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Dissertation

Fragmentation processes in continuum with applications

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Contents

Introduction

Large systems of individuals interacting with each other and with the environment are being studied in life sciences such as biology, chemistry, physics, ecology, genetics, oceanology, as well as in economics, social sciences, etc. There exists a great amount of works dedicated to the mathematical theory of such objects characterized by the level of sophistication that varies from simple heuristic modeling to advanced approaches based on the use of deep real-world models supported by appropriate numerical methods and computer simulation techniques [9, 10]. Among the most important issues studied in these works are various aspects of the time evolution of such systems. The aim of the present thesis is to contribute to the development of the dynamical theory of infinite particle systems undergoing fragmentation. First theoretical works on coagulation-fragmentation processes can be traced back to works by Marian Smoluchowski appeared at the beginning of the XX century. Such processes are considered, see [6], as basic acts of nature, which points to the great importance of their mathematical theory. Fragmentation can also be interpreted as branching, cf. [12], the first works on which go back to the Galton-Watson theory of the extinction of families [27]. Their modern development is mostly conducted in the framework of the theory of stochastic processes [21, Chapt. 4], among which one can distinguish works [13, 14] on branching in particle systems.

Studying branching in particle systems is usually restricted to those dwelling in a compact habitat [21, Chapt. 4], or to finite particle systems [13, 14]. As mentioned above, this work is dedicated to studying infinite particle systems, which seems to be the first instance of the theory of this kind. Namely, the following two models are introduced and studied. In the first model, [38, 39], an infinite population of point entities is placed in $\mathbb{R}^d$. Each entity undergoes binary fission with disappearance afterward. It is also subject to a random death caused by crowding – local competition. By this, the particles interact with each other. The pure states of the system are locally finite simple configurations $\gamma \subset \mathbb{R}^d$ and the general states are probability measures on the space Γ of all such configurations. This model may be considered as a branching version of the Bolker-Pacala model [16, 32, 34]. In the proposed model, an entity, with trait $x \in \mathbb{R}^d$, undergoes the following: (a) independent fission with rate $b(x|y_1, y_2)$ in the course of which the particle gives birth to two new particles with traits $y_1, y_2 \in \mathbb{R}^d$ and disappears afterwards; (b) state-dependent death (disappearance) with rate $m(x) + \sum_{y\in\gamma\setminus x} a(x-y)$. When dealing with infinite configurations, one usually imposes a priori restrictions on the properties of probability measures modeling states of the system. The main idea of this is to pass to considering dynamics on spaces of finite configurations by employing so called correlation measures and functions [43]. In this work, this is done by introducing sub-Poissonian measures $\mathcal{P}_{\exp}(\Gamma)$, see Definition 1.2.3 below. Then the ultimate goal is to construct the evolution $t \mapsto \mu_t \in \mathcal{P}_{\exp}(\Gamma)$ of this model. The first step in this direction is performed by considering its finite version, cf. [37]. Here I apply the Thieme-Voigt perturbation technique [50] adapted to our purposes, see Section 1.5, to obtain the evolution of states in the Banach space of signed measures with finite variation. Thereafter, this construction is used to achieve the main result mentioned above. This is done in Theorem 2.3.5 and Corollary 2.3.6. A characteristic feature of this result is the use of the evolution equations for correlation functions in the corresponding L^∞-type Banach spaces. In view of this, the standard semigroup methods cannot be directly applied here. To deal with this, I construct a certain (sun-dual) C_0-semigroup in appropriated Banach space, which I use to obtain a family of linear bounded operators acting from smaller to bigger spaces, see Lemma 4.1.3, which gives a classical solution of the mentioned equation. I demonstrate that the local competition – interaction explicitly taken into account – can produce a global regulating effect, i.e. by Theorem 2.3.5 the evolution of measure is obtained by identifying the measure μ_t with the solution of the evolution correlation functions equation. Moreover, I prove that μ_t is the sub-Poissonian state for all $t > 0$ (continuation), which means that the evolution of measures preserves the sub-Poissonicity of the states and hence the self-regulation takes place.

The second model presented in this thesis was introduced and studied in [40]. Its preliminary version was introduced in [49]. In contrast to the first model, here the particles do not interact with each other, however, the basic space where they are placed in is a general locally compact Polish space. Each particle produces at random a finite cloud of new particles, and disappears afterwards. An infinite system of such point particles – an infinite 'cloud' – is placed in a locally compact Polish space X in such a way that each compact $\Lambda \subset X$ contains only finitely many elements of the cloud, but multiple locations of particles are possible. Here I also employ probability measures as states of the system, the evolution of which is described by the corresponding Fokker-Planck equation directly, i.e., without calling correlation functions. The branching mechanism here is presented by a branching (probability) kernel $b_x(d\xi)$, which describes the distribution of offsprings (constituting cloud ξ) of the particle located at x. Models of this type (and even much more complex) are well-known [13, 14], but in the finite-system version. To deal with infinite configurations, similarly as in the first model I impose a restriction on the support of the considered states $\mu \in \mathcal{P}(\Gamma)$ by imposing a condition on the branching kernel. This allows for passing to tempered configurations, the set of which, Γ^{ψ}, equipped with a certain topology becomes a Polish space (Proposition 1.1.4). Then I define the Kolmogorov operator L as a closed linear operator with domain $\mathcal{D}(L)$ in an appropriate space of continuous functions. First important result here is Theorem 3.3.1 which gives the unique classical solvability of the Kolmogorov equation. I obtain the solution by constructing a C_0-semigroup generated by $(L, \mathcal{D}(L))$. The key step of this is solving a nonlinear evolution equation in the space of bounded continuous function on X defined by the branching kernel, see Lemma 3.2.2. Thereby, in Theorem 3.3.3, I prove the existence and uniqueness of the solution of the Fokker - Planck equation.

Statements similar to Theorems 2.3.5, 3.3.1 and 3.3.3 give a microscopic description of the dynamics of the system where we take into account the traits of each individual particle. To establish the connection of such the description with phenomenological theories, i.e. the meso- and macroscopic description [5,47,48], we use scaling techniques, which is also a subject of the dissertation. I make a comparison of such descriptions for fission-death model. I rescaled the interaction between entities, cf. [26], and obtain the kinetic equation correspond to fission-death model, and also to the Bolker-Pacala-type model, which means that only microscopic level give us a fully precise description of the considered system.

The work has the following structure. In Chapter 1 I introduce phase space, spaces of functions and measures, and provide information and the analysis in such spaces. I deal with function acting on such spaces and present the corresponding evolution equations, like Kolmogorov equation and Fokker-Planck equation. At the end of the chapter, I add elements of semigroup theory. In Chapters 2 and 3, I introduce the fission - death model and the free branching model that are the subject of the dissertation and present main results connected to these models, including consideration about both finite and infinite systems. The final Chapter contains the proofs of the most important results of the thesis.

Chapter 1

Preliminaries

In this chapter, I present the main technical aspects of the work and comments, in particular, regarding the spaces in which I consider models of fragmentation, measures and functions in these spaces and the corresponding evolution equations. A detailed description of these aspects can be found in [2, 35, 43] and the literature quoted in these articles.

1.1 Configuration spaces

Simple configurations in $\mathbb{R}^d$

The Euclidean space $\mathbb{R}^d$, $d \geq 1$, is supposed to be equipped with the usual norm topology. By $\mathcal{B}(\mathbb{R}^d)$ we denote the corresponding Borel σ-field of subsets of $\mathbb{R}^d$. In the approach used in this work the phase space is the *configuration space*, i.e. the set

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d\},$$

where $\gamma_\Lambda := \gamma \cap \Lambda$ and $|\cdot|$ stands for cardinality. The elements of Γ are called *configurations*. We associate Γ with the subset of the space of all positive Radon measures on $\mathbb{R}^d$ by using the representation

$$\gamma = \sum_{x \in \gamma} \delta_x,$$

where δ_x is the Dirac measure centered at x. That is, the atomic measure with a single atom at $x \in \mathbb{R}$ with mass one. For a given

$B \in \mathcal{B}(\mathbb{R}^d)$, it takes values

$$\delta_x(B) = \mathbb{I}_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbb{I}_B$ is the indicator of B. Hence, we may equip Γ with a measurable structure, which we do by introducing a topology. Let K be any nonempty set and $Z = \{\zeta_i\}_{i \in I}$ be a family of maps $\zeta : K \to \mathbb{R}$ indexed by an arbitrary set I. The topology on K *induced by* Z is the weakest topology that makes continuous all the maps ζ_i. Let $C_{\mathrm{cs}}(\mathbb{R}^d)$ stand for the set of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ which have compact support. For $f \in C_{\mathrm{cs}}(\mathbb{R}^d)$, we define the map

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x)\gamma(dx) = \sum_{x \in \gamma} f(x) \in \mathbb{R}.$$

Definition 1.1.1. The vague topology on Γ is the topology induces by the family $\{\langle f, \cdot \rangle : f \in C_{\mathrm{cs}}(\mathbb{R}^d)\}$.

By $\mathcal{B}(\Gamma)$ we then denote the Borel $\sigma-$field of subsets of Γ.

The topology on Γ admits a metrization, see [36] and [44]. Let $B_r(x)$ denote the open ball with radius $r \in \mathbb{R}_+$ and centre $x \in \mathbb{R}^d$. Then, for any subset $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, the neighbourhood of A is defined by $A^\varepsilon := \bigcup_{a \in A} B_\varepsilon(a)$. For any $\eta_1, \eta_2 \in \Gamma_0$ let us first define the Prohorov distance $d : \Gamma_0 \times \Gamma_0 \to \mathbb{R}_+$ by the formula

$$\begin{aligned} d(\eta_1, \eta_2) := \inf\{\varepsilon > 0 :\ & \eta_1(A) \leq \eta_2(A^\varepsilon) + \varepsilon \text{ and } \eta_2(A) \leq \eta_1(A^\varepsilon) + \varepsilon \\ & \text{for all closed } A \in \mathbb{R}\}. \end{aligned} \tag{1.1}$$

Using (1.1), for any $\gamma_1, \gamma_2 \in \Gamma$ we introduce the metric D on Γ

$$D(\gamma_1, \gamma_2) := \int_0^\infty e^{-r} \frac{d(\gamma_1^{(r)}, \gamma_2^{(r)})}{1 + d(\gamma_1^{(r)}, \gamma_2^{(r)})} dr, \tag{1.2}$$

where $\gamma^{(r)}(A) = \gamma(A \cap B_r(0))$ for all measurable $A \subset \mathbb{R}^d$. Then we obtain the following two results.

Theorem 1.1.2. [44, Theorem 1.1] *Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in Γ and $\gamma \in \Gamma$. Then the following statements are equivalent:*

(i) $D(\gamma_k, \gamma) \to 0$ *as* $k \to \infty$*;*

(ii) $\displaystyle\int_{\mathbb{R}^d} f(x)\gamma_k(dx) \to \int_{\mathbb{R}^d} f(x)\gamma(dx)$ *as* $k \to \infty$ *for all bounded continuous functions* f *on* $\mathbb{R}^d$ *with compact support;*

(iii) *there exists an increasing sequence* $(r_n)_{n\in\mathbb{N}}$ *with* $r_n \to \infty$ *as* $n \to \infty$ *such that* $d(\gamma_k^{(r_n)}, \gamma^{(r_n)}) \to 0$ *as* $k \to \infty$ *for all* $n \in \mathbb{N}$*;*

(iv) $\gamma_k(A) \to \gamma(A)$ *as* $k \to \infty$ *for all bounded sets* $A \in \mathbb{R}^\Gamma$ *such that* $\gamma(\partial A) = 0$*, where* ∂A *is the boundary of* A*.*

Theorem 1.1.3. [44, Theorem 1.2] *The space* (Γ, D) *has the properties:*

(i) Γ *is a complete and separable metric space (Polish space) when it is equipped with the distance function* D *defined in* (1.2)*.*

(ii) *The Borel* $\sigma-$*field* $\mathcal{B}(\Gamma)$ *is the smallest* $\sigma-$*field that makes all mappings* $\Phi_A : \Gamma \to \mathbb{N} \cup \{\infty\}$*,* $A \in \mathcal{B}(\mathbb{R}^d)$*, measurable, where* $\Psi_A(\gamma) = \gamma(A)$*.*

Thereby, $(\Gamma, \mathcal{B}(\Gamma))$ is a standard Borel space. By $\mathcal{P}(\Gamma)$ we denote the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$.

Let Λ be compact. Set $\Gamma_\Lambda = \{\gamma : \gamma \subset \Lambda\}$ and define the following sub-field of $\mathcal{B}(\Gamma)$:

$$\mathcal{B}(\Gamma_\Lambda) = \{\mathbb{A} \cap \Gamma_\Lambda : \mathbb{A} \in \mathcal{B}(\Gamma)\}.$$

Hence, $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is also a standard Borel space. By $\mathcal{B}_\Lambda(\Gamma)$ we denote the smallest sub-σ-field of $\mathcal{B}(\Gamma)$ that contains all *cylinder* events

$$\mathbb{C}_{\mathbb{A}} = \{\gamma \in \Gamma : \gamma_\Lambda \in \mathbb{A}\}. \tag{1.3}$$

For a compact Λ, we define the map

$$\Gamma \ni \gamma \mapsto N_\Lambda(\gamma) = |\gamma_\Lambda| \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Then $\mathcal{B}(\Gamma)$ is generated by the family of sets

$$\Gamma^{\Lambda,n} := \{\gamma \in \Gamma : N_\Lambda(\gamma) = n\}, \qquad n \in \mathbb{N}_0, \quad \Lambda - \text{compact}. \tag{1.4}$$

By setting $\Lambda = \mathbb{R}^d$ in (1.4), we obtain the set Γ^n of all $n-$point configurations and then we define the set of finite configurations by the formula

$$\Gamma_0 := \bigcup_{n=0}^{\infty} \Gamma^n \in \mathcal{B}(\Gamma).$$

Each Γ^n is equipped with the topology related to the Euclidean topology of the underlying space $\mathbb{R}^d$ and then $\Gamma_0 \in \mathcal{B}(\Gamma)$. Hence, the $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is a standard Borel space with corresponding Borel σ-field of subsets of Γ_0 coincides with the σ-field

$$\mathcal{B}(\Gamma_0) = \{A \cap \Gamma_0 : A \in \mathcal{B}(\Gamma)\}.$$

Multiple configurations in a locally compact Polish space

In this part, the trait space X is just a locally compact Polish space. By $C_{\mathrm{b}}(X)$ we denote the set of all continuous and bounded functions $f : X \to \mathbb{R}$ and by $C_{\mathrm{b}}^+(X)$ - the set of positive elements of $C_{\mathrm{b}}(X)$. Then by $C_0^+(X)$ we denote the set of all $f \in C_{\mathrm{b}}^+(X)$ which satisfy: (a) $f(x) > 0$ for all $x \in X$; (b) for each $\varepsilon > 0$, one finds a compact $\Lambda_\varepsilon \subset X$ such that $f(x) < \varepsilon$ whenever $x \in X \setminus \Lambda_\varepsilon$.

As in [43], by a configuration γ we mean a finite or countably infinite, unordered system of points placed in $\mathbb{R}^d$, in which several points may have the same location. As in the previous case, the set Γ is equipped with the vague (weak-hash) topology – the weakest topology that makes continuous all the maps $\gamma \mapsto \sum_{x\in\gamma} g(x)$, $g \in C_{\mathrm{cs}}(\mathbb{R}^d)$. Here by writing $\sum_{x\in\gamma} g(x)$ we understand $\sum_i g(x_i)$ for a certain enumeration of the elements of γ. Clearly, such sums are independent of the enumeration choice, see [43]. In the same sense, we shall understand sum of this kind

$$\sum_{x\in\gamma} \sum_{x\in\gamma\setminus x} \cdots$$

The vague topology is separable and consistent with a complete metric, which makes Γ a Polish space.

In dealing with infinite configurations, we may restrict ourselves to those ones that have a priori prescribed properties. Here we do this by employing a function $\psi \in C_{\mathrm{b}}^+(X)$, $\psi(x) \le 1$, for which we set

$$\Psi(\gamma) = \sum_{x\in\gamma} \psi(x). \tag{1.5}$$

Then the set of tempered configurations is defined as

$$\Gamma^\psi = \{\gamma \in \Gamma : \Psi(\gamma) < \infty\},$$

cf. [21, page 41]. It is clear that

$$\Gamma^{\psi'} \supset \Gamma^\psi, \qquad \text{whenever} \quad \psi' \leq \psi. \tag{1.6}$$

By this observation we get a possibility to vary Γ^ψ from Γ (by taking $\psi \in C_0^+(X)$) to Γ_0, corresponding to $\psi \equiv 1$. If $\psi \in C_0^+(X)$, then Γ^ψ is a proper subset of Γ and subset of Γ_0. As an example one can take $X = \mathbb{R}$ and $\psi(x) = e^{-\alpha|x|}$, $\alpha > 0$. Then the configuration $\mathbb{N} \subset \mathbb{R}$ is in Γ^ψ, whereas $\{\log n : n \in \mathbb{N}\}$ is not if $\alpha \leq 1$.

For each $\gamma \in \Gamma^\psi$, the measure

$$\nu_\gamma = \sum_{x\in\gamma} \psi(x)\delta_x \tag{1.7}$$

is finite. Thus, one can equip Γ^ψ with the topology defined as the weakest one that makes continuous all the maps

$$\Gamma^\psi \ni \gamma \mapsto \sum_{x\in\gamma} g(x)\psi(x), \qquad g \in C_{\mathrm{b}}(X). \tag{1.8}$$

Similarly as in Proposition 2.7 and Corollary 2.8 of [41], one can prove the following.

