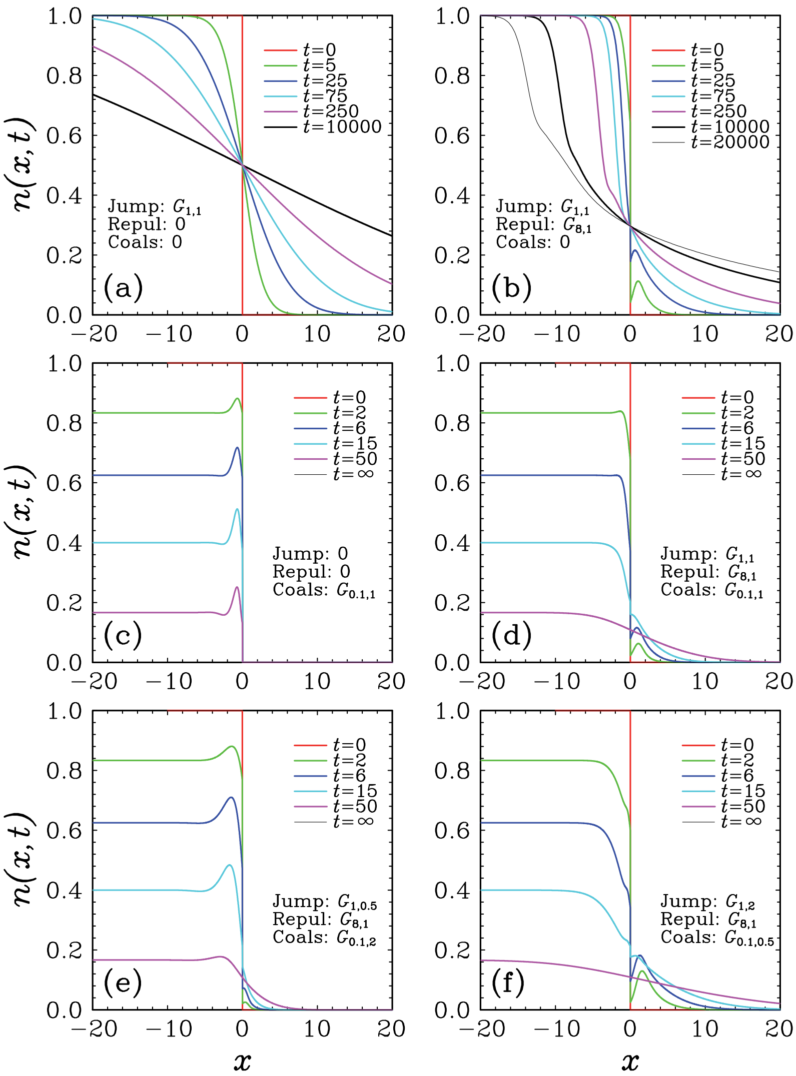

FIGURE 8. Time evolution of density for initial condition H_1. The jump, repulsion, and coalescence kernels are Gaussians with different intensities and ranges (see the legends inside). Initially, the system is considered on the finite interval $[-10, 10]$ and further its size gradually increases according to the automatically adjusted approach. [29]

Note that for the inverse initial step function $\rho_0(x) = H_{n_0}(-x)$, the results $n(x,t)$ presented in Fig. 8 should be inversed to obtain solutions without resolving the kinetic equation, i.e. $n(x,t) = n^*(-x,t)$, where n^* is obtained approximated solution in the case of $\rho_0 = H_1$. This statement is quite general and remains in force not only for step functions, but for any other asymmetric initial condition. This follows from the structure of kinetic equation (6.2) and the symmetry of kernel functions.

A very interesting case is the dynamics of the initial step distribution H_1 in the presence of strongly repulsive jumps, when shifted Gaussian kernels are employed. In Figure 9(a) the results of simulations are presented with shifting parameter $s = 2$ for jump kernel in the form of $G_{1,1,2}$ and $s = 4$ for repulsion intensity $G_{10,1,4}$. It leads to the emergence of self-propagating spatial inhomogeneity in the form of thin peaks. They propagate to the left with increasing amplitude and to the right in the form