Proposition 1.1.4. *Let $\psi \in C_{\mathrm{b}}^+(X)$ be separated away from zero, i.e. $\psi(x) > 0$ for x in a compact $\Lambda \subset X$. Then with the topology defined in (1.8), Γ^ψ is a Polish space, continuously embedded in Γ. Thus, $\mathcal{B}(\Gamma^\psi) = \{A \in \mathcal{B}(\Gamma) : A \subset \Gamma^\psi\}$.*

Proof. First we note that the set of measures $\{\nu_\gamma : \gamma \in \Gamma^\psi\}$ is a subset of the space $\mathcal{N}$ of all positive finite Borel measures on X, which is a Polish space with the weak topology. Let us prove that Γ^ψ is a closed subset of $\mathcal{N}$. To this end, we take a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma^\psi$ such that $\{\nu_{\gamma_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric of $\mathcal{N}$ that makes this space complete. Let $\nu \in \mathcal{N}$ be its limit, and hence

$$\sum_{x\in\gamma_n} g(x)\psi(x) \to \nu(g), \qquad n \to +\infty, \tag{1.9}$$

holding for all $g \in C_{\rm b}(X)$, in particular for $g \in C_{\rm cs}(X)$. Since ψ is separated away from zero, each $h \in C_{\rm cs}(X)$ can be written in the form $h(x) = g(x)\psi(x)$ with $g \in C_{\rm cs}(X)$. Hence, the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ vaguely converges to some locally finite counting Borel measure $\nu^{\#}$ on X as the set $\mathcal{N}^{\#}$ of all such measures with the vague topology is a Polish space, see [20, Proposition 9.1.IV, page 6]. The limiting counting measure can be associated with a certain $\gamma \in \Gamma$, i.e., $\nu(h) = \nu^{\#}_{\gamma}(h) = \sum_{x\in\gamma} h(x)$, holding for all $h \in C_{\rm cs}(X)$. To prove that this γ lies in Γ^{ψ}, we take an ascending sequence of compact $\Lambda_m \subset X$, i.e., $\Lambda_m \subset \Lambda_{m+1}$, $m \in \mathbb{N}$, such that each $x \in X$ is contained in some Λ_m. Then we take $g^{(m)} \in C_{\rm cs}(X)$ such that $g^{(m)}(x) = 1$ for $x \in \Lambda_m$, and $g^{(m)}(x) = 0$ for $x \in X \setminus \Lambda_{m+1}$, which is possible by Urysohn's lemma. Then

$$\nu^{\#}_{\gamma}(h^{(m)}) = \sum_{x\in\gamma} g^{(m)}(x)\psi(x) = \nu(g^{(m)}) \leq \nu(X),$$

$$h^{(m)}(x) = g^{(m)}(x)\psi(x).$$

Now we pass here to the limit $m \to +\infty$ and obtain (by the Beppo - Levi theorem) that $\Psi(\gamma) \leq \nu(X)$, which yields, $\gamma \in \Gamma^{\psi}$. Thus, ν in (1.9) is equal to $\nu^{\#}_{\gamma}$, which yields that $\{\nu_\gamma : \gamma \in \Gamma^{\psi}\}$ is closed in $\mathcal{N}$, and thereby is also Polish, see [18, Proposition 8.1.2, page 240]. In view of the aforementioned identification γ with ν_γ, the latter proves the first half of the statement. The stated continuity of the embedding $\Gamma^{\psi} \hookrightarrow \Gamma$ is immediate. Then the conclusion concerning the σ-fields follows by Kuratowski's theorem, see [45, Theorem 3.9, page 21]. □

Remark 1.1.5. The continuity of the embedding $\Gamma^{\psi} \hookrightarrow \Gamma$ allows one to establish the following fact:

$$\mathcal{P}(\Gamma^{\psi}) = \{\mu \in \mathcal{P}(\Gamma) : \mu(\Gamma^{\psi}) = 1\}. \tag{1.10}$$

That is, each $\mu \in \mathcal{P}(\Gamma)$ possessing the property $\mu(\Gamma^{\psi}) = 1$ can be redefined as a probability measure on Γ^{ψ}. Therefore, by restricting ourselves to tempered configurations – members of Γ^{ψ} – we exclude from our consideration all those $\mu \in \mathcal{P}(\Gamma)$ that fail to satisfy the mentioned support condition.

Let $B_{\rm b}(X)$ denote the set of all bounded and measurable functions $f : X \to \mathbb{R}$. Following [24, page 11], we say that a sequence $\{h_n\}_{n\in\mathbb{N}} \subset B_{\rm b}(X)$ converges to a certain $h \in B_{\rm b}(X)$ *boundedly* and *pointwise* if:

(a) $\sup_n \|h_n\| < \infty$; (b) $h_n(x) \to h(x)$ for each $x \in X$. In this case, we write $h_n \stackrel{bp}{\to} h$. A subset, $\mathcal{H} \subset B_{\rm b}(X)$, is said to be *bp*-closed, if $\{h_n\} \subset \mathcal{H}$ and $h_n \stackrel{bp}{\to} h$ imply $h \in \mathcal{H}$. The *bp*-closure of $\mathcal{H} \subset B_{\rm b}(X)$ is the smallest *bp*-closed subset of $B_{\rm b}(X)$ that contains $\mathcal{H}$. An $\mathcal{H}'$ is *bp*-dense in $\mathcal{H}$, if the latter is the smallest *bp*-closed set that contains $\mathcal{H}'$.

Let $\mathcal{F}$ be a family of functions $f : X \to \mathbb{R}$. By $\sigma\mathcal{F}$ we denote the smallest sub-field of $\mathcal{B}(X)$ such that each $f \in \mathcal{F}$ is $\sigma\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. The weak topology of $\mathcal{P}(X)$ is defined as the weakest one that makes continuous all the maps $\mu \mapsto \mu(f)$, $f \in C_{\rm b}(X)$. With this topology $\mathcal{P}(X)$ can also be turned into a Polish space. By writing $\mu_n \Rightarrow \mu$, $n \to +\infty$, we mean that $\{\mu_n\}_{n\in\mathbb{N}}$ weakly converges to μ. A family $\mathcal{F}$ of functions $f : X \to \mathbb{R}$ is called *separating* if $\mu_1(f) = \mu_2(f)$, holding for all $f \in \mathcal{F}$, implies $\mu_1 = \mu_2$ for each pair $\mu_1, \mu_2 \in \mathcal{P}(X)$. $\mathcal{F}$ is said to separate the points of X if for each distinct $x, y \in X$, one finds $f \in \mathcal{F}$ with the property $f(x) \neq f(y)$. If $\mathcal{F}$ separates points and is closed with respect multiplication, it is separating, see [24, Theorem 4.5, page 113]. A family $\mathcal{F}$ is called *convergence determining* if $\mu_n(f) \to \mu(f)$, $f \in \mathcal{F}$, implies $\mu_n \Rightarrow \mu$. The following is known, see [24, Proposition 4.2, page 111] and/or [21, Lemmas 3.2.1, 3.2.3, pages 41, 42].

Proposition 1.1.6. *For each Polish space X, there exists a countable family $\mathcal{H} \subset C_{\rm b}^{+}(X)$ that has the following properties: (a) the linear span of $\mathcal{H}$ is bp-dense in $B_{\rm b}(X)$; (b) $\mathcal{B}(X) = \sigma\mathcal{H}$; (c) it contains the unit function $u(x) \equiv 1$ and is closed with respect to addition; (d) it is separating; (e) it is convergence determining.*

Let $\mathcal{V} = \{v_l\}_{l\in\mathbb{N}} \subset C_{\rm b}^{+}(X)$ be a family of functions with the property as in Proposition 1.1.6. We may and will assume that each $v_l \in \mathcal{V}$ satisfies $\inf_X v_l(\hat{x}) \geq c_{0,l} > 0$ for an appropriate $c_{0,l}$, cf. [21, Remark 3.2.3, page 42]. Indeed, if this is not the case, instead of v_l one can take $\tilde{v}_l := v_l + c_{0,l}$. Then the family $\{\tilde{v}_l\}_{l\in\mathbb{N}}$ has all the properties we need. For $\gamma \in \Gamma^{\psi}$, we have, cf. (1.7), $\nu_\gamma(v_l) = \sum_{x\in\gamma} v_l(x)\psi(x)$. Then the topology mentioned in Proposition 1.1.4 is metrizable with the metric

$$\upsilon_*(\gamma, \gamma') = \sum_{l=0}^{\infty} \frac{2^{-l}\,|\nu_\gamma(v_l) - \nu_{\gamma'}(v_l)|}{1 + |\nu_\gamma(v_l) - \nu_{\gamma'}(v_l)|}. \tag{1.11}$$

For $\mu \in \mathcal{P}(\Gamma^\psi)$, its Laplace transform is defined by the expression

$$\mathfrak{L}_\mu(g) = \mu(G^g), \qquad g \in C_{\mathrm{b}}^+(X) \tag{1.12}$$

$$G^g(\gamma) := \exp\left(-\nu_\gamma(g)\right) = \exp\left(-\sum_{x\in\gamma} g(x)\psi(x)\right).$$

The following is known, see [21, Lemma 3.2.5 and Theorem 3.2.6, page 43].

Proposition 1.1.7. *Let $\mathcal{V}$ be the family of functions used in (1.11). Then:*

(i) $\mathcal{B}(\Gamma^\psi) = \sigma\{G^v : v \in \mathcal{V}\}$*;*

(ii) $B_{\mathrm{b}}(\Gamma^\psi)$ *is the bp-closure of the linear span of* $\{G^v : v \in \mathcal{V}\}$*;*

(iii) $\{G^v : v \in \mathcal{V}\}$ *is separating;*

(iv) $\{G^v : v \in \mathcal{V}\}$ *is convergence determining.*

The proof of claim (iv) is essentially based on the concrete choice of the metric (1.11), by which one shows that the family $\{G^v : v \in \mathcal{V}\}$ is strongly separating, cf. [24, page 113]. In the sequel, we will use the following functions

$$\phi(x) = 1 - \theta(x) = \exp\left(-g(x)\psi(x)\right), \tag{1.13}$$

with $g \in C_{\mathrm{b}}^+(X)$.

1.2 Functions and measures on configuration spaces

In this section, all the statements and fact hold true for both multiple and single configurations, and X stands for the habitat, that includes also the case $X = \mathbb{R}^d$. We write $\mathbb{R}^d$ rather than X to stress that we mean exactly this habitat.

Recall that $C_{\mathrm{cs}}(X)$ denotes the set of all compactly supported continuous numerical functions on X. By Θ we denote the set of all those $\theta \in C_{\mathrm{cs}}(X)$ which take values in $(-1, 0]$. Then the map

$$\Gamma \ni \gamma \mapsto F^{\theta}(\gamma) := \prod_{x \in \gamma}(1+\theta(x)) = \exp\left(\sum_{x\in\gamma}\log(1+\theta(x))\right), \qquad \theta \in \Theta, \tag{1.14}$$

is (vaguely) continuous and satisfies $0 < F^{\theta}(\gamma) \le 1$ for all γ. The upper bound is evident, and the continuity follows by the continuity of $\gamma \mapsto \sum_{x\in\gamma}\log(1+\theta(x))$. The set Θ has the following properties:

(a) for each pair of distinct $\gamma, \gamma' \in \Gamma$, there exists $\theta \in \Theta$ such that $F^{\theta}(\gamma) \neq F^{\theta}(\gamma')$;

(b) for each pair $\theta, \theta' \in \Theta$, the point-wise combination $\theta + \theta' + \theta\theta'$ is also in Θ;

(c) the zero function belongs to Θ.

The mentioned properties yield that $\{F^{\theta} : \theta \in \Theta\}$ is a separating family, see [1, Proposition 1.3.28, page 113]. Moreover, for each $\theta \in \Theta$, $\mu(F^{\theta}) = \mu^{\Lambda_\theta}(F^{\theta})$, where a compact Λ_θ is such that $\theta(x) = 0$ for $x \in \Lambda_\theta^c := \mathbb{R}^d \setminus \Lambda_\theta$.

A function $G : \Gamma_0 \to \mathbb{R}$ is $\mathcal{B}(\Gamma_0)/\mathcal{B}(\mathbb{R})$-measurable if and only if, for each $n \in \mathbb{N}$, there exists a collection of symmetric Borel functions $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, such that

$$G(\eta) = G^{(n)}(x_1, \ldots, x_n), \qquad \text{for} \ \ \eta = \{x_1, \ldots, x_n\}. \tag{1.15}$$

Definition 1.2.1. A measurable function $G : \Gamma_0 \to \mathbb{R}$ is said to have *bounded support* if: (a) there exists a compact $\Lambda \subset \mathbb{R}^d$ such that $G(\eta) = 0$ whenever $\eta \cap \Lambda \neq \eta$; (b) there exists $N \in \mathbb{N}$ such that $G(\eta) = 0$ whenever $|\eta| > N$. By $B_{\mathrm{bs}}(\Gamma_0)$ we denote the set of all bounded functions with bounded support.

Definition 1.2.2. $F : \Gamma \to \mathbb{R}$ is a *cylinder function* if there exists a compact Λ and a $\mathcal{B}(\Gamma_\Lambda)/\mathcal{B}(\mathbb{R})$-measurable function G such that $F(\gamma) = G(\gamma_\Lambda)$, $\gamma \in \Gamma$.

For a compact Λ and $\mathbb{A} \in \mathcal{B}(\Gamma_\Lambda)$ let $\mathcal{B}_\Lambda(\Gamma)$ be the sub-σ-field of $\mathcal{B}(\Gamma)$ generated by all cylinder sets $\mathbb{C}_{\mathbb{A}}$ defined in (1.3). By Definition

1.2.2, a cylinder function $F : \Gamma \to \mathbb{R}$ is $\mathcal{B}_\Lambda(\Gamma)/\mathcal{B}(\mathbb{R})$-measurable for some compact Λ. For a compact Λ and a given $\mu \in \mathcal{P}(\Gamma)$, we define the *projection* of μ on Γ_Λ by the relation

$$\mu(\mathbb{C}_{\mathbb{A}}) = \mu^\Lambda(\mathbb{A}) \tag{1.16}$$

which determines $\mu^\Lambda \in \mathcal{P}(\Gamma_\Lambda)$. Note that all such projections $\{\mu^\Lambda\}_\Lambda$ of a given $\mu \in \mathcal{P}(\Gamma)$ are consistent in the Kolmogorov sense.

In the sequel, we consider probability measures on Γ as state of the system. Each $\mu \in \mathcal{P}(\Gamma)$ is characterized by its values on the sets (1.4). A homogeneous Poisson measure $\pi_\varkappa \in \mathcal{P}(\Gamma)$ with density $\varkappa > 0$ (see, e.g., [30]) is then defined by the following expression.

$$\pi_\varkappa(\Gamma^{\Lambda,n}) = \frac{(\varkappa|\Lambda|)^n}{n!} \exp\left(-\varkappa|\Lambda|\right), \tag{1.17}$$

where $|\Lambda|$ stands for the Lebesgue measure of Λ.

For a given $\mu \in \mathcal{P}(\Gamma)$, we introduce the Bogolubov functional as

$$B_\mu(\theta) = \mu(F^\theta) = \int_\Gamma F^\theta d\mu, \qquad \theta \in \Theta. \tag{1.18}$$

For the Poisson measure it has the following form

$$B_{\pi_\varkappa}(\theta) = \exp\left(\varkappa \int_{\mathbb{R}^d} \theta(x)dx\right).$$

Obviously $\pi_\varkappa(F^\theta)$ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d)$. Having this in mind we introduce the following class of measures.

Definition 1.2.3. The set of *sub-Poissonian measures* $\mathcal{P}_{\exp}(\Gamma)$ consists of all those $\mu \in \mathcal{P}(\Gamma)$ for each of which $\mu(F^\theta)$ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d)$.

It can by shown that the measure $\pi_\varkappa$ has the property: $\pi_\varkappa(\Gamma_0) = 0$, and for each $\mu \in \mathcal{P}_{\exp}(\Gamma)$, there exists $\varkappa > 0$ such that

$$\mu(\Gamma^{\Lambda,n}) \le \pi_\varkappa(\Gamma^{\Lambda,n}), \tag{1.19}$$

holding for all compact Λ and $n \in \mathbb{N}$.

By Definition 1.2.3, see [35], a given μ belongs to $\mathcal{P}_{\exp}(\Gamma)$ if and only if $\mu(F^\theta)$ can be written down in the form

$$\mu(F^\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \ldots, x_n)\theta(x_1)\cdots\theta(x_n)dx_1\cdots dx_n, \tag{1.20}$$

where $k_\mu^{(n)}$ - called n-th order *correlation function* of the state μ- is a positive and symmetric element of $L^\infty((\mathbb{R}^d)^n)$. Moreover, for the collection $\{k_\mu^{(n)}\}_{n\in\mathbb{N}}$ satisfies *Ruelle's bound*

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n} \leq \varkappa^n, \qquad n \in \mathbb{N}, \tag{1.21}$$

holding with some $\varkappa > 0$. Note that $k_{\pi_\varkappa}^{(n)} = \varkappa^n$.This implies the form of $B_{\pi_\varkappa}$, and so $k_\mu^{(n)}(x_1, \ldots, x_n) \leq \varkappa^n$, cf. (1.19).

Recall that $B_{\mathrm{bs}}(\Gamma_0)$ stands for the set of all bounded functions with bounded support.

Definition 1.2.4. The *Lebesgue-Poisson measure* λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by the integrals

$$\int_{\Gamma_0} G(\eta)\lambda(d\eta) = G(\varnothing) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots x_n)dx_1\cdots dx_n, \tag{1.22}$$

holding for all $G \in B_{\mathrm{bs}}(\Gamma_0)$.

For $G \in B_{\mathrm{bs}}(\Gamma_0)$, we set

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \qquad \gamma \in \Gamma, \tag{1.23}$$

where $\eta \Subset \gamma$ means that $\eta \subset \gamma$ and η is finite. For each $G \in B_{\mathrm{bs}}(\Gamma_0)$, by Λ_G and N_G we denote the smallest Λ and N with the properties mentioned in Definition 1.2.1, and use the notations $C_G = \sup_{\eta\in\Gamma_0} |G(\eta)|$. Then, cf. Definition 1.2.1,

$$|(KG)(\gamma)| \leq C_G \left(1 + |\gamma \cap \Lambda_G|\right)^{N_G}, \qquad G \in B_{\mathrm{bs}}(\Gamma_0).$$

Like in (1.15), we introduce the real valued function $k_\mu : \Gamma_0 \to \mathbb{R}$ such that $k_\mu(\eta) = k^{(n)}(x_1, \ldots, x_n)$ for $\eta = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, and

$k_\mu(\varnothing) = 1$. Then we rewrite (1.20) as follows

$$\mu(F^\theta) = \int_{\Gamma_0} k_\mu(\eta) e(\theta;\eta)\lambda(d\eta), \qquad e(\theta;\eta) := \prod_{x\in\eta}\theta(x), \tag{1.24}$$

where λ is the Lebesgue-Poisson measure introduced in (1.22). In the sequel, we call k_μ the *correlation function* corresponding to the measure μ.

For $\mu \in \mathcal{P}_{\rm exp}(\Gamma)$ and a compact Λ, the projection μ^Λ, as a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$, is absolutely continuous with respect to the Lebesgue-Poisson measure λ, see [35]. Hence, we may write

$$\mu^\Lambda(d\eta) = R^\Lambda_\mu(\eta)\lambda(d\eta). \tag{1.25}$$

Then the Radon-Nikodym derivative R^Λ_μ and the correlation function k_μ are related to each other by

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R^\Lambda_\mu(\eta\cup\xi)\lambda(d\xi), \qquad \eta\in\Gamma_\Lambda, \quad \Lambda \text{ - compact.} \tag{1.26}$$

For each $G \in B_{\rm bs}(\Gamma_0)$ and $k : \Gamma_0 \to \mathbb{R}$ such that $k^{(n)} \in L^\infty((\mathbb{R}^d)^n)$ the integral

$$\langle\!\langle G, k\rangle\!\rangle := \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta) \tag{1.27}$$

exists. Hence, by (1.20), (1.23), (1.25) and (1.27) we obtain

$$\int_\Gamma (KG)(\gamma)\mu(d\gamma) = \langle\!\langle G, k_\mu\rangle\!\rangle \tag{1.28}$$

holding for all $G \in B_{\rm bs}(\Gamma_0)$ and $\mu \in \mathcal{P}_{\rm exp}(\Gamma)$. Set

$$B^\star_{\rm bs}(\Gamma_0) = \{G \in B_{\rm bs}(\Gamma_0) : (KG)(\gamma) \geq 0 \ \text{ for all } \ \gamma\in\Gamma\}. \tag{1.29}$$

By [35, Theorems 6.1, 6.2 and Remark 6.3] we know that the following Proposition is true.

Proposition 1.2.5. *Let a measurable function $k : \Gamma_0 \to \mathbb{R}$ have the following properties:*

$$(a) \qquad \langle\!\langle G, k_\mu\rangle\!\rangle \geq 0, \qquad \text{for all } \ G\in B^\star_{\rm bs}(\Gamma_0);$$

$$(b) \qquad k(\varnothing) = 1;$$

$$(c) \qquad k(\eta) \leq C^{|\eta|};$$

with (c) holding for some $C > 0$ and λ-almost all $\eta\in\Gamma_0$. Then there exists a unique $\mu \in \mathcal{P}_{\rm exp}(\Gamma)$ such that k is its correlation function.

Finally, we present the inequality which will extensively be used in the sequel

$$n^p e^{-\sigma n} \le \left(\frac{p}{e\sigma}\right)^p, \qquad p \ge 1, \quad \sigma > 0, \quad n \in \mathbb{N}, \tag{1.30}$$

and two useful formulas holding for appropriate functions $g : \mathbb{R}^d \to \mathbb{R}$ and $G : \Gamma_0 \to \mathbb{R}$.

$$\forall x \in \gamma \qquad \sum_{\eta \Subset \gamma} \prod_{z \in \eta} g(z) = (1 + g(x)) \sum_{\eta \Subset \gamma \setminus x} \prod_{z \in \eta} g(z), \tag{1.31}$$

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi, \eta, \eta \setminus \xi) \lambda(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\xi, \eta \cup \xi, \eta) \lambda(d\xi) \lambda(d\eta), \tag{1.32}$$

see [25], Lemma 2.4.

1.3 Evolution equations

Here and in the sequel, we use the notation, cf. (1.18),

$$\mu(F) = \int F d\mu.$$

The Markov evolutions which are studied in this work are described by the (backward) Kolmogorov equation

$$\frac{d}{dt} F_t =: \dot{F}_t = LF_t, \qquad F_t|_{t=0} = F_0, \tag{1.33}$$

where $\dot{F}_t$ denotes the time derivative and $F_t : \Gamma \to \mathbb{R}$ is an *observable*. The expression LF determines the model, i.e. contains all informations about it.

Recall that we consider probability measures on Γ as states of the system. The evolution of such states $\mu_0 \mapsto \mu_t$ is defined by the Fokker-Planck equation

$$\dot{\mu}_t = L^\mu \mu_t, \qquad \mu_t|_{t=0} = \mu_0. \tag{1.34}$$

Both evolutions, (1.33) and (1.34), are in the duality $\mu_0(F_t) = \mu_t(F_0)$. This means that L^μ is obtained from L by the following rule

$$\int_{\Gamma_0} (LF)(\gamma) \mu_t(d\gamma) = \int_{\Gamma_0} F(\gamma)(L^\mu \mu_t)(d\gamma). \tag{1.35}$$

For infinite systems, the direct use of (1.33) and (1.34) is rather impossible, so we proceed as follows. Having in mind (1.24), instead of the Fokker-Planck equation (1.34) we may consider the following evolution equation for correlation functions

$$\frac{d}{dt}k_t = L^{\Delta}k_t, \qquad k_t|_{t=0} = k_{\mu_0}, \tag{1.36}$$

where operator L^{Δ} is obtained from L in (1.33) by the rule

$$\int_{\Gamma}(LF)(\gamma)\mu(d\gamma) = \int_{\Gamma_0}(L^{\Delta}k_{\mu})(\eta)\left(\prod_{x\in\eta}\theta(x)\right)\lambda(d\eta) \tag{1.37}$$

In the case of finite configurations measures of our interest are absolutely continuous with respect to λ defined in (1.22). Set, cf. (1.25),

$$\mu_t(d\gamma) = R_t(\gamma)\lambda(d\gamma).$$

Then one can also transform (1.34) into the problem

$$\frac{d}{dt}R_t(\eta) = (L^{\dagger}R_t)(\eta), \quad R_t|_{t=0} = R_0.$$

according to

$$(L^{\mu}\mu)(d\eta) = (L^{\dagger}R_{\mu})(\eta)\lambda(d\eta). \tag{1.38}$$

1.4 Spaces of measures and functions

We introduce here spaces in which equations (1.33), (1.34) and (1.36) will be considered. We start with the space of signed measures with bounded variation, where the equation (1.34) can be defined. By $\mathcal{M}$ we denote the Banach space of all signed measures on $(\Gamma_0, \mathcal{B}(\Gamma_0))$. Let $\mathcal{M}^+$ stands for the cone of positive elements of $\mathcal{M}$, which is generating, i.e. $\mathcal{M} = \mathcal{M}^+ - \mathcal{M}^+$, see [18] and [50] for more details. Using the Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$, $\mu^{\pm} \in \mathcal{M}^+$, we define the norm of $\mu \in \mathcal{M}$ as

$$\|\mu\|_{\mathcal{M}} = \mu^+(\Gamma_0) + \mu^-(\Gamma_0).$$

Then $\mathcal{P}(\Gamma_0)$ is a subset of $\mathcal{M}^+$ consisting of those μ for which $\|\mu\|_{\mathcal{M}} = 1$. Note that, the linear functional

$$\varphi_{\mathcal{M}}(\mu) := \mu(\Gamma_0) = \int_{\Gamma_0} \mu(d\eta) = \mu^+(\Gamma_0) - \mu^-(\Gamma_0)$$

has the property $\varphi_{\mathcal{M}}(\mu) = \|\mu\|_{\mathcal{M}}$ for each $\mu \in \mathcal{M}^+$; hence, $\|\cdot\|_{\mathcal{M}}$ is additive on the cone $\mathcal{M}^+$. Therefore $\mathcal{M}$ is an AL-space, see [50].

For an increasing function $\chi : \mathbb{N}_0 \to [0, +\infty)$, we set

$$\mathcal{M}_\chi = \left\{ \mu \in \mathcal{M} : \int_{\Gamma_0} \chi(|\eta|)\mu^\pm(d\eta) < \infty \right\}, \qquad \mathcal{M}^+_\chi = \mathcal{M}_\chi \cap \mathcal{M}^+, \tag{1.39}$$

and introduce

$$\begin{aligned} \varphi_{\mathcal{M}_\chi}(\mu) &= \int_{\Gamma_0} \chi(|\eta|)\mu(d\eta) \qquad (1.40) \\ &= \int_{\Gamma_0} \chi(|\eta|)\mu^+(d\eta) - \int_{\Gamma_0} \chi(|\eta|)\mu^-(d\eta), \qquad \mu \in \mathcal{M}_\chi. \end{aligned}$$

Regarding χ we assume that $\mathcal{M}_\chi$ is a proper subset of $\mathcal{M}$ and the corresponding embedding is continuous.

Along with the space $\mathcal{M}$ we will consider its subspace consisting of all measures absolutely continuous with respect to the Lebesgue-Poisson measure, which is

$$\mathcal{R} := L^1(\Gamma_0, \lambda).$$

As we did before, we define the functional

$$\varphi_{\mathcal{R}}(R) = \int_{\Gamma_0} R(\eta)\lambda(d\eta).$$

Let $\mathcal{R}^+$ and $\mathcal{R}^+_1$ stands for the cone of positive elements of $\mathcal{R}$ and the probability densities, respectively. Then $\varphi_{\mathcal{R}}(R) = \|R\|_{\mathcal{R}}$ for $R \in \mathcal{R}^+$ and hence $\mathcal{R}$ is AL-space too. Similarly, as in (1.39) and (1.40), for an increasing function $\chi : \mathbb{N}_0 \to [0, +\infty)$, we set

$$\mathcal{R}_\chi = \left\{ R \in \mathcal{R} : \int_{\Gamma_0} \chi(|\eta|)|R(\eta)|\lambda(d\eta) < \infty \right\}, \tag{1.41}$$

$$\varphi_{\mathcal{R}_\chi}(R) = \int_{\Gamma_0} \chi(|\eta|)R(\eta)\lambda(d\eta), \qquad R \in \mathcal{R}_\chi,$$

$$\mathcal{R}^+_\chi = \mathcal{R}_\chi \cap \mathcal{R}^+, \qquad \mathcal{R}^+_{\chi,1} = \{R \in \mathcal{R}^+_\chi : \varphi_{\mathcal{R}}(R) = 1\}.$$

Let us now consider the L^∞-like Banach spaces of functions $\mathcal{K}_\alpha$. Having in mind that the correlation functions of measures from the set $\mathcal{P}_{\exp}(\Gamma)$ satisfy (1.21), for appropriated non-increasing function $w(|\eta|)$ and $\alpha \in \mathbb{R}$, we introduce the following norms

$$\|k\|_{\alpha,w} = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} w(|\eta|)e^{-\alpha|\eta|}|k(\eta)|, \tag{1.42}$$

and the corresponding spaces

$$\mathcal{K}_\alpha = \{k : \Gamma_0 \to \mathbb{R} : \|k\|_{\alpha,w} < \infty\}, \tag{1.43}$$

equipped with the usual point-wise linear operations. For $\alpha' < \alpha$, we have that $\|k\|_{\alpha'} \geq \|k\|_\alpha$. Therefore, $\mathcal{K}_{\alpha'} \hookrightarrow \mathcal{K}_\alpha$, where "$\hookrightarrow$" denotes continuous embedding. In view of this, we have obtained an ascending scale of Banach spaces $\{\mathcal{K}_\alpha\}_{\alpha\in\mathbb{R}}$.

For $\alpha \in \mathbb{R}$, we set, cf. (1.28), (1.29) and Proposition 1.2.5,

$$\mathcal{K}^\star_\alpha = \{k \in \mathcal{K}_\alpha : \langle\!\langle G, k\rangle\!\rangle \geq 0 \ \text{ and } \ k(\varnothing) = 1\}. \tag{1.44}$$

Note that $\mathcal{K}^\star_\alpha$ is a proper subset of the

$$\mathcal{K}^+_\alpha := \{k \in \mathcal{K}_\alpha : k(\eta) \geq 0\}. \tag{1.45}$$

1.5 Thieme - Voigt perturbation theory

In this section, we present elements of the semigroup theory [46] and the Thieme - Voigt perturbation theory [50], see also [4]. Here, we adapted the theory to our purposes. Let $\mathcal{E}$ be either $\mathcal{M}$ or $\mathcal{R}$, and $\|\cdot\|_{\mathcal{E}}$ stand for the corresponding norm. The sets $\mathcal{E}^+$, $\mathcal{E}^+_1$, $\mathcal{E}_\chi$, $\mathcal{E}^+_\chi$, $\mathcal{E}^+_{\chi,1}$, and the functionals $\varphi_{\mathcal{E}}$, $\varphi_{\mathcal{E}_\chi}$ are defined analogously to the sets and functional defined in Section 1.4, i.e., they should coincide with the corresponding objects introduced in Section 1.4 if $\mathcal{E}$ is replaced by $\mathcal{M}$ or $\mathcal{R}$ (by $\mathcal{M}^+_1$ we then understand $\mathcal{P}(\Gamma_0)$). Let $\mathcal{D} \subset \mathcal{E}$ be a linear subspace, $\mathcal{D}^+ = \mathcal{D} \cap \mathcal{E}^+$ and $(A, \mathcal{D})$, $(B, \mathcal{D})$ be operators in $\mathcal{E}$. Set also $\mathcal{D}_\chi = \{u \in \mathcal{D} \cap \mathcal{E}_\chi : Au \in \mathcal{E}_\chi\}$ and denote by A_χ the trace of A in $\mathcal{E}_\chi$, i.e., the restriction of A to $\mathcal{D}_\chi$. Recall that a C_0-semigroup of bounded linear operators $S = \{S(t)\}_{t\geq 0}$ in $\mathcal{E}$ is called *positive* if $S(t) : \mathcal{E}^+ \to \mathcal{E}^+$ for each $t \geq 0$. A *sub-stochastic* (resp. *stochastic*) semigroup in $\mathcal{E}$ is a positive C_0-semigroup such that $\varphi_{\mathcal{E}}(S(t)u) \leq \varphi_{\mathcal{E}}(u)$ (resp. $\varphi_{\mathcal{E}}(S(t)u) = \varphi_{\mathcal{E}}(u)$) whenever $u \in \mathcal{E}^+$.

Proposition 1.5.1. [50, Proposition 2.2] *Let* $(A, \mathcal{D})$ *be the generator of a positive* C_0*-semigroup in* $\mathcal{E}$*, and* $(B, \mathcal{D})$ *be positive, i.e.,* $B : \mathcal{D}^+ \to \mathcal{E}^+$*. Suppose also that*

$$\forall u \in \mathcal{D}^+ \qquad \varphi_{\mathcal{E}}((A + B)u) \leq 0. \tag{1.46}$$

Then, for each $r \in (0, 1)$*, the operator* $(A + rB, \mathcal{D})$ *is the generator of a sub-stochastic semigroup in* $\mathcal{E}$*.*

Proposition 1.5.2. [50, Proposition 2.7] *Assume that:*

(i) $-A : \mathcal{D}^+ \to \mathcal{E}^+$ *and* $B : \mathcal{D}^+ \to \mathcal{E}^+$*;*

(ii) $(A, \mathcal{D})$ *be the generator of a sub-stochastic semigroup* $S = \{S(t)\}_{t\geq 0}$ *on* $\mathcal{E}$ *such that* $S(t) : \mathcal{E}_\chi \to \mathcal{E}_\chi$ *for all* $t \geq 0$ *and the restrictions* $S(t)|_{\mathcal{E}_\chi}$ *constitute a* C_0*-semigroup on* $\mathcal{E}_\chi$ *generated by* $(A_\chi, \mathcal{D}_\chi)$*;*

(iii) $B : \mathcal{D}_\chi \to \mathcal{E}_\chi$ *and* $\varphi_{\mathcal{E}}\left((A + B)u\right) = 0$*, for* $u \in \mathcal{D}^+$*;*

(iv) there exist $c > 0$ *and* $\varepsilon > 0$ *such that*

$$\varphi_{\mathcal{E}_\chi}\left((A + B)u\right) \leq c\varphi_{\mathcal{E}_\chi}(u) - \varepsilon \|Au\|_{\mathcal{E}}, \qquad \text{for} \;\; u \in \mathcal{D}_\chi \cap \mathcal{E}^+.$$

Then the closure of $(A + B, \mathcal{D})$ *in* $\mathcal{E}$ *is the generator of a stochastic semigroup* $S_{\mathcal{E}} = \{S_{\mathcal{E}}(t)\}_{t\geq 0}$ *in* $\mathcal{E}$ *which leaves* $\mathcal{E}_\chi$ *invariant. The restrictions* $S_{\mathcal{E}_\chi}(t) := S_{\mathcal{E}}(t)|_{\mathcal{E}_\chi}$*,* $t \geq 0$*, constitute a* C_0*-semigroup* $S_{\mathcal{E}_\chi}$ *in* $\mathcal{E}_\chi$ *generated by the trace of the generator of* $S_{\mathcal{E}}$ *in* $\mathcal{E}_\chi$*.*

Chapter 2

Fission-death system with competition

This chapter is devoted to the evolution of the system of interacting point entities with traits $x \in \mathbb{R}^d$, introduced and studied in [38, 39]. In the model, each entity is subject to a state-dependent death (with rate that includes a competition term) and independent fission, in the course of which each entity produces two descendants and simultaneously disappears. The states of the system are probability measures on the corresponding configuration space. We also perform a multi-scale analysis of this system.

2.1 The model and the basic assumptions

The Markov evolution of the model is described by the Kolmogorov equation (1.33), in which L specifies the model. In the considered case it has the following form

$$(LF)(\gamma) = \sum_{x\in\gamma} \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y) \right) [F(\gamma \setminus x) - F(\gamma)] \tag{2.1}$$

$$+ \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) [F(\gamma \setminus x \cup \{y_1, y_2\}) - F(\gamma)] \, dy_1 dy_2.$$

In expressions like $\gamma \cup x$, we treat x as a singleton configuration $\{x\}$. The first term in (2.1) corresponds to the death of the particle with

trait x occurring:
(i) independently with rate $m(x) \geq 0$;
(ii) under the influence (competition) of the other particles in γ occurring with rate

$$E^a(x, \gamma \setminus x) := \sum_{y \in \gamma \setminus x} a(x - y) \geq 0. \tag{2.2}$$

The second term in (2.1) is responsible for an independent fission with rate $b(x|y_1, y_2) \geq 0$.

Assumption 1. The nonnegative measurable functions a, b and m are subject to the following:

(i) a is integrable and bounded; hence, we may set

$$\sup_{x \in \mathbb{R}^d} a(x) = a^*, \qquad \int_{\mathbb{R}^d} a(x) dx = \langle a \rangle.$$

(ii) There exist positive r and a_* such that $a(x) \geq a_*$ whenever $|x| \leq r$.

(iii) For each $x \in \mathbb{R}^d$, $b(x|y_1, y_2)dy_1dy_2$ is a symmetric finite measure on $(\mathbb{R}^d)^2$; hence, we may set

$$\langle b \rangle = \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) dy_1 dy_2,$$

where, for simplicity, we restrict the consideration to the translation invariant case where the right-hand side of the latter formula is independent of x. The mentioned symmetry means that $b(x|y_1, y_2) = b(x|y_2, y_1)$ for all values of the arguments.

(iv) For each $y_1, y_2 \in \mathbb{R}^d$, $b(\cdot|y_1, y_2)$ is integrable, and hence we may set

$$\beta(y_1 - y_2) = \int_{\mathbb{R}^d} b(x|y_1, y_2) dx,$$

where the function $\beta \geq 0$ has the property $\int_{\mathbb{R}^d} \beta(x) dx = \langle b \rangle$ and is supposed to be such that

$$\sup_{x \in \mathbb{R}^d} \beta(x) =: \beta^* < \infty.$$

Our consideration includes also the case where b is a distribution. In particular, b may take the form

$$b(x|y_1, y_2) = \frac{1}{2}(\delta(x - y_1) + \delta(x - y_2))\beta(y_1 - y_2)$$

which corresponds to the Bolker-Pacala model [32].