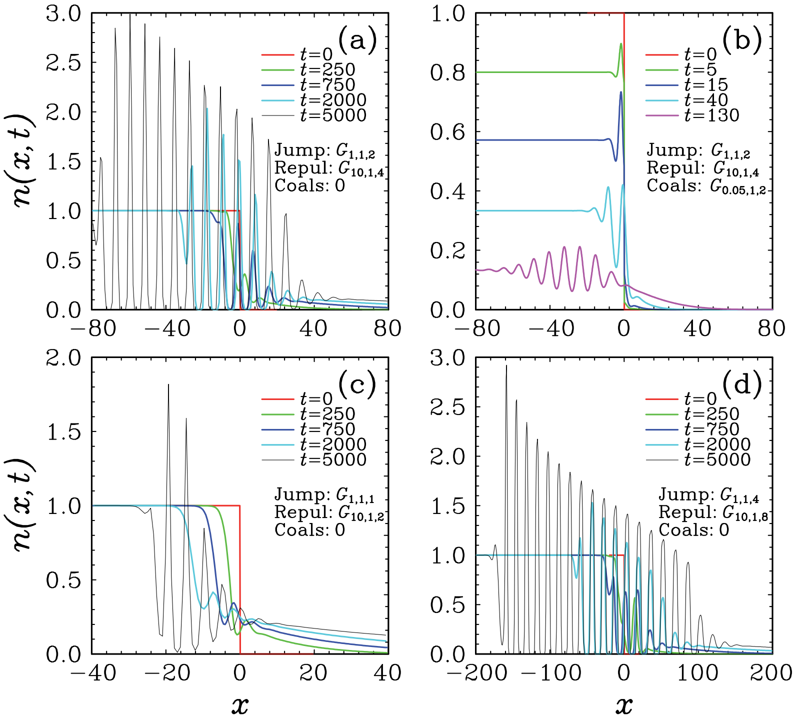

FIGURE 9. Dynamics of the system starting from unit step function H_1 in the cases of pure repulsive jumps (a,c,d) and repulsive jumps with presence of coalescence (b). The interactions are described by the shifted Gaussian jump $G_{1,1,s}$, repulsion $G_{10,1,s'}$ and coalescence $G_{0.05,1,2}$ kernels with different kernel shifts: (a) $s = 2$, $s' = 4$; (b) $s = 2$, $s' = 4$ (c) $s = 1$, $s' = 2$; and (d) $s = 4$, $s' = 8$. Initially the system is considered on the interval $[-20, 20]$ and further its size gradually increases according to the automatically adjusted system-size approach. [29]

of damped oscillation. The inclusion of coalescence even with a slight intensity of $\mu_b = 0.05$ drastically changes the situation, see Fig. 9(b). Here, the oscillations are much weaker, almost imperceptible for positive x and visibly reduced for negative x. The spatial inhomogeneity persists until the density of the system gets reduced to a very low level.

This case is presented in more details in Figure 10. Notice how the density drawn for small times (T=0, T = 8) is truncated at $x = -20$ and $x = -40$ respectively. It shows how the system size was gradually being enlarged. It should be understood that the density for smaller x is equal to its value on the truncation. The oscillations are getting visibly damped for longer times (compare $T = 192$ vs $T = 256$ and $T = 320$).

When shifting parameters are decreased, the process of inhomogeneity emergence is much slower with amplitudes of peaks and distance between them reduced, see Figure 9(c). In contrast, in the presence of increased shift parameters, the density propagation is accelerated, with high amplitude of the oscillations and the distance between peaks, see Figure 9(d).

6.2.4. *Preservation of mass.* In this section we consider another form of coalescence intensity c_1. Recall that previously it was defined as (6.1), which guaranteed that the result of two merging particles lies exactly in the middle between them. Here, we take

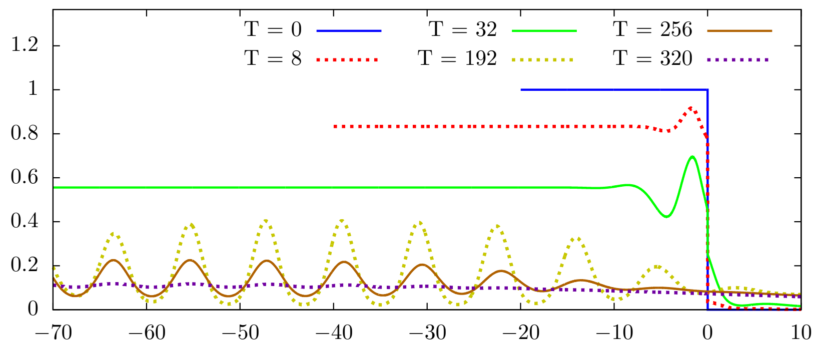

FIGURE 10. Jumps with shifted repulsion kernel in the presence of coalescence. Density on [-70, 10] for H_1 initial condition. Jump kernel $G_{1,1,2}$ with repulsion potential $G_{10;1;4}$ and $G_{0.05,1,2}$ coalescence kernel. [22]

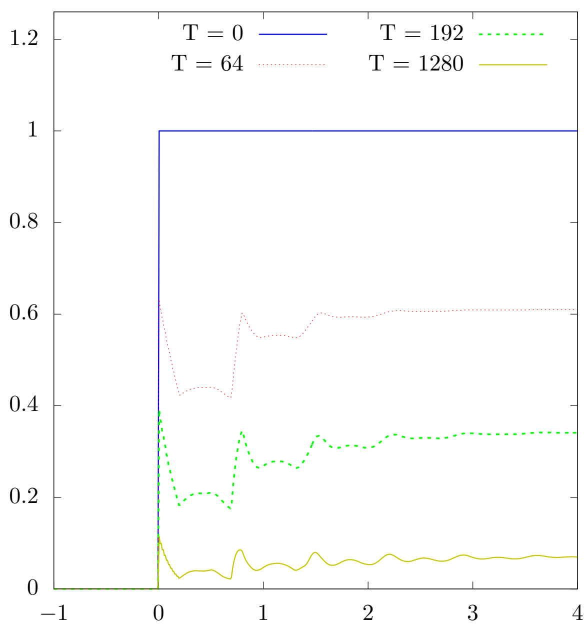

FIGURE 11. System with initial density in the form of inversed H_1. Coalescence in the mass-preserving form with kernel $G_{0.02,0.2}$ with no jumps. Emergence of propagating inhomogeneity. [22]

instead

$$c_1(x,y,z) = b(x-y)\delta(\ln(e^x + e^y) - z). \tag{6.13}$$

This modification can be interesting from the point of view of applications. It can correspond to a system in which the coordinates of points are not related to their spatial location, but with the logarithm of mass of the corresponding entities. It means that element x has mass of e^x. Then, the proposed form of the intensity ensures that in the process of coalescence the mass of the two merging entities is preserved.

In Figure 11 an example of behaviour of the system with coalescence in the form (6.13) is presented. One can see that even in the absence of jumps an interesting inhomogeneous spatial structure can emerge, which propagates to the right. Due to the coalescence the density seems to tend to zero for every x, which may indicate dishonesty of the system.

Addition of jumps prevents persistence of the observed irregular structure and allows density for negative arguments to grow. Jumps can be interpreted here as random fluctuation of mass of the described entities.

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