Remark 2.1.1. The function β describes the dispersal of newborn twins, which are subject to the competition described by a. As in the Bolker-Pacala model, the following two cases may occur:

- *short dispersal:* there exists $\omega > 0$ such that $a(x) \geq \omega\beta(x)$ holding for all $x \in \mathbb{R}^d$;
- *long dispersal:* for each $\omega > 0$, there exists $x \in \mathbb{R}^d$ such that $a(x) < \omega\beta(x)$.

For $\eta \in \Gamma_0$, we set, cf. (2.2),

$$\begin{aligned} E^a(\eta) &= \sum_{x\in\eta} E^a(x, \eta \setminus x) = \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a(x-y), \qquad (2.3) \\ E^b(\eta) &= \sum_{x\in\eta}\sum_{y\in\eta\setminus x} \beta(x-y) = \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\int_{\mathbb{R}^d} b(z|x,y)dz. \end{aligned}$$

The properties of the model mentioned in (ii) and (iv) of Assumption 1 imply the following fact.

Proposition 2.1.2. *There exist $\omega > 0$ and $\upsilon \geq 0$ such that*

$$\upsilon|\eta| + E^a(\eta) \geq \omega E^b(\eta), \qquad (2.4)$$

holding for each $\eta \in \Gamma_0$.

The inequality above can be rewritten in the form

$$\Phi_\omega(\eta) := \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\left[a(x-y) - \omega\int_{\mathbb{R}^d} b(z|x,y)dz\right] \geq -\upsilon|\eta|. \qquad (2.5)$$

Proposition 2.1.3. *Assume that (2.5) holds true for some $\omega_0 > 0$ and $\upsilon_0 > 0$. Then for each $\omega < \omega_0$, it holds also for $\upsilon = \upsilon_0\omega/\omega_0$.*

Proof. For $\omega \in [0, \omega_0]$ by adding and subtracting $\frac{\omega}{\omega_0} E^a(\eta)$ we obtain

$$\Phi_\omega(\eta) = \frac{\omega}{\omega_0}\left[\left(\frac{\omega_0}{\omega} - 1\right) E^a(\eta) + \Phi_{\omega_0}(\eta)\right] \geq -\frac{\omega}{\omega_0}\upsilon_0|\eta|.$$

□

Proof of Proposition 2.1.2. According to Assumption 1, β is Riemann integrable, then for arbitrary $\varepsilon > 0$, one can divide $\mathbb{R}^d$ into equal cubic cells E_l, $l \in \mathbb{N}$, of small enough side $h > 0$ such that the following holds

$$h^d \sum_{l=1}^{+\infty} \beta_l \leq \langle\beta\rangle + \varepsilon, \qquad \beta_l := \sup_{x \in E_l} \beta(x). \tag{2.6}$$

For $r > 0$, set $K_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$, $x \in \mathbb{R}^d$, and

$$a_r = \inf_{x \in K_{2r}(0)} a(x). \tag{2.7}$$

Then we fix ε and pick $r > 0$ such that $a_r > 0$. For fixed r, h and ε as above, we prove the proposition by the induction in the number of points in η. By (2.5) we rewrite inequality (2.4) in the form

$$U_\omega(\eta) := \upsilon|\eta| + \Phi_\omega(\eta) \geq 0, \tag{2.8}$$

and for some $x \in \eta$, consider

$$\begin{aligned} U_\omega(x, \eta \setminus x) &:= U_\omega(\eta) - U_\omega(\eta \setminus x) \\ &= \upsilon + 2\left(\sum_{y \in \eta \setminus x} a(x - y) - \omega \sum_{y \in \eta \setminus x} \beta(x - y)\right). \end{aligned}$$

Set $c_d = |K_1|$ and let $\Delta(d)$ be the packing constant for regid balls in $\mathbb{R}^d$, cf. [28]. Then set

$$\delta(\beta) = \max\{\beta^*; (\langle b\rangle + \varepsilon) g_d(h, r)\}, \tag{2.9}$$

$$g_d(h, r) = \frac{\Delta(d)}{c_d}\left(\frac{h + 2r}{hr}\right)^d.$$

Next,assume that υ and ω satisfy the following, cf. (2.7),

$$\omega \leq \min\{\frac{\upsilon}{2\delta(\beta)}; \frac{a_r}{\delta(\beta)}\}. \tag{2.10}$$

Let us show that

(i) for each $\eta = \{x, y\}$, (2.10) implies (2.8);

(ii) for each η, one finds $x \in \eta$ such that $U_\omega(x, \eta \setminus x) \geq 0$ whenever (2.10) holds.

To prove (i) by (2.10) and (2.9) we get

$$\begin{aligned} U_\omega(\{x, y\}) &= 2\upsilon + 2a(x-y) - 2\omega\beta(x-y) \\ &\geq (\upsilon - 2\omega\beta^*) + 2a(x-y) \geq 0. \end{aligned}$$

To prove (ii), for $y \in \eta$, we set

$$s = \max_{y\in\eta} |\eta \cap K_{2r}(y)|. \tag{2.11}$$

Let also $x \in \eta$ be such that $|\eta \cap K_{2r}(x)| = s$. For this x, by $E_l(x)$, $l \in \mathbb{N}$, we denote the corresponding translates of E_l which appear in (2.6). Set $\eta_l = \eta \cap E_l(x)$ and let $l_* \in \mathbb{N}$ be such that $\eta \subset \bigcup_{l\leq l_*} E_l(x)$ which is possible since η is finite. For a given l, a subset $\zeta_l \subset \eta_l$ is called $r-$admissible if for each distinct $y, z \in \zeta_l$, one has that $K_r(y) \cap K_r(z) = \emptyset$. Such a subset ζ_l is called maximal $r-$admissible if $|\zeta_l| \geq |\zeta'|$ for any other $r-$admissible ζ'_l. It is clear that

$$\eta_l \subset \bigcup_{z\in\zeta_l} K_{2r}(z). \tag{2.12}$$

Otherwise, one finds $y \in \eta_l$ such that $|y - z| \geq 2r$, for each $z \in \zeta_l$, which yields that ζ_l is not maximal. Since all the balls $K_r(z)$, $z \in \zeta_l$, are contained in the $h-$extended cell

$$E_l^h(x) := \{y \in \mathbb{R}^d : \inf_{z\in E_l(x)} |y - z| \leq h\},$$

their maximum number - and hence $|\zeta_l|$ - can be estimated as follows

$$|\zeta_l| \leq \Delta(d) V(E_l^h(x))/c_d r^d = h^d \frac{\Delta(d)}{c_d} \left(\frac{h+2r}{hr}\right)^d = h^d g_d(h, r), \tag{2.13}$$

where c_d and $\Delta(d)$ are as in (2.9). Then by (2.11) and (2.12) we get

$$\sum_{y\in\eta\setminus x} \beta(x-y) \leq \sum_{l=1}^{l_*} \sum_{z\in\zeta_l} \sum_{y\in K_{2r}(z)\cap\eta_l} \beta_l.$$

The cardinality of $K_{2r}(z) \cap \eta_l$ does not exceed s, see (2.11), whereas the cardinality of ζ_l safisfies (2.13). Then

$$\sum_{y\in\eta\setminus x} \beta(x-y) \leq s g_d(h,r) \sum_{l=1}^{\infty} \beta_l h^d \leq s g_d(h,r)(\langle b\rangle + \varepsilon) \leq s\delta. \quad (2.14)$$

On other hand, by (2.7) and (2.11) we get

$$\sum_{y\in\eta\setminus x} a(x-y) \geq \sum_{y\in(\eta\setminus x)\cap K_{2r}(x)} a(x-y) \geq (s-1)a_r.$$

We use this estimate and (2.14) in (2.8) and obtain

$$U_\omega(x,\eta\setminus x) \geq 2\delta\left[\left(\frac{\upsilon}{2\delta} - \omega\right) + (s-1)\left(\frac{a_r}{\delta} - \omega\right)\right] \geq 0,$$

see (2.10). Thus, (ii) also holds and the proof follows by the induction in $|\eta|$.

□

2.2 Evolution of states of the finite system

Here we assume that the initial state in the Fokker-Plank equation (1.34) has the property $\mu_0(\Gamma_0) = 1$, that is, the system in μ_0 is finite. Then the evolution related to (1.34) will be constructed in the Banach space of signed measures with bounded variation introduced in Section 1.4, where the generator L^μ can be defined as an unbounded linear operator and C_0-semigroup techniques can be applied.

We consider the following types of function $\chi(n)$ that defines the spaces $\mathcal{M}_\chi$, cf. (1.39),
(a) $\chi_m(n) := (1+n)^m$, $m \in \mathbb{N}$,
(b) $\chi^\kappa(n) := e^{\kappa n}$, $\kappa > 0$.

Let us then set, cf. Assumption 1 and (2.3),

$$\Psi(\eta) = M(\eta) + E^a(\eta) + \langle b\rangle|\eta|, \qquad M(\eta) := \sum_{x\in\eta} m(x) \leq m^*|\eta|, \quad (2.15)$$

and then

$$\mathcal{D} = \left\{ \mu \in \mathcal{M} : \int_{\Gamma_0} \Psi(\eta)\mu^{\pm}(d\eta) < \infty \right\}. \tag{2.16}$$

By (2.3) we have that $\Psi(\eta) \leq C|\eta|^2$ for an appropriate $C > 0$; hence, $\mathcal{M}_{\chi_2} \subset \mathcal{D}$.

Proposition 2.2.1. *For $\mu \in \mathcal{D}$, we set $L^\mu = A + B$, where*

$$(A\mu)(d\eta) = -\Psi(\eta)\mu(d\eta), \qquad (B\mu)(d\eta) = \int_{\Gamma_0} \Xi(d\eta|\xi)\mu(d\xi), \tag{2.17}$$

where for $\mathbb{A} \in \mathcal{B}(\Gamma_0)$ the measure kernel Ξ is

$$\begin{aligned} \Xi(\mathbb{A}|\xi) &= \sum_{x \in \xi} (m(x) + E^a(x, \xi \setminus x)) \mathbb{1}_{\mathbb{A}}(\xi \setminus x) \qquad (2.18) \\ &+ \sum_{x \in \xi} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \mathbb{1}_{\mathbb{A}}(\xi \setminus x \cup \{y_1, y_2\}) dy_1 dy_2, \end{aligned}$$

and $\mathbb{1}_{\mathbb{A}}$ is the indicator of $\mathbb{A}$. See also Assumption 1 and (2.15)*.*

By direct inspection one checks that L^μ satisfies $\mu(LF) = (L^\mu\mu)(F)$ holding for all $\mu \in \mathcal{D}$ and appropriate $F : \Gamma_0 \to [0, +\infty)$, see (2.1).

By a global solution of (1.34) in $\mathcal{M}$ with $\mu_0 \in \mathcal{D}$ we understand a continuous map $[0, +\infty) \ni t \mapsto \mu_t \in \mathcal{D} \subset \mathcal{M}$, which is continuously differentiable in $\mathcal{M}$ on $(0, +\infty)$ and is such that both equalities in (1.34) hold. Our main result here is as follows.

Theorem 2.2.2. *The problem in (1.34) with $\mu_0 \in \mathcal{D}$ has a unique global solution $\mu_t \in \mathcal{M}$, which has the following properties:*

(a) for each $m \in \mathbb{N}$, $\mu_t \in \mathcal{M}_{\chi_m} \cap \mathcal{P}(\Gamma_0)$ for all $t > 0$ whenever $\mu_0 \in \mathcal{M}_{\chi_m} \cap \mathcal{P}(\Gamma_0)$;

(b) for each $\kappa > 0$ and $\kappa' \in (0, \kappa)$, $\mu_t \in \mathcal{M}_{\chi^{\kappa'}} \cap \mathcal{P}(\Gamma_0)$ for all $t \in (0, T(\kappa, \kappa'))$ whenever $\mu_0 \in \mathcal{M}_{\chi^\kappa} \cap \mathcal{P}(\Gamma_0)$, where

$$T(\kappa, \kappa') = \frac{\kappa - \kappa'}{\langle b \rangle} e^{-\kappa}; \tag{2.19}$$

(c) for all $t > 0$, $\mu_t(d\eta) = R_t(\eta)\lambda(d\eta)$ whenever $\mu_0(d\eta) = R_0(\eta)\lambda(d\eta)$.

2.3 Evolution of states of the infinite system

In this section, we construct the evolution of states $\mu_0 \to \mu_t$ assuming that the system in μ_0 is infinite. In particular, we have to consider also infinite configurations for which sums as those in, e.g., (2.18) may not exist. Hence, the direct use of L^μ as linear operator in appropriated Banach spaces is rather impossible and the method developed in Section 2.2 does not work anymore. Instead, we will try to obtain evolution $\mu_0 \to \mu_t$ from the evolution $B_0 \to B_t$, where $B_0(\theta) = \mu_0(F^\theta)$ and the initial state μ_0 is taken in the set $\mathcal{P}_{\rm exp}(\Gamma)$, see (1.18) and Definition 1.2.3. In view of (1.24), the evolution $B_0 \to B_t$ can be constructed by employing correlation functions, the evolution of which will be performed in the following three steps: (a) constructing $k_0 \to k_t$ for $t < T$ (for some $T < \infty$) by solving equation (1.36); (b) proving that k_t is the correlation function of a unique $\mu_t \in \mathcal{P}_{\rm exp}(\Gamma)$; (c) continuing k_t to all $t > 0$. Using the relation (1.37) between operators L and L^Δ we obtain the following.

Proposition 2.3.1. *Operator* L^Δ *in* (1.36) *has the form*

$$
\begin{aligned}
L^\Delta &= A_1^\Delta + A_2^\Delta + B_1^\Delta + B_2^\Delta, \qquad (2.20)\\
(A_1^\Delta k)(\eta) &= -\Psi(\eta) k(\eta),\\
(A_2^\Delta k)(\eta) &= \int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} k(\eta\cup x\setminus\{y_1,y_2\}) b(x|y_1,y_2)dx,\\
(B_1^\Delta k)(\eta) &= -\int_{\mathbb{R}^d} k(\eta\cup x) E^a(x,\eta)dx,\\
(B_2^\Delta k)(\eta) &= 2\int_{(\mathbb{R}^d)^2} \sum_{y_1\in\eta} k(\eta\cup x\setminus y_1) b(x|y_1,y_2)dy_2dx,
\end{aligned}
$$

where Ψ *is given in (2.15).*

Our aim is to employ the scale of Banach spaces in which we can define linear operator acting as in (2.20). Thus we use the norm introduced in (1.42) with $w(|\eta|) = 1$ and the corresponding Banach spaces $\mathcal{K}_\alpha$, see (1.43). This yields that

$$
|k(\eta)| \leq e^{\alpha|\eta|}\|k\|_\alpha, \quad \alpha\in\mathbb{R}. \qquad (2.21)
$$

First, for a given $\alpha \in \mathbb{R}$, we define an unbounded operator $(L^\Delta_\alpha, \mathcal{D}^\Delta_\alpha)$, where

$$\mathcal{D}^\Delta_\alpha = \{k \in \mathcal{K}_\alpha : \Psi k \in \mathcal{K}_\alpha\}. \tag{2.22}$$

Thus, $A^\Delta_1 : \mathcal{D}^\Delta_\alpha \to \mathcal{K}_\alpha$. Furthermore, for each $k \in \mathcal{D}^\Delta_\alpha$, there exists $C > 0$ such that $(1 + \Psi(\eta))|k(\eta)| \leq e^{\alpha|\eta|}C$. We apply the latter fact and item (iv) of Assumption 1 to get

$$\left|(A^\Delta_2 k)(\eta)\right| \leq \frac{Ce^{-\alpha+\alpha|\eta|}}{1+\Psi(\eta)} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \beta(y_1 - y_2) \leq C\beta^* e^{-\alpha+\alpha|\eta|},$$

which means that also A^Δ_2 maps $\mathcal{D}^\Delta_\alpha$ to $\mathcal{K}_\alpha$. In a similar way, we prove that $B^\Delta_i : \mathcal{D}^\Delta_\alpha \to \mathcal{K}_\alpha$, $i = 1, 2$. Thus, the expression in (2.20) can be used to define $(L^\Delta_\alpha, \mathcal{D}^\Delta_\alpha)$. By means of the inequality (1.30) one readily proves that

$$\forall \alpha' < \alpha \qquad \mathcal{K}_{\alpha'} \subset \mathcal{D}^\Delta_\alpha. \tag{2.23}$$

Proposition 2.3.2. *For $\alpha' < \alpha$ and L^Δ as in* (2.20)*, we define a bounded linear operator $L^\Delta_{\alpha\alpha'} : \mathcal{K}_{\alpha'} \to \mathcal{K}_\alpha$, the operator norm of which satisfies*

$$\|L^\Delta\|_{\alpha\alpha'} \leq \left(4\frac{m^* + \langle b\rangle + a^* + \beta^* e^{-\alpha'}}{e^2(\alpha-\alpha')^2} + \frac{\langle a\rangle e^{\alpha'} + 2\langle b\rangle}{e(\alpha - \alpha')} \right). \tag{2.24}$$

Then, in view of (2.23), we have that each $k \in \mathcal{K}_{\alpha'}$ lies in $\mathcal{D}^\Delta_\alpha$, and

$$L^\Delta_{\alpha\alpha'} k = L^\Delta_\alpha k. \tag{2.25}$$

In the sequel, we consider two types of operators with the action as in (2.20): (a) unbounded operators $(L^\Delta_\alpha, \mathcal{D}(L^\Delta_\alpha))$, $\alpha \in \mathbb{R}$, with domains as in (2.22); (b) bounded operators $L^\Delta_{\alpha\alpha'}$ satisfying (2.24) and related to the operator L^Δ_α by (2.25), i.e., $L^\Delta_{\alpha\alpha'}$ can be considered as the restriction of L^Δ_α to $\mathcal{K}_{\alpha'}$.

Let us fix some $\alpha_1 \in \mathbb{R}$. Then take $\alpha_2 > \alpha_1$ and consider the following Cauchy problem in space $\mathcal{K}_{\alpha_2}$, cf. (1.43),

$$\dot{k}_t = L^\Delta_{\alpha_2} k_t, \qquad k_t|_{t=0} = k_0 \in \mathcal{K}_{\alpha_1}. \tag{2.26}$$

By its solution on a time interval $[0, T)$ we mean a continuous (in $\mathcal{K}_{\alpha_2}$) map $[0, T) \ni t \mapsto k_t \in \mathcal{D}^\Delta_{\alpha_2}$, which is continuously differentiable on

$(0,T)$ and satisfies both equalities in (2.26). For $\alpha, \alpha' \in \mathbb{R}$ such that $\alpha' < \alpha$ and for $\upsilon \geq 0$ as in Proposition 2.1.2, we set

$$T(\alpha, \alpha') = \frac{\alpha - \alpha'}{2\langle\beta\rangle + \upsilon + \langle a\rangle e^{\alpha}}. \tag{2.27}$$

Lemma 2.3.3. *Let ω and υ be as in Proposition 2.1.2. Then for each $\alpha_1 > -\log\omega$ and an arbitrary $k_0 \in \mathcal{K}_{\alpha_1}$, the problem in (2.26) has a unique solution $k_t \in \mathcal{D}^{\Delta}_{\alpha_2}$ on the time interval $[0, T(\alpha_2, \alpha_1))$.*

In contrast to the case of finite configurations described in Theorem 2.2.2, the construction of a C_0-semigroup that solves (2.26) is rather hopeless. In view of this, we will try to find the solution of the equation (2.26) in the following steps:

(i) the operator L^{Δ} will be written in the form $L^{\Delta} = A^{\Delta}_{\upsilon} + B^{\Delta}_{\upsilon}$ in such a way that $A^{\Delta}_{\upsilon} := A^{\Delta}_{1,\upsilon} + A^{\Delta}_{2}$ will be used to construct a certain (sun-dual) C_0-semigroup in $\mathcal{K}_{\alpha_2}$;

(ii) this semigroup and $B^{\Delta}_{\upsilon} := B^{\Delta}_{1} + B^{\Delta}_{2,\upsilon}$, will be used to construct the family of operators $\{Q_{\alpha\alpha'}(t) : t \in [0, T(\alpha, \alpha'))\}$, see (2.27), such that $Q_{\alpha\alpha'}(t) \in \mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_{\alpha})$ and $k_t = Q_{\alpha_2\alpha_1}(t)k_0$ is the solution of (2.26). By $\mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_{\alpha})$ we denote the Banach space of all bounded linear operators acting from $\mathcal{K}_{\alpha'}$ to $\mathcal{K}_{\alpha}$

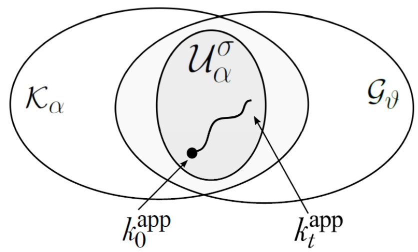
Our next aim is to show that the solution obtained in Lemma 2.3.3 has the property $k_t = k_{\mu_t}$ for a unique $\mu_t \in \mathcal{P}_{\exp}(\Gamma)$. We call this *identification* since it allows us to identify the mentioned solutions as correlation functions of sub-Poissonian states.

Recall that υ and ω appear in Proposition 2.1.2 and $\mathcal{K}^{\star}_{\alpha}$ is defined in (1.44).

Lemma 2.3.4 (Identification)**.** *For each $\alpha_2 > \alpha_1 > -\log\omega$, it follows that $Q_{\alpha_2\alpha_1}(t) = Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{\upsilon}) : \mathcal{K}^{\star}_{\alpha_1} \to \mathcal{K}^{\star}_{\alpha_2}$ for all $t \in [0, \tau(\alpha_2, \alpha_1)]$ with $\tau(\alpha_2, \alpha_1) = T(\alpha_2, \alpha_1)/3$.*

The proof of this lemma consists in the following steps:

(i) constructing an approximation k^{app}_t of $k_t = Q_{\alpha_2\alpha_1}(t)k_0$, $k_0 \in \mathcal{K}^{\star}_{\alpha_1}$, such that $\langle\!\langle G, k^{\text{app}}_t \rangle\!\rangle \geq 0$ for all $G \in B^{\star}_{\text{bs}}(\Gamma_0)$, which includes also constructing an auxiliary model;



(ii) proving that $\langle\!\langle G, k_t^{\rm app}\rangle\!\rangle \to \langle\!\langle G, k_t\rangle\!\rangle$ as the approximations are eliminated.

The figure above provides an illustration to the idea of how to realize step (i). The origin of the inequality in question is in (1.28) and (1.29). To relate k_t with a positive measure one uses local approximations of μ_0, the densities of which (not necessarily normalized) evolve $R_0^{\rm app} \to R_t^{\rm app}$ in L_1-like spaces according to Theorem 2.3.5. These approximations are tailored in such a way that the corresponding correlation functions (1.26) (that have the desired property by construction) also evolve $q_0^{\rm app} \to q_t^{\rm app}$ in L_1-like spaces $\mathcal{G}_\vartheta$. The technique developed in Sect. 4.1.5 allows for proving that $\langle\!\langle G, k_t^{\rm app}\rangle\!\rangle$ converges to $\langle\!\langle G, k_t\rangle\!\rangle$ only if $k_t^{\rm app} = Q_{\alpha\alpha_0}(t)q_0^{\rm app}$. That is, at this stage there is no connection between the evolutions $q_0^{\rm app} \to q_t^{\rm app}$ and $q_0^{\rm app} \to k_t^{\rm app}$ as they take place in (different) spaces, $\mathcal{G}_\vartheta$ and $\mathcal{K}_\alpha$, respectively. It turns out, that these spaces have an intersection $\mathcal{U}_\alpha^\sigma$ constructed with the help of some objects dependent on a parameter $\sigma > 0$. To employ this fact we use auxiliary models (indexed by σ), for which we prove that both evolutions $q_0^{\rm app} \to q_t^{\rm app}$ and $q_0^{\rm app} \to k_t^{\rm app}$ take place in $\mathcal{U}_\alpha^\sigma$ and thus coincide. That is $q_t^{\rm app} = k_t^{\rm app}$ for $t \leq \tau$ with some positive τ, that yields the desired positivity of $k_t^{\rm app}$. Then step (ii) includes also taking the limit $\sigma \to 0^+$.

In Section 1.4, for $\alpha \in \mathbb{R}$, we defined spaces $\mathcal{K}_\alpha^\star$ and $\mathcal{K}_\alpha^+$, see (1.44) and (1.45). Since the spaces set in (1.43) form an ascending scale, we have that $k \in \mathcal{K}_{\alpha_0}$ lies in all $\mathcal{K}_\alpha$ with $\alpha > \alpha_0$. Our main result is given by the following statement.

Theorem 2.3.5. *For each $\mu_0 \in \mathcal{P}_{\rm exp}(\Gamma_0)$, one can choose real α_0 and c such that $k_{\mu_0} \in \mathcal{K}_{\alpha_0}$ and there exists a unique map $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}_{\alpha_t}^\star$ with $\alpha_t = \alpha_0 + ct$ and $k_0 = k_{\mu_0}$, which has the following properties:*

(i) For each $T > 0$, and all $t \in [0,T)$, the map

$$[0,T) \ni t \mapsto k_t \in \mathcal{K}_{\alpha_t} \subset \mathcal{D}(L^{\Delta}_{\alpha_T}) \subset \mathcal{K}_{\alpha_T}$$

is continuous on $[0,T)$ and continuously differentiable on $(0,T)$ in $\mathcal{K}_{\alpha_T}$.

(ii) For all $t \in (0,T)$ it satisfies

$$\dot{k}_t = L^{\Delta}_{\alpha_T} k_t.$$

Corollary 2.3.6. *Let $k_t \in \mathcal{K}^{\star}_{\alpha_t}$, $t \geq 0$, be as in Theorem 2.3.5, and then $\mu_t \in \mathcal{P}_{\exp}(\Gamma)$ be the measure corresponding to this k_t according to Proposition 1.2.5. Then the map $t \mapsto \mu_t$ is such that*

1. *for each compact Λ and $t \geq 0$, μ_t^{Λ} lies in the domain $\mathcal{D} \subset \mathcal{M}$ defined in (2.16);*

2. *for each $\theta \in \Theta$, the map $[0,+\infty) \ni t \mapsto \mu_t(F^{\theta})$ is continuous and continuously differentiable on $(0,+\infty)$ and the following holds*

$$\frac{d}{dt}\mu_t(F^{\theta}) = (L^* \mu_t^{\Lambda_\theta})(F^{\theta}) = \langle\!\langle e(\theta,\cdot), L^{\Delta}_{\alpha_T} k_t \rangle\!\rangle, \tag{2.28}$$

where the latter equality holds for all $T > t$, see (1.24) and (1.27).

The main part of the proof of these statements contained the following: (a) constructing the evolution $k_{\mu_0} \mapsto k_t$ for t belonging to a bounded interval (Lemma 2.3.3); (b) proving that k_t belongs to $\mathcal{K}^{\star}_{\alpha}$ with an appropriate α, which with the help of Proposition 1.2.5 will allow to associate it with a unique $\mu \in \mathcal{P}_{\exp}(\Gamma)$ (Identification Lemma 2.3.4); (c) proving that k_t lies in $\mathcal{K}_{\alpha_t}$ on the mentioned time interval, which will be used to continue k_t to all $t > 0$.

2.4 Mesoscopic description

Along with the microscopic theory based on the equation (1.33), we also consider its relation with phenomenological models through the mesoscopic description of the system obtained by the so-called Vlasov scaling, see [26] and [11]. Here, the scale is described by a parameter

$\varepsilon \in (0,1]$, in such a way that $\varepsilon = 1$ corresponds to the microscopic level. In the scaling limit $\varepsilon \to 0$, the corpuscular structure of the system disappears and it turns into a medium described by the density. Having in mind, that any Poissonian state π_ϱ is fully characterized by the density, see (1.17), we introduce the following definition, cf. [8,11].

Definition 2.4.1. A state $\mu \in \mathcal{P}_{\exp}(\Gamma)$ is said to be Poisson - approximable if: (i) there exist $\alpha \in \mathbb{R}$ and $\varrho \in L^\infty(\mathbb{R}^d)$ such that both k_μ and k_{π_ϱ} lie in $\mathcal{K}_\alpha$; (ii) for each $\varepsilon \in (0,1]$, there exists $q^{(\varepsilon)}$ such that $q^{(1)} = k_\mu$ and $\|q^{(\varepsilon)} - k_{\pi_\varrho}\|_\alpha \to 0$ as $\varepsilon \to 0$.

Our aim is to show that the evolution $\mu_0 \mapsto \mu_t$ from Theorem 2.3.5 preserves the Poisson-approximability defined above, respecting the density function ϱ_t obtained from the kinetic equation

$$\frac{d}{dt}\varrho_t = -m\varrho_t - (a * \varrho_t)\varrho_t + (b * \varrho_t), \qquad \varrho_t|_{t=0} = \varrho_0. \tag{2.29}$$

where for $\mathbf{B}$ being either β or a we set

$$(\mathbf{B} * \varrho_t)(x) = \int_{\mathbb{R}^d} \mathbf{B}(x-y)\varrho_t(y)dy.$$

Theorem 2.4.2. *Let k_t be the solution of equation* (1.36) *with initial condition $k_t|_{t=0} = k_{\mu_0} \in \mathcal{K}_{\alpha_0}$ obtained in Theorem 2.3.5 and ϱ_t be the solution of* (2.29). *Let also μ_0 be Poisson-approximable by the measure π_ϱ, i.e. there exist $\alpha_0 \in \mathbb{R}$ and $q_0^{(\varepsilon)}$ such that $k_{\mu_0} = q_0^{(1)}$ and $\|k_{\pi_\varrho} - q_0^{(\varepsilon)}\|_{\alpha_0} \to 0$ as $\varepsilon \to 0$. Then there exists $\alpha^* > \alpha_0$ and $T > 0$ such that the following holds*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|q_t^{(\varepsilon)} - k_{\pi_{\varrho_t}}\|_{\alpha^*} = 0.$$

Chapter 3

Free branching in the continuum

In this chapter we discuss the model of fragmentation of an infinite system of point particles introduced in [49] and study in [40] placed in locally compact Polish space X. Here, each 'particle' produces at random a finite 'cloud' (possibly empty) of new particles, and disappears afterwards. In contrast to the fission - death model presented in Chapter 2, here particles undergo the free branching. The main result of this chapter is the construction of the solutions of both Kolmogorov and Fokker - Planck evolution equations.

3.1 The model

The model is determined by the Kolmogorov equation (1.33), where the generator L has the following form

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\Gamma} [F(\gamma \setminus x \cup \xi) - F(\gamma)] b_x(d\xi). \tag{3.1}$$

The branching kernel b_x is a map $(X, \mathcal{N}) \ni (x, \Xi) \mapsto b_x(\Xi) \in [0,1]$ such that each b_x is a probability measure on $\mathcal{N}$ and $x \mapsto b_x(\Xi)$ is measurable for each $\Xi \in \mathcal{B}(\mathcal{N})$. We assume that, for each $x \in X$, b_x is a probability measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$. Its correlation measure β_x is defined by the integrals

$$\int_{\Gamma_0}\left(\sum_{\eta\subset\xi} G(\eta)\right) b_x(d\xi) = \int_{\Gamma_0} G(\eta)\beta_x(d\eta) \tag{3.2}$$

$$= G(\varnothing) + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{X^n} G^{(n)}(x_1,\ldots,x_n)\beta_x^{(n)}(dx_1,\ldots,dx_n),$$

with G running through a separating family. As such one, we can take the family of all bounded Borel functions $G:\Gamma_0\to\mathbb{R}$ such that $G^{(n)}\equiv 0$ whenever $n>N$, see (1.15). For $n\in\mathbb{N}_0$, we set $\Gamma^n=\{\xi\in\Gamma_0:|\xi|=n\}$. Then $b_x(\Gamma^n)$ is the probability of producing n offsprings by the particle located at x. Note that $\delta(x):=b_x(\Gamma^0)$ is just the death probability, and

$$n(x):=\int_{\Gamma_0}|\xi| b_x(d\xi) = \sum_{n=1}^{\infty} n b_x(\Gamma_n) = \beta_x^{(1)}(X) \tag{3.3}$$

is the expected number of offsprings of the particle located at x.

For ϕ as in (1.13), we define

$$(\Phi\phi)(x) = \int_{\Gamma_0}\left(\prod_{y\in\xi}\phi(y)\right) b_x(d\xi). \tag{3.4}$$

Clearly, $0\le(\Phi\phi)(x)\le 1$ for each $x\in X$. Recall that we use ψ in (1.5) in defining tempered configurations.

Assumption 2. The probability kernel b is subject to the following conditions:

(i) $\Phi\phi\in C_{\mathrm{b}}(X)$ for each ϕ as in (1.13);

(ii) $\sup_{x\in X} n(x) =: n_* < \infty$;

(iii) the death probability δ satisfies $\delta(x)\ge 1-\psi(x)\ge\delta_*>0$, holding for all $x\in X$;

(iv) there exists $m>0$ such that, for all $x\in X$, the following holds

$$\int_X \psi(y)\beta_x^{(1)}(dy)\le n(x)m\psi(x). \tag{3.5}$$

By (1.13), (3.5) and Jensen's inequality we get

$$-\log(\Phi\phi)(x) \le \int_{\Gamma_0}\left(-\log\prod_{y\in\xi}\phi(y)\right)b_x(d\xi)$$
$$= \int_X g(x)\psi(y)\beta_x^{(1)}(dy) \le \left(\sup_{x\in X} g(x)\right)n(x)m\psi(x).$$

Note that by (1.12) and (3.4) it follows that

$$(\Phi\phi)(x) = \int_{\Gamma_0} G^g(\xi)b_x(d\xi) = \mathfrak{L}_{b_x}(g).$$

Then assumption (i) can be reformulated as the continuity of the map $X \ni x \mapsto \mathfrak{L}_{b_x}(g) \in \mathbb{R}$, holding for all $g \in C_{\rm b}^{+}(X)$. The remaining assumptions are supposed to control the production of new particles, of which (ii) and (iii) are related to the properties of $b_x(\Gamma^n)$, $n \in \mathbb{N}_0$, see (3.3). In general, (ii) and (iii) may be quite independent as the choice of $\delta(x)$ leaves enough possibilities to modify $n(x)$. However, in some cases, $\delta(x)$ and $n(x)$ can be expressed through each other. For instance, if b_x is a Poisson measure, then $\delta(x) = e^{-n(x)}$. In this case, (ii) follows by (iii) with $n_* = -\log\delta_*$. The role of (iv) is to control the dispersal of offsprings, and thus the nonlocality of the process. To illustrate its role, we take $X = \mathbb{R}$ and

$$\bar{\beta}_x(dy) := \beta_x^{(1)}(dy)/n(x) = \frac{1}{2r}\mathbb{I}_{[x-r,x+r]}(y)dy, \qquad r > 0.$$

Then $\psi(y) = e^{-\alpha|y|}$ satisfies

$$\int_X \psi(y)\bar{\beta}_x(dy) \le \left(\frac{e^{\alpha r} - e^{-\alpha r}}{2\alpha r}\right)\psi(x),$$

which yields (3.5) with $m = \sinh(\alpha r)/\alpha r > 1$. Note that this m can be made arbitrarily close to one by taking small enough either r or α. The former corresponds to a short dispersal, whereas by choosing small α one makes Γ^ψ – and hence $\mathcal{P}(\Gamma^\psi)$ – smaller, cf. (1.6) and (1.10).

3.2 The Kolmogorov Operator

3.2.1 Solving the log-Laplace equation

Our aim now is to prepare solving (1.34), which we begin by making precise the definition of the Kolmogorov operator. To this end, however, we have to study the following nonlinear equation. For $\phi \in C_{\rm b}(X)$ and $x \in X$ define $\theta(x) := 1 - \phi(x)$ and then

$$C_\psi(X) = \{\phi \in C_{\rm b}(X) : \forall x \in X \quad 0 < c_\phi \psi(x) \leq 1 - \phi(x) \leq 1 - \delta(x)\}\,, \tag{3.6}$$

i.e., each $\theta = 1 - \phi$ has its own lower bound, whereas the upper bound is one and the same for all such functions. Notably, by item (iii) of Assumption 2 it follows that each $\phi \in C_\psi(X)$ satisfies

$$\phi(x) \geq 1 - \psi(x) \geq \delta_*. \tag{3.7}$$

Let us prove that $(\Phi\phi)(x) \geq \delta(x)$, holding for each $\phi \in C_\psi(X)$. Indeed, by (3.4) we have

$$\begin{aligned}(\Phi\phi)(x) &= b_x(\Gamma^0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} \prod_{j=1}^{n} \phi(y_j) b_x^{(n)}(dy_1, \dots dy_n) \\ &\geq \delta(x) \geq 1 - \psi(x) \geq \delta_*,\end{aligned} \tag{3.8}$$

see item (iii) of Assumption 2. Moreover, by (1.13) and (3.7) it follows that

$$g(x) \leq -\frac{1}{\psi(x)} \log(1 - \psi(x)) = \sum_{n=1}^{\infty} \frac{[\psi(x)]^{n-1}}{n} \leq -\frac{\log(1-\delta_*)}{1-\delta_*} =: g_*. \tag{3.9}$$

Both (3.7) and (3.9) holding for all $x \in X$.

Now for $T > 0$, by $\mathcal{C}^T$ we denote the Banach space of continuous maps $[0,T] \ni t \mapsto \varphi_t \in C_{\rm b}(X)$, equipped with the norm

$$\|\varphi\|_T = \sup_{t\in[0,T]} \sup_{x\in X} |\varphi_t(x)|. \tag{3.10}$$

We also set

$$\mathcal{C}_\psi^T = \{\varphi \in \mathcal{C}^T : \varphi_t \in C_\psi(X),\ t \in [0,T]\},$$

and

$$\mathcal{C}_\psi^T(\phi) = \{\varphi \in \mathcal{C}_\psi^T : \varphi_0 = \phi\}, \qquad \phi \in C_\psi(X).$$

Obviously, the latter is a closed subset of $\mathcal{C}^T$. Thereafter, we define

$$(K\varphi)_t(x) = \varphi_0(x)e^{-t} + \int_0^t e^{-(t-s)}(\Phi\varphi_s)(x)ds. \tag{3.11}$$

Proposition 3.2.1. *Let n_* introduced in Assumption 2 and T satisfy $n_*(1 - e^{-T}) < 1$. Then for each $\phi \in C_\psi(X)$, the map K has a unique fixed point $\varphi \in \mathcal{C}_\psi^T(\phi)$.*

Now we consider the following nonlinear equation

$$\frac{\partial}{\partial t}\phi_t(x) = -\phi_t(x) + (\Phi\phi_t)(x), \qquad \phi_0 = \phi. \tag{3.12}$$

In a sense, it is a nonlocal analog of the log-Laplace equation – a standard object in the theory of branching processes, see, e.g., [21, page 61]. By a solution of (3.12) we will understand a map $\mathbb{R}_+ \ni t \mapsto \phi_t \in C_{\rm b}(X)$ which is everywhere continuously differentiable and satisfies both equalities mentioned therein.

Lemma 3.2.2. *For each $\phi \in C_\psi(X)$, (3.12) has a unique solution $t \mapsto \phi_t \in C_\psi(X)$ which satisfies*

$$c_\phi(t)\psi(x) := e^{-t}c_\phi\psi(x) \le 1 - \phi_t(x) =: \theta_t(x) \le \psi(x). \tag{3.13}$$

For $n_* < 1$, this solution tends to $\phi_\infty(x) \equiv 1$ in the norm of $C_{\rm b}(X)$.

Remark 3.2.3. By Lemma 3.2.2 and its proof (see (4.120)), it follows that the solution (3.12) – which is a nonlinear Cauchy problem in the Banach space $C_{\rm b}(X)$ – is given by a continuous semigroup of nonlinear operators, say $\{\rho_t\}_{t\ge 0}$, in the form $\phi_t = \rho_t(\phi_0)$, $\phi_t \in C_\psi(X)$. If one writes $\phi_t \in C_\psi(X)$ in the form $\phi_t(x) = \exp(-g_t(x)\psi((x))$, see (1.13), then the map $g \mapsto g_t$ also has the flow property, and hence can be considered as the continuous semigroup of nonlinear operators $\{r_t\}_{t\ge 0}$ such that $g_t = r_t(g_0)$. It is known as the log-Laplace semigroup, see [21, page 60].

We conclude this subsection by establishing the following useful properties of the solution ϕ_t.

Lemma 3.2.4. *Let $\phi_t = 1 - \theta_t$ be the solution as in Lemma 3.2.2. Then, for each $t \geq 0$, $u > 0$ and all $x \in X$, the following holds*

$$(a) \qquad |\phi_{t+u}(x) - \phi_t(x)| = |\theta_{t+u}(x) - \theta_t(x)| \leq 2u\psi(x), \tag{3.14}$$

$$(b) \qquad |g_{t+u}(x) - g_t(x)| \leq 2u/\delta_*,$$

$$(c) \qquad |(\Phi\phi_{t+u})(x) - (\Phi\phi_t)(x)| \leq 2un_* m\psi(x).$$

3.2.2 Basic estimates

In defining L, we employ a number of estimates which we derive now. To simplify our notations, for $\phi \in C_\psi(X)$ we set, see (1.13),

$$F^\phi(\gamma) = \prod_{x\in\gamma} \phi(x) = \exp\left(- \sum_{x\in\gamma} g(x)\psi(x) \right) = G^g(\gamma), \tag{3.15}$$

where $G^g(\gamma)$ is as in (1.12).

Proposition 3.2.5. *Let F^ϕ be as in (3.15) with $\phi \in C_\psi(X)$, see (3.6). Then, for each $\gamma \in \Gamma^\psi$, the following holds*

$$\left| LF^\phi(\gamma) \right| \leq \frac{2}{e\delta_* c_\phi}, \tag{3.16}$$

where c_ϕ defines the lower bound in (3.6). By (3.16) it follows that $LF^\phi \in C_{\mathrm{b}}(\Gamma^\psi)$.

As in (3.6) we do not restrict the lower bounds, the right-hand side of (3.16) can be arbitrarily large for small enough c_ϕ.

Lemma 3.2.6. *For a given $\phi \in C_\psi(X)$, let ϕ_t be the solution of (3.12), see Lemma 3.2.2. Then, for each $t \geq 0$, $u > 0$ and $\gamma \in \Gamma^\psi$, the following holds*

$$\left| F^{\phi_{t+u}}(\gamma) - F^{\phi_t}(\gamma) \right| \leq \frac{2ue^{t+u}}{e\delta_* c_\phi}.$$

Lemma 3.2.7. *Let ϕ, t and u be as in Lemma 3.2.6. Then there exists $C_\phi > 0$ such that, for all $\gamma \in \Gamma^\psi$, the following holds*

$$\left| (LF^{\phi_{t+u}})(\gamma) - (LF^{\phi_t})(\gamma) \right| \leq C_\phi u e^{2(t+u)}. \tag{3.17}$$

3.2.3 The domain and the resolvent

We begin by introducing

$$E(\Gamma^\psi) = \overline{E^0(\Gamma^\psi)}, \qquad E^0(\Gamma^\psi) := \text{l.s.}\{F^\phi : \phi \in C_\psi(X)\}, \tag{3.18}$$

where l.s. denotes linear span and the closure is taken in the Banach space $C_\mathrm{b}(\Gamma^\psi)$, i.e., in

$$\|F\| := \sup_{\gamma\in\Gamma^\psi} |F(\gamma)|.$$

With this norm, $E(\Gamma^\psi)$ becomes a separable Banach space.

Remark 3.2.8. The set $E^0(\Gamma^\psi)$, and hence also $E(\Gamma^\psi)$, have all the properties stated in Proposition 1.1.7. This follows by the fact that the family $\{G^v : v \in \mathcal{V}\}$ mentioned therein is a subset of $E^0(\Gamma^\psi)$, see (3.15).

Since the map $t \mapsto F^{\phi_t} \in C_\mathrm{b}(\Gamma^\psi)$ is continuous and bounded (by one), for each $\lambda > 0$ the Bochner integral, see [3, Sect. 1.1, pages 6–15]

$$F^\phi_\lambda = \int_0^{+\infty} e^{-\lambda t} F^{\phi_t} dt, \qquad \phi \in C_\psi(X), \tag{3.19}$$

is the limit of the corresponding Riemannian integral sums. Hence, $F^\phi_\lambda \in E(\Gamma^\psi)$ for each $\lambda > 0$ and $\phi \in C_\psi(X)$. Naturally, in (3.19) ϕ_t stands for the solution of (3.12), see Lemma 3.2.2. Furthermore, by (3.16) and (3.17) it follows that the map $t \mapsto LF^{\phi_t} \in C_\mathrm{b}(\Gamma^\psi)$ is continuous and absolutely $e^{-\lambda t}dt$-integrable for all $\lambda > 1$. This observation leads us to the following fact.

Lemma 3.2.9. *For each $\phi \in C_\psi(X)$ and $\lambda > 1$, the following holds*

$$LF^\phi_\lambda = \int_0^{+\infty} e^{-\lambda t} LF^{\phi_t} dt = -F^\phi + \lambda F^\phi_\lambda. \tag{3.20}$$

Proof. The first equality in (3.20) follows by the absolute integrability of $t \mapsto LF^{\phi_t} \in C_\mathrm{b}(\Gamma^\psi)$ as just discussed. The second one is obtained by integrating by parts. □

Set

$$\mathcal{D}^0(L) = \text{l.s.}\{F^\phi_\lambda : \lambda > .1,\ \phi \in C_\psi(X)\}. \tag{3.21}$$

As has just discussed, we know that

$$\mathcal{D}^0(L) \subset E(\Gamma^\psi) \quad \text{and} \quad L : \mathcal{D}^0(L) \to E(\Gamma^\psi),$$

where the latter follows by (3.20). In view of this, we can introduce

$$\|F\|_L = \|F\| + \|LF\|, \qquad F \in \mathcal{D}^0(L), \tag{3.22}$$

i.e., $\|\cdot\|_L$ is the corresponding graph-norm. Thereby, we define

$$\mathcal{D}(L) = \overline{\mathcal{D}^0(L)}^L, \tag{3.23}$$

where the closure is taken in the norm set in (3.22).

Lemma 3.2.10. *It follows that $E^0(\Gamma^\psi) \subset \mathcal{D}(L)$. Therefore, $\mathcal{D}(L)$ has all the properties mentioned in Proposition 1.1.7.*

Corollary 3.2.11. *The operator $(L, \mathcal{D}(L))$ is closed and densely defined in the Banach space $E(\Gamma^\psi)$. Its resolvent set contains $(0, +\infty)$.*

3.3 The Result

3.3.1 Solving the Kolmogorov equation

Now we are prepared to solve the Kolmogorov equation (1.33), which we define as a Cauchy problem in the Banach space $E(\Gamma^\psi)$, see (3.18). For a given $F \in \mathcal{D}(L)$, by its solution we understand a map $[0, +\infty) \ni t \mapsto F_t \in \mathcal{D}(L)$, continuously differentiable in $E(\Gamma^\psi)$, such that both equalities in (1.33) hold true. That is, we are going to deal with classical solutions of (1.33), cf. [3, page 108].

Theorem 3.3.1. *For each $F_0 \in \mathcal{D}(L)$, the Cauchy problem (1.33) has a unique classical solution $t \mapsto F_t \in \mathcal{D}(L)$. For $n_* < 1$, this solution satisfies $F_t(\gamma) \to F_\infty(\gamma)$, where $F_\infty(\gamma) \equiv 1$ and the convergence is to hold for each $\gamma \in \Gamma^\psi$.*

Since $F^\phi \in \mathcal{D}(L)$, see Lemma 3.2.10, it might be quite natural to expect that the map $t \mapsto F^{\phi_t}$ is a solution of the Kolmogorov equation with the initial condition F^ϕ. It is indeed the case. To show this, we write

$$\lambda F_\lambda^{\phi_t} = S(t) \lambda F_\lambda^\phi,$$

and pass here to the limit $\lambda \to +\infty$. Since $S(t)$ is a bounded operator and $\|\lambda F^\phi_\lambda - F^\phi\|_L \to 0$ as $\lambda \to +\infty$, we can do this and obtain the conclusion in question, i.e.,

$$F^{\phi_t} = S(t)F^\phi, \qquad t \geq 0, \quad \phi \in C_\psi(X). \tag{3.24}$$

3.3.2 Solving the Fokker-Planck equation

Now we may turn to the probabilistic part of the topic. Here it would be reasonable to recall that we use probability measures on Γ^ψ as states of the studied branching system.

Definition 3.3.2. By a solution of the Fokker-Planck equation (1.34) we understand a map $\mathbb{R}_+ \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma^\psi)$ possessing the following properties: (a) for each $F \in B_{\rm b}(\Gamma^\psi)$, the map $\mathbb{R}_+ \ni t \mapsto \mu_t(F) \in \mathbb{R}$ is measurable; (b) the equality in (1.34) holds for all $F \in \mathcal{D}(L)$, where the latter is defined in (3.23).

Theorem 3.3.3. *For each $\mu_0 \in \mathcal{P}(\Gamma^\psi)$, the Fokker-Planck equation (1.34) has a unique solution in the sense of the definition given above. Moreover, this solution is weakly continuous, i.e., $\mu_t \Rightarrow \mu_s$ as $t \to s \in \mathbb{R}_+$. In the subcritical case, $\mu_t \Rightarrow \mu_\infty$ as $t \to +\infty$, where μ_∞ is the measure supported on the singleton subset of Γ^ψ consisting of the empty configuration, i.e., $\mu_\infty(\Gamma^0) = 1$*

The proof of this theorem is based, in particular, on the following fact.

Lemma 3.3.4. *Let a map $t \mapsto \mu_t$ satisfy condition (b) of Definition 3.3.2. Then it also satisfies (a), and hence is a solution of (1.34).*

The proof of this statement in turn is based on the following result, which has its own value.

Proposition 3.3.5. *Let $t \mapsto \mu_t \in \mathcal{P}(\Gamma^\psi)$ satisfy (1.34) for all t_1, t_2 and $F \in \mathcal{D}(L)$. Then, for each $F \in \mathcal{D}^0(L)$, the map $t \mapsto \mu_t(F) \in \mathbb{R}$ is Lipschitz-continuous. The same is true also for $F \in E^0(\Gamma^\psi)$, see (3.18).*

A direct consequence of Theorem 3.3.3 is the existence of a Markov process with values in Γ^ψ, that may be constructed by means of the

Markov transition function p^t_γ, see [24, pages 156, 157], determined by its values on $\{F^\phi : \phi \in C_\psi(X)\}$, cf. Remark 3.2.8. These values are given by the following formula

$$p^\gamma_t(F^\theta) = F^{\phi_t}(\gamma), \qquad \gamma \in \Gamma^\psi, \quad \phi_t = S(t)\phi,$$

see (4.134). Then the uniqueness stated in Theorem 3.3.3 can be used to prove that such a process is unique up to modifications. Another observation is that, in our model, branching is the only evolutionary act, whereas papers on branching in finite particle systems, e.g., [5, 14, 22, 23], assume more such acts, e.g., diffusion in X. Such generalizations can also be done in our setting.

Chapter 4

Proofs

In this chapter, we present the proofs of the most important theorems that have appeared in previous chapters.

4.1 Proofs regarding the fission - death model

4.1.1 Proof of Proposition 2.2.1

To prove that operator L^{μ} can be written down as (2.17) we use the rule (1.35) which transform operator L in (2.1) into operator in question. Hence, the first summand in (2.1), using properties of Dirac measure, is convert into the following.

$$\begin{aligned}
&\int_{\Gamma_0} \sum_{x\in\gamma} \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y) \right) \left[F(\gamma \setminus x) - F(\gamma) \right] \mu(d\gamma) \\
&\qquad = \int_{\Gamma_0} \sum_{x\in\gamma} \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y) \right) F(\gamma \setminus x) \mu(d\gamma) \\
&\qquad\quad - \int_{\Gamma_0} \sum_{x\in\gamma} \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y) \right) F(\gamma) \mu(d\gamma) \\
&\qquad = \int_{\Gamma_0} \int_{\Gamma_0} \sum_{x\in\gamma} \left(m(x) + \sum_{y\in\gamma\setminus x} a(x-y) \right) F(\eta) \delta_{\gamma\setminus x}(d\eta) \mu(d\gamma)
\end{aligned}$$

$$-\int_{\Gamma_0}\sum_{x\in\gamma}\left(m(x)+\sum_{y\in\gamma\setminus x}a(x-y)\right)F(\gamma)\mu(d\gamma).$$

Then the first part of operator L^μ has the form

$$\begin{aligned}(L_1^\mu\mu)(d\eta) &= \int_{\Gamma_0}\sum_{x\in\gamma}\left(m(x)+\sum_{y\in\gamma\setminus x}a(x-y)\right)\mathbb{I}_{d\eta}(\gamma\setminus x)\mu(d\gamma)\\ &\quad -\int_{\Gamma_0}\sum_{x\in\eta}\left(m(x)+\sum_{y\in\eta\setminus x}a(x-y)\right)\mu(d\eta) \qquad (4.1)\end{aligned}$$

We calculate the second part of the operator in the similar way, and hence, the second part of L^μ is

$$\begin{aligned}(L_2^\mu\mu)(d\eta) &= \int_{\Gamma_0}\sum_{x\in\gamma}\int_{(\mathbb{R}^d)^2} b(x|y_1,y_2)\mathbb{I}_{d\eta}(\gamma\setminus x\cup\{y_1,y_2\})dy_1dy_2\mu(d\gamma)\\ &\quad -\int_{\Gamma_0}\sum_{x\in\eta}\int_{(\mathbb{R}^d)^2} b(x|y_1,y_2)dy_1dy_2\mu(d\eta) \qquad (4.2)\end{aligned}$$

By (4.1) and (4.2) we get $L^\mu = L_1^\mu + L_2^\mu = A+B$, where A and B are given in (2.17). □

4.1.2 Proof of Theorem 2.2.2

To prove Theorem 2.2.2 we use the Thieme-Voigt perturbation technique [50], the basic elements of which we present in Section 1.5 in the form adapted to our purposes.

Proof of Theorem 2.2.2. Along with $L^\mu = A+B$ defined in (2.15), (2.17) and (2.18) we consider the corresponding operator in $\mathcal{R}$, defined accordingly to the rule (1.38). Then $L^\dagger = A^\dagger + B^\dagger$ with

$$\begin{aligned}(A^\dagger R)(\eta) &= -\Psi(\eta)R(\eta), \qquad (4.3)\\ (B^\dagger R)(\eta) &= \int_{\mathbb{R}^d}\left(m(x)+E^a(x,\eta)\right)R(\eta\cup x)dx\\ &\quad +\int_{\mathbb{R}^d}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1} b(x|y_1,y_2)R(\eta\cup x\setminus\{y_1,y_2\})dx,\end{aligned}$$

the domain of which is, cf. (2.16),

$$\mathcal{D}^\dagger = \left\{ R \in \mathcal{R} : \int_{\Gamma_0} \Psi(\eta)|R(\eta)|\lambda(d\eta) < \infty \right\}. \tag{4.4}$$

For $R \in \mathcal{D}^\dagger \cap \mathcal{R}^+$, by (1.32) and (2.15) we obtain from (4.3)

$$\begin{aligned} \varphi_{\mathcal{R}}(B^\dagger R) &= \int_{\Gamma_0} \left(\sum_{x\in\eta} [m(x) + E^a(x, \eta \setminus x)] \right) R(\eta)\lambda(d\eta) \qquad (4.5) \\ &\quad + \int_{\Gamma_0} \left(\sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) dy_1 dy_2 \right) R(\eta)\lambda(d\eta) \\ &= \int_{\Gamma_0} \Psi(\eta) R(\eta)\lambda(d\eta) = -\varphi_{\mathcal{R}}(A^\dagger R). \end{aligned}$$

By (4.4) and (4.5) we then get that: (a) $B^\dagger : \mathcal{D}^\dagger \to \mathcal{R}$ and $B^\dagger : \mathcal{R}^+ \cap \mathcal{D}^\dagger \to \mathcal{R}^+$; (b) $\varphi_{\mathcal{R}}((A^\dagger + B^\dagger)R) = 0$ for each $R \in \mathcal{R}^+ \cap \mathcal{D}^\dagger$. In the same way, we prove that the operators defined in (2.16) and (2.17) satisfy: (a) $B : \mathcal{D} \to \mathcal{M}$ and $B : \mathcal{D}^+ \to \mathcal{M}^+$; (b) $\varphi_{\mathcal{M}}((A+B)\mu) = 0$ for each $\mu \in \mathcal{D}^+$. Thus, both pairs $(A, \mathcal{D})$, $(B, \mathcal{D})$ and $(A^\dagger, \mathcal{D}^\dagger)$, $(B^\dagger, \mathcal{D}^\dagger)$ satisfy item (i) of Proposition 1.5.2.

We proceed further by setting

$$\begin{aligned} (S(t)\mu)(d\eta) &= \exp\left(-t\Psi(\eta)\right) \mu(d\eta), \quad \mu \in \mathcal{M}, \quad t > 0, \qquad (4.6) \\ (S^\dagger(t)R)(\eta) &= \exp\left(-t\Psi(\eta)\right) R(\eta), \quad R \in \mathcal{R}. \end{aligned}$$

Obviously, $S = \{S(t)\}_{t\geq 0}$ and $S^\dagger = \{S^\dagger(t)\}_{t\geq 0}$ are sub-stochastic semigroups on $\mathcal{M}$ and $\mathcal{R}$, respectively. They are generated respectively by $(A, \mathcal{D})$ and $(A^\dagger, \mathcal{D}^\dagger)$. Let $\mathcal{M}_\chi$ be either $\mathcal{M}_{\chi^\kappa}$ or $\mathcal{M}_{\chi_m}$ and $\mathcal{R}_\chi$ be either $\mathcal{R}_{\chi^\kappa}$ or $\mathcal{R}_{\chi_m}$, as in Theorem 2.2.2. Clearly, the restrictions $S(t)|_{\mathcal{M}_\chi}$ and $S^\dagger(t)|_{\mathcal{R}_\chi}$ constitute positive C_0-semigroups. Likewise, $B : \mathcal{D}_\chi \to \mathcal{M}_\chi$ and $B^\dagger : \mathcal{D}^\dagger_\chi \to \mathcal{R}_\chi$. Thus, the conditions in items (ii) and (iii) of Proposition 1.5.2 are satisfied in both cases.

Now we turn to item (iv) of Proposition 1.5.2. By the definitions of functionals introduced in Section 1.4, we have

$$\begin{aligned} \varphi_{\mathcal{M}_\chi}((A + B)\mu) &= \varphi_{\mathcal{M}_\chi}(L^\mu \mu) = \int_{\Gamma_0} (LF_\chi)(\eta)\mu(d\eta), \\ \varphi_{\mathcal{R}_\chi}((A^\dagger + B^\dagger)R) &= \varphi_{\mathcal{R}_\chi}(L^\dagger R) = \int_{\Gamma_0} (LF_\chi)(\eta)R(\eta)\lambda(d\eta), \end{aligned}$$

where $F_\chi(\eta)$ is either equal to $F_{\chi^\kappa}(\eta) = e^{\kappa|\eta|}$ or $F_{\chi_m}(\eta) = (1+|\eta|)^m$. Then the condition in item (iv) of Proposition 1.5.2 is satisfied if, for some positive c and ε and all η, the following holds

$$(LF_\chi)(\eta) + \varepsilon\Psi(\eta) \leq c\chi(|\eta|). \tag{4.7}$$

For $F_\chi(\eta) = F_{\chi_m}(\eta)$, $m \in \mathbb{N}$, by (2.1) we have, cf. (2.15),

$$\begin{aligned} (LF_{\chi_m})(\eta) &= -\left(M(\eta) + E^a(\eta)\right)\epsilon_m(|\eta|) + \langle b\rangle|\eta|\epsilon_m(|\eta|+1), \qquad (4.8)\\ \epsilon_m(n) &:= (n+1)^m - n^m = (n+1)^{m-1} + (n+1)^{m-2}n \\ &\quad + \cdots + n^{m-1} \leq m(n+1)^{m-1}. \end{aligned}$$

For $F_\chi(\eta) = F_{\chi^\kappa}(\eta)$, we have

$$(LF_{\chi^\kappa})(\eta) = -\left(M(\eta) + E^a(\eta)\right)e^{\kappa|\eta|}(1-e^{-1}) + \langle b\rangle|\eta|e^{\kappa|\eta|}(e-1).$$

By (4.8) the condition in (4.7) takes the form

$$-\left(M(\eta) + E^a(\eta)\right)\left(\epsilon_m(|\eta|) - \varepsilon\right) + \langle b\rangle|\eta|\left(\epsilon_m(|\eta|+1) + \varepsilon\right) \leq c\left(|\eta|+1\right)^m. \tag{4.9}$$

since $\epsilon_m(|\eta|) \geq 1$. For $\varepsilon < 1$, the validity of (4.9) will follow whenever c satisfies

$$c \geq m\langle b\rangle\left(2^{m-1}+1\right).$$

Hence, for $\chi = \chi_m$, all the conditions of Proposition 1.5.2 are met for both choices of $\mathcal{E}$ and the corresponding operators. Therefore, we have two semigroups: $S_\mathcal{M}$ and $S_\mathcal{R}$, with the properties described in the mentioned statement. Then $\mu_t = S_\mathcal{M}(t)\mu_0$ is the unique solution of the Fokker-Planck equation with $\mu_0 \in \mathcal{D}$, which proves claim (a) of Theorem 2.2.2. At the same time, $R_t = S_\mathcal{R}(t)R_0(\eta)$ is the unique solution of

$$\dot{R}_t = L^\dagger R_t, \qquad R_t|_{t=0} = R_{\mu_0} \in \mathcal{D}^\dagger. \tag{4.10}$$

By (4.4) we have that $R_{\mu_0} \in \mathcal{D}^\dagger$ and $\mu_0 \in \mathcal{D}$ are equivalent. By direct inspection one checks that $\mu_t(d\eta) = R_t(\eta)\lambda(d\eta)$ solves (1.34) if R_t solves (4.10). Then the unique solution $\mu_t = S_\mathcal{M}(t)\mu_0$ of (1.34) has the mentioned form, which proves claim (c).

To complete the proof we fix $\kappa > 0$ and consider the trace of A in $\mathcal{M}_{\chi^\kappa}$, cf. (2.17), defined on the domain

$$\mathcal{D}_\kappa := \left\{\mu \in \mathcal{M}_{\chi^\kappa} : \int_{\Gamma_0} \Psi(\eta)e^{\kappa|\eta|}\mu^\pm(d\eta) < \infty\right\}.$$

First, we split B into the sum $B_1 + B_2$, where for $\mathbb{A} \in \mathcal{B}(\Gamma_0)$ we set, cf. (2.18),

$$(B_1\mu)(\mathbb{A}) = \int_{\Gamma_0} \left(\sum_{x\in\eta} [m(x) + E^a(x, \eta\setminus x)] \mathbb{I}_{\mathbb{A}}(\eta \setminus x) \right) \mu(d\eta), \quad (4.11)$$

and

$$(B_2\mu)(\mathbb{A}) = \int_{\Gamma_0} \left(\sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \mathbb{I}_{\mathbb{A}}(\eta \setminus x \cup \{y_1, y_2\}) dy_1 dy_2 \right) \mu(d\eta). \quad (4.12)$$

For $\mu \in \mathcal{D}^+_\kappa := \mathcal{D}_\kappa \cap \mathcal{M}^+$, from (4.11) we have

$$\begin{aligned} \varphi_\kappa(B_1\mu) &= \int_{\Gamma_0} e^{\kappa|\xi|} \int_{\Gamma_0} \sum_{x\in\eta} [m(x) + E^a(x, \eta\setminus x)] \delta_{\eta\setminus x}(d\xi) \mu(d\eta) \\ &= \int_{\Gamma_0} e^{\kappa(|\eta|-1)} \left(M(\eta) + E^a(\eta)\right) \mu(d\eta) \quad (4.13) \\ &\leq -e^{-\kappa} \varphi_\kappa(A\mu). \end{aligned}$$

For $r = e^{-\kappa}$, by (4.13) we have that $\varphi_\kappa(A + r^{-1}B_1\mu) \leq 0$ for each $\mu \in \mathcal{D}^+_\kappa$. Then by Proposition 1.5.1 we obtain that $(A + B_1, \mathcal{D}_\kappa)$ generates a sub-stochastic semigroup S_κ on $\mathcal{M}_{\chi^\kappa}$. For $\kappa' \in (0, \kappa)$, let us show now that B_2 acts as a bounded linear operator from $\mathcal{M}_{\chi^\kappa}$ to $\mathcal{M}_{\chi^{\kappa'}}$. In view of the Hahn-Jordan decomposition, it is enough to consider the action of B_2 on positive elements of $\mathcal{M}_{\chi^\kappa}$. Since B_2 is positive, cf. (4.12), for $\mu \in \mathcal{M}^+_{\chi^\kappa}$, we have

$$\begin{aligned} \|B_2\mu\|_{\mathcal{M}_{\chi^{\kappa'}}} &= \int_{\Gamma_0} e^{\kappa'|\xi|} \int_{\Gamma_0} \sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \delta_{\eta\setminus x \cup\{y_1,y_2\}}(d\xi) dy_1 dy_2 \mu(d\eta) \\ &= e^{\kappa'} \int_{\Gamma_0} e^{\kappa'|\eta|} \sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) dy_1 dy_2 \mu(d\eta) \\ &= e^{\kappa'} \langle b \rangle \int_{\Gamma_0} |\eta| e^{-(\kappa-\kappa')|\eta|} e^{\kappa|\eta|} \mu(d\eta) \quad (4.14) \\ &\leq \frac{e^{\kappa'}\langle b \rangle}{e(\kappa - \kappa')} \|\mu\|_{\mathcal{M}_{\chi^\kappa}}. \end{aligned}$$

Let $(B_2)_{\kappa'\kappa} : \mathcal{M}^+_{\chi^\kappa} \to \mathcal{M}^+_{\chi^{\kappa'}}$ be the bounded linear operator as just described. For fixed $\kappa > 0$, $\kappa' \in (0, \kappa)$ and $n \in \mathbb{N}$, we set

$$\kappa_l = \kappa - (\kappa - \kappa')l/n, \qquad l = 0, 1, \ldots, n. \tag{4.15}$$

By means of (4.14) and (4.15) we then have the following estimate of the operator norm of $(B_2)_{\kappa_{l+1}\kappa_l}$

$$\|(B_2)_{\kappa_{l+1}\kappa_l}\| \leq \frac{e^\kappa n\langle b\rangle}{e(\kappa - \kappa')}. \tag{4.16}$$

Next, for $t > 0$ and $0 \leq t_n \leq \cdots \leq t_0 = t$, we consider the following bounded linear operator acting from $\mathcal{M}_{\chi^\kappa}$ to $\mathcal{M}_{\chi^{\kappa'}}$

$$T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n) = S_{\kappa_n}(t - t_1)(B_2)_{\kappa_n\kappa_{n-1}} S_{\kappa_{n-1}}(t_1 - t_2) \cdots (B_2)_{\kappa_1\kappa} S_\kappa(t_n),$$

where S_{κ_l} is the sub-stochastic semigroup in $\mathcal{M}_{\chi^{\kappa_l}}$ generated by $(A + B_1, \mathcal{D}_{\kappa_l})$. By the latter fact we have that $T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n) : \mathcal{M}_{\chi^\kappa} \to \mathcal{D}_{\kappa'}$ and

$$\begin{aligned} \frac{d}{dt} T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n) &= (A + B_1) T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n), \\ T^{(n)}_{\kappa'\kappa}(t, t, t_2, \ldots, t_n) &= (B_2)_{\kappa'\kappa_{n-1}} T^{(n-1)}_{\kappa_{n-1}\kappa}(t, t_2, \ldots, t_n). \end{aligned} \tag{4.17}$$

As $(B_2)_{\kappa'\kappa_{n-1}}$ is the restriction of $(B_2, \mathcal{D}_{\kappa'})$ to $\mathcal{M}_{\chi^{\kappa_{n-1}}} \subset \mathcal{D}_{\kappa'}$ and $T^{(n-1)}_{\kappa'\kappa}(t, t_2, t_2, \ldots, t_n) : \mathcal{M}_{\chi^\kappa} \to \mathcal{D}_{\kappa'}$, the second line in (4.17) can be rewritten as

$$T^{(n)}_{\kappa'\kappa}(t, t, t_2, \ldots, t_n) = B_2 T^{(n-1)}_{\kappa'\kappa}(t, t_2, \ldots, t_n). \tag{4.18}$$

On the other hand, since all the semigroups S_{κ_l} are sub-stochastic and $(B_2)_{\kappa'\kappa}$ are positive, by (4.16) we get the following estimate of its operator norm

$$\|T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n)\| \leq \left(\frac{e^\kappa n\langle b\rangle}{e(\kappa - \kappa')}\right)^n. \tag{4.19}$$

We also set $T^{(0)}_{\kappa'\kappa}(t) = S_{\kappa'}(t)|_{\mathcal{M}_{\chi^\kappa}}$, and then consider

$$Q_{\kappa'\kappa}(t) := \sum_{n=0}^\infty \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} T^{(n)}_{\kappa'\kappa}(t, t_1, t_2, \ldots, t_n) dt_n dt_{n-1} \cdots dt_1. \tag{4.20}$$

By (4.19) we conclude that the series in (4.20) converges uniformly on compact subsets of $[0, T(\kappa, \kappa'))$, see (2.19), to a continuously differentiable function

$$(0, T(\kappa, \kappa')) \ni t \mapsto Q_{\kappa'\kappa}(t) \in \mathcal{L}(\mathcal{M}_{\chi^{\kappa}}, \mathcal{M}_{\chi^{\kappa'}}),$$

where the latter is the Banach space of all bounded linear operators acting from $\mathcal{M}_{\chi^{\kappa}}$ to $\mathcal{M}_{\chi^{\kappa'}}$. By (4.17) and (4.18) we obtain

$$\frac{d}{dt} Q_{\kappa'\kappa}(t) = (A + B_1 + B_2) Q_{\kappa'\kappa}(t) = L^{\mu} Q_{\kappa'\kappa}(t). \tag{4.21}$$

Thus, assuming that $\mu_0 \in \mathcal{M}_{\chi^{\kappa}}$ we get that $\tilde{\mu}_t := Q_{\kappa'\kappa}(t)\mu_0$, for $t \in [0, T(\kappa, \kappa'))$, lies in $\mathcal{M}_{\chi^{\kappa'}}$ and solves (1.34). Therefore, $\tilde{\mu}_t$ coincides with $\mu_t = S_{\mathcal{M}}(t)\mu_0$, which completes the proof. □

4.1.3 Proof of Proposition 2.3.1

Our aim here is to transform the Kolmogorov equation (1.33) with L given by (2.1) into the problem (1.36). Recall, that generators L and L^{Δ} are related to each other by the formula (1.37). The observable F is given by the expression, cf. (1.14),

$$F(\gamma) = \prod_{x \in \gamma} (1 + \theta(x)), \qquad \theta \in \Theta.$$

Let $(LF)(\gamma) = (L_1 F)(\gamma) + (L_2 F)(\gamma)$, where

$$\begin{aligned} (L_1 F)(\gamma) &= \sum_{x \in \gamma} \left(m(x) + \sum_{y \in \gamma \setminus x} a(x - y) \right) [F(\gamma \setminus x) - F(\gamma)], \\ (L_2 F)(\gamma) &= \sum_{x \in \gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) [F(\gamma \setminus x \cup \{y_1, y_2\}) - F(\gamma)] \, dy_1 dy_2. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Gamma} (L_2 F)(\gamma) \mu(d\gamma) = &\int_{\Gamma} \sum_{x \in \gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \\ &\times \left[\prod_{z \in \gamma \setminus x \cup \{y_1, y_2\}} (1 + \theta(z)) - \prod_{z \in \gamma} (1 + \theta(z)) \right] dy_1 dy_2 \mu(d\gamma) \end{aligned}$$

$$= \int_\Gamma \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2)\tau(\gamma, x, y_1, y_2)dy_1dy_2\mu(d\gamma),$$

where

$$\begin{aligned} \tau(\gamma, x, y_1, y_2) &= (1+\theta(y_1))(1+\theta(y_2)) \prod_{z\in\gamma\setminus x}(1+\theta(z)) \\ &\quad -(1+\theta(x)) \prod_{z\in\gamma\setminus x}(1+\theta(z)) \\ &= \prod_{z\in\gamma\setminus x}(1+\theta(z)) \left[\theta(y_1)+\theta(y_2)+\theta(y_1)\theta(y_2)-\theta(x)\right]. \end{aligned}$$

Then we may write $(L_2F)(\gamma) = 2(L_{2,1}F)(\gamma)+(L_{2,2}F)(\gamma)-(L_{2,3}F)(\gamma)$, where

$$\begin{aligned} (L_{2,1}F)(\gamma) &= \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\gamma\setminus x}(1+\theta(z))\theta(y_1)dy_1dy_2 \\ (L_{2,2}F)(\gamma) &= \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\gamma\setminus x}(1+\theta(z))\theta(y_1)\theta(y_2)dy_1dy_2 \\ (L_{2,3}F)(\gamma) &= \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\gamma\setminus x}(1+\theta(z))\theta(x)dy_1dy_2 \end{aligned}$$

Hence, by direct calculations based on the definition of Lebesque-Poisson measure and formulas (1.24), (1.31) and (1.32) we get the following.

$$\int_\Gamma (L_{2,1}F)(\gamma)\mu(d\gamma) =$$

$$\begin{aligned} &= \int_\Gamma \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\gamma\setminus x}(1+\theta(z))\theta(y_1)dy_1dy_2\mu(d\gamma) \\ &= \int_\Gamma \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \sum_{\eta\in\subset} \prod_{z\in\gamma\setminus x} \theta(z)\theta(y_1)dy_1dy_2\mu(d\gamma) \\ &= \int_\Gamma \sum_{x\in\gamma} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \sum_{\eta\subset\gamma} \prod_{z\in\gamma\setminus x\cup y_1} \theta(z)dy_1dy_2\mu(d\gamma) \\ &= \int_\Gamma \sum_{\eta\subset\gamma} \sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\eta\setminus x\cup y_1} \theta(z)dy_1dy_2\mu(d\gamma) \end{aligned}$$

$$= \int_{\Gamma_0} \sum_{x\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\eta\setminus x\cup y_1} \theta(z) dy_1 dy_2\, k_\mu(\eta)\, \lambda(d\gamma)$$

$$= \int_{\Gamma_0} \sum_{y_1\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\eta} \theta(z) k_\mu(\eta \setminus y_1 \cup x) dy_2 dx \lambda(d\gamma)$$

$$= \int_{\Gamma_0} (L^\Delta_{2,1} k)(\eta)\lambda(d\eta),$$

where

$$(L^\Delta_{2,1} k)(\eta) = \sum_{y_1\in\eta} \int_{(\mathbb{R}^d)^2} b(x|y_1, y_2) \prod_{z\in\eta} \theta(z) k_\mu(\eta \setminus y_1 \cup x) dy_2 dx.$$

In analogical way, we may obtain parts $L^\Delta_{2,2}$, $L^\Delta_{2,3}$ and L^Δ_1, which yields (2.20).

4.1.4 Proof of Proposition 2.3.2

For $\alpha' < \alpha$, by means of (1.30) and the inequality (2.21), we obtain from (2.20) the following estimates

$$\begin{aligned} \|A^\Delta_1 k\|_\alpha &\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-\alpha|\eta|} \Psi(\eta)|k(\eta)| \\ &\leq \left((m^* + \langle b\rangle + a^*) \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left[|\eta|^2 e^{-(\alpha-\alpha')|\eta|} \right] \right) \|k\|_{\alpha'} \\ &= \frac{4(m^* + \langle b\rangle + a^*)}{e^2(\alpha-\alpha')^2} \|k\|_{\alpha'}, \end{aligned}$$

$$\begin{aligned} \|A^\Delta_2 k\|_\alpha &\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-\alpha|\eta|} \int_{\Gamma_0} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} b(x|y_1, y_2)|k(\eta \cup x \setminus \{y_1, y_2\})| dx \\ &\leq \left(\beta^* \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left[|\eta|^2 e^{\alpha'(|\eta|-1)} e^{-\alpha|\eta|} \right] \right) \|k\|_{\alpha'} \\ &= \frac{4\beta^* e^{-\alpha'}}{e^2(\alpha-\alpha')^2} \|k\|_{\alpha'}, \end{aligned}$$

$$\begin{aligned}\|B_1^\Delta k\|_\alpha &\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-\alpha|\eta|}\int_{\mathbb{R}^d} E^a(x,\eta)|k(\eta\cup x)|dx \\ &\leq \left(\langle a\rangle \operatorname*{ess\,sup}_{\eta\in\Gamma_0}\left[|\eta|e^{-\alpha'(|\eta|+1)}e^{-\alpha|\eta|}\right]\right)\|k\|_{\alpha'} \\ &= \frac{\langle a\rangle e^{\alpha'}}{e(\alpha-\alpha')}\|k\|_{\alpha'},\end{aligned}$$

$$\begin{aligned}\|B_2^\Delta k\|_\alpha &\leq 2\operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-\alpha|\eta|}\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta} b(x|y_1,y_2)|k(\eta\cup x\setminus y_1)|dy_2dx \\ &\leq \left(\langle a\rangle \operatorname*{ess\,sup}_{\eta\in\Gamma_0}\left[|\eta|e^{-(\alpha-\alpha')|\eta|}\right]\right)\|k\|_{\alpha'} = \frac{2\langle b\rangle}{e(\alpha-\alpha')}\|k\|_{\alpha'},\end{aligned}$$

The above estimations yield

$$\|L^\Delta k\|_\alpha \leq \left(4\frac{m^*+\langle b\rangle+a^*+\beta^*e^{-\alpha'}}{e^2(\alpha-\alpha')^2}+\frac{\langle a\rangle e^{\alpha'}+2\langle b\rangle}{e(\alpha-\alpha')}\right)\|k\|_{\alpha'}. \quad (4.22)$$

Then we may define a bounded operator $L^\Delta_{\alpha\alpha'}:\mathcal{K}_{\alpha'}\to\mathcal{K}_\alpha$ with the norm (2.24) which can be estimated by means of (4.22).

4.1.5 Proof Lemma 2.3.3

We begin by constructing a semigroup, predual to that mentioned in item (i) in the sketch of the proof of Lemma 2.3.3.

The predual semigroup. For $\alpha\in\mathbb{R}$, the space predual to $\mathcal{K}_\alpha$ is

$$\mathcal{G}_\alpha := L^1(\Gamma_0, e^{\alpha|\cdot|}d\lambda), \quad (4.23)$$

which for $\alpha>0$ coincides with $\mathcal{R}_\chi$ defined in (1.41) with $\chi(n)=e^{\alpha n}$. Here, however, we allow α to be any real number. The norm in $\mathcal{G}_\alpha$ is

$$|G|_\alpha = \int_{\Gamma_0}|G(\eta)|e^{\alpha|\eta|}\lambda(d\eta). \quad (4.24)$$

Clearly, $|G|_{\alpha'} \leq |G|_\alpha$ whenever $\alpha' < \alpha$. Then $\mathcal{G}_\alpha \hookrightarrow \mathcal{G}_{\alpha'}$, and this embedding is also dense. In order to use Proposition 2.1.2 we modify the operators introduced in (2.20) by adding and subtracting the term $\upsilon|\eta|$. This will lead also to the corresponding modification of the predual operators. Thus, for an appropriate $G : \Gamma_0 \to \mathbb{R}$ we set, cf. (2.15),

$$\begin{aligned}
(A_{1,\upsilon}G)(\eta) &= -\Psi_\upsilon(\eta)G(\eta) = -\left(\upsilon|\eta| + E^a(\eta) + M(\eta) + \langle b\rangle|\eta|\right) G(\eta), \\
(A_2G)(\eta) &= \sum_{x\in\eta} \int_{(\mathbb{R})^2} G(\eta \setminus x \cup y_1 \cup y_2) b(x|y_1, y_2) dy_1 dy_2, \qquad (4.25) \\
\mathcal{D}_\alpha &= \{G :\in \mathcal{G}_\alpha : \Psi_\upsilon G \in \mathcal{G}_\alpha\}.
\end{aligned}$$

By Proposition 2.1.2 we have that

$$\Psi_\upsilon(\eta) \geq \omega E^b(\eta). \qquad (4.26)$$

The operator $(A_{1,\upsilon}, \mathcal{D}_\alpha)$ is the generator of the semigroup $S_{0,\alpha} = \{S_{0,\alpha}\}_{t\geq 0}$ of multiplication operators which act in $\mathcal{G}_\alpha$ as follows, cf. (4.6),

$$(S_{0,\alpha}(t)G)(\eta) = \exp\left(-t\Psi_\upsilon(\eta)\right) G(\eta). \qquad (4.27)$$

Let $\mathcal{G}_\alpha^+$ be the cone of positive elements of $\mathcal{G}_\alpha$ The semigroup defined in (4.27) is obviously *sub-stochastic*. Set $\mathcal{D}_\alpha^+ = \mathcal{D}_\alpha \cap \mathcal{G}_\alpha^+$. By (1.32), (4.24) and (4.25) we get

$$\begin{aligned}
|A_2G|_\alpha &= \int_{\Gamma_0} e^{\alpha|\eta|} |(A_2G)(\eta)| \lambda(d\eta) \qquad (4.28) \\
&\leq \int_{\Gamma_0} e^{\alpha|\eta|} \int_{(\mathbb{R}^d)^2} \sum_{x\in\eta} |G(\eta \setminus x \cup y_1 \cup y_2)| b(x|y_1, y_2) dy_1 dy_2 \lambda(d\eta) \\
&= \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} e^{\alpha(|\eta|-1)} |G(\eta)| b(x|y_1, y_2) dx \lambda(d\eta) \\
&= e^{-\alpha} \int_{\Gamma_0} e^{\alpha|\eta|} E_b(\eta) |G(\eta)| \lambda(d\eta) \\
&\leq (e^{-\alpha}/\omega) |A_{1,\upsilon}G|_\alpha.
\end{aligned}$$

The latter was obtained by (4.26), see also (2.3). The next statement summarizes the construction of the predual semigroup in question.

Lemma 4.1.1. *Let υ and ω be as in Proposition 2.1.2 and $A_{1,\upsilon}$, A_2 and $\mathcal{D}_\alpha$ be as in (4.25). Then for each $\alpha > -\log\omega$, the operator $(A_\upsilon, \mathcal{D}_\alpha) := (A_{1,\upsilon} + A_2, \mathcal{D}_\alpha)$ is the generator of a sub-stochastic semigroup $S_\alpha = \{S_\alpha(t)\}_{t\geq 0}$ on $\mathcal{G}_\alpha$.*

Proof. We apply Proposition 1.5.1 with $\mathcal{E} = \mathcal{G}_\alpha$, $\mathcal{D} = \mathcal{D}_\alpha$ and $A = A_{1,\upsilon}$. For some $r \in (0,1)$ (which will be chosen below), we set $B = r^{-1}A_2$, which is clearly positive. By (4.28) B is defined on $\mathcal{D}_\alpha$. To show that (1.46) holds we take $G \in \mathcal{D}_\alpha^+$ and proceed as in (4.28). That is,

$$\int_{\Gamma_0} \left((A_{1,\upsilon} + r^{-1}A_2)G\right)(\eta) e^{\alpha|\eta|}\lambda(d\eta) = -\int_{\Gamma_0} \Psi_\upsilon(\eta) G(\eta) e^{\alpha|\eta|}\lambda(d\eta)$$

$$+ r^{-1}\int_{\Gamma_0}\sum_{x\in\eta}\int_{(\mathbb{R}^d)^2} G(\eta\setminus x \cup \{y_1,y_2\}) b(x|y_1,y_2) e^{\alpha|\eta|} dy_1 dy_2 \lambda(d\eta)$$

$$\leq -\int_{\Gamma_0}\left(\upsilon|\eta| + E^a(\eta) - r^{-1}e^{-\alpha}E^b(\eta)\right) G(\eta) e^{\alpha|\eta|}\lambda(d\eta).$$

Now, for $\alpha > -\log\omega$, we can pick $r \in (0,1)$ in such a way that $r^{-1}e^{-\alpha} \leq \omega$, which by Proposition 2.1.2 implies that (1.46) holds for this choice. Then the operator $A_{1,\upsilon} + r(r^{-1}A_2)$ satisfies Proposition 1.5.1 by which the proof follows. □

By the very definition of the sub-stochasticity of S_α we have that $|S_\alpha(t)G|_\alpha \leq |G|_\alpha$ whenever $G \in \mathcal{G}_\alpha^+$. Let us show now that the same estimate holds also for all $G \in \mathcal{G}_\alpha$. Each such G in a unique way can be decomposed $G = G^+ - G^-$ with $G^\pm \in \mathcal{G}_\alpha^+$. Moreover, by (4.24) we have that

$$|G|_\alpha = \int_{\Gamma_0} e^{\alpha|\eta|}\left(G^+(\eta) + G^-(\eta)\right)\lambda(d\eta) = |G^+|_\alpha + |G^-|_\alpha.$$

Then

$$\begin{aligned} |S_\alpha(t)G|_\alpha &= |S_\alpha(t)(G^+ - G^-)|_\alpha \leq |S_\alpha(t)G^+|_\alpha + |S_\alpha(t)G^-|_\alpha \\ &\leq |G^+|_\alpha + |G^-|_\alpha = |G|_\alpha. \end{aligned} \tag{4.29}$$

The sun-dual semigroup. Let $S_\alpha(t)$ be an element of the semigroup as in Lemma 4.1.1. Then its adjoint $S_\alpha^*(t)$ is a bounded linear operator in $\mathcal{K}_\alpha$. Clearly, $\{S_\alpha^*(t)\}_{t\geq 0}$ is a semigroup. However, it is not strongly continuous and hence cannot be directly used to construct (classical) solutions of differential equations. This obstacle is usually circumvented as follows, see [46]. Set, cf. (1.27),

$$\mathcal{D}_\alpha^* = \{k \in \mathcal{K}_\alpha : \exists \hat{k} \in \mathcal{K}_\alpha \ \forall G \in \mathcal{D}_\alpha \ \langle\!\langle A_\upsilon G, k \rangle\!\rangle = \langle\!\langle G, \hat{k} \rangle\!\rangle \}.$$

Then the operator $(A_\upsilon^*, \mathcal{D}_\alpha^*)$ is adjoint to $(A_\upsilon, \mathcal{D}_\alpha)$. It acts as follows

$$\begin{aligned} (A_\upsilon^* k)(\eta) &= -\Psi_\upsilon(\eta) k(\eta) \\ &+ \int_{\mathbb{R}^d} \sum_{y_1 \in \eta} \sum_{y_2 \in \eta \setminus y_1} k(\eta \cup x \setminus \{y_1, y_2\}) b(x|y_1, y_2) dx. \end{aligned}$$

By direct inspection one obtains that $\mathcal{K}_{\alpha'} \subset \mathcal{D}_\alpha^*$ whenever $\alpha' < \alpha$. Let $\mathcal{Q}_\alpha$ be the closure of $\mathcal{D}_\alpha^*$ in $\mathcal{K}_\alpha$. Then we have

$$\mathcal{K}_{\alpha'} \subset \mathcal{D}_\alpha^* \subset \mathcal{Q}_\alpha \subsetneq \mathcal{K}_\alpha, \qquad \alpha' < \alpha. \tag{4.30}$$

Now we set

$$\mathcal{D}_\alpha^\odot = \{k \in \mathcal{D}_\alpha^* : A_\upsilon^* k \in \mathcal{Q}_\alpha\},$$

and denote by $A_\upsilon^\odot$ the restriction of A_υ^* to $\mathcal{D}_\alpha^\odot$. Then $(A_\upsilon^\odot, \mathcal{D}_\alpha^\odot)$ is the generator of a C_0-semigroup, which we denote by $S_\alpha^\odot = \{S_\alpha^\odot(t)\}_{t \geq 0}$. This is the semigroup which we have aimed to construct. It has the following property, see [46, Lemma 10.1].

Proposition 4.1.2. *for each $k \in \mathcal{Q}_\alpha$ and $t \geq 0$, it follows that $\|S_\alpha^\odot(t)k\|_\alpha = \|S_\alpha^*(t)k\|_\alpha \leq \|k\|_\alpha$. Moreover, for each $\alpha' < \alpha$ and $k \in \mathcal{K}_{\alpha'}$, the map $[0, +\infty) \ni t \mapsto S_\alpha^\odot(t)k \in \mathcal{Q}_\alpha$ is continuous.*

The estimate $\|S_\alpha^*(t)k\|_\alpha \leq \|k\|_\alpha$ is obtained by means of (4.29). The continuity follows by (4.30) and the fact that $S_\alpha^\odot$ is a C_0-semigroup.

The resolving operators. Now we construct the family of operators $\{Q_{\alpha\alpha'}(t)\}$ such that the solution of (2.26) is obtained in the form $k_t = Q_{\alpha_2\alpha_1}(t)k_0$. This construction, in which we employ $S^\odot$, resembles the one used to get (4.20). We begin by rearranging the operators in (2.20) as follows

$$L^\Delta = A^\Delta + B^\Delta = A_\upsilon^\Delta + B_\upsilon^\Delta, \tag{4.31}$$

where $A^{\Delta}_{\upsilon} = A^{\Delta}_{1,\upsilon} + A^{\Delta}_{2}$, see (4.25), and

$$
\begin{aligned}
B^{\Delta}_{\upsilon} &= B^{\Delta}_{1} + B^{\Delta}_{2,\upsilon}, \qquad (4.32)\\
(B^{\Delta}_{2,\upsilon}k)(\eta) &= (B^{\Delta}_{2}k)(\eta) + \upsilon|\eta|k(\eta)\\
&= 2\int_{(\mathbb{R}^d)^2} \sum_{y_1\in\eta} b(x|y_1,y_2)k(\eta\cup x\setminus y_1)dxdy_2 + \upsilon|\eta|k(\eta),
\end{aligned}
$$

whereas B^{Δ}_{1} is as in (2.20). By means of (4.32), for $\alpha \in \mathbb{R}$ and $\alpha' < \alpha$, we define $(B^{\Delta}_{\upsilon})_{\alpha\alpha'} \in \mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_{\alpha})$ the norm of which can be estimated similarly as in (2.24), which yields

$$
\|(B^{\Delta}_{\upsilon})_{\alpha\alpha'}\| \le \frac{2\langle b\rangle + \upsilon + \langle a\rangle e^{\alpha'}}{e(\alpha-\alpha')}. \qquad (4.33)
$$

Now let $\mathbf{B}$ be either B^{Δ}_{υ} or $B^{\Delta}_{2,\upsilon}$, and $\mathbf{B}_{\alpha\alpha'}$ be the corresponding bounded operator. Then, cf. (4.33),

$$
\|\mathbf{B}_{\alpha\alpha'}\| \le \frac{\varpi(\alpha;\mathbf{B})}{e(\alpha-\alpha')}, \qquad (4.34)
$$

where

$$
\varpi(\alpha; B^{\Delta}_{\upsilon}) = 2\langle b\rangle + \upsilon + \langle a\rangle e^{\alpha}, \quad \varpi(\alpha; B^{\Delta}_{2,\upsilon}) = 2\langle b\rangle + \upsilon. \qquad (4.35)
$$

For some α_1, α_2 such that $\alpha_1 < \alpha_2$, we then set $\Sigma_{\alpha_2\alpha_1}(t) = S^{\odot}_{\alpha_2}(t)|_{\mathcal{K}_{\alpha_1}}$, $t > 0$, where $S^{\odot}_{\alpha}$ is the sub-stochastic semigroup as in Proposition 4.1.2. We also set $\Sigma_{\alpha_2\alpha_1}(0)$ to be the embedding operator $\mathcal{K}_{\alpha_1} \to \mathcal{K}_{\alpha_2}$. Hence, see Proposition 4.1.2, the operator norm satisfies

$$
\|\Sigma_{\alpha_2\alpha_1}(t)\| \le 1, \qquad t \ge 0. \qquad (4.36)
$$

We also have

$$
\begin{aligned}
\Sigma_{\alpha_2\alpha_1}(t) &= \Sigma_{\alpha_2\alpha_1}(0)S^{\odot}_{\alpha_1}(t), \qquad (4.37)\\
\Sigma_{\alpha_3\alpha_1}(t+s) &= \Sigma_{\alpha_3\alpha_2}(t)\Sigma_{\alpha_2\alpha_1}(s), \qquad \alpha_3 > \alpha_2,
\end{aligned}
$$

holding for all $t, s \ge 0$. Moreover,

$$
\frac{d}{dt}\Sigma_{\alpha_2\alpha_1}(t) = A^{\Delta}_{\upsilon}\Sigma_{\alpha_2\alpha_1}(t),
$$

which follows by Lemma 4.1.1 and the construction of the semigroup $S^{\odot}_{\alpha}$. Now we set

$$T(\alpha_2, \alpha_1; \mathbf{B}) = \frac{\alpha_2 - \alpha_1}{\varpi(\alpha_2; \mathbf{B})}, \tag{4.38}$$

see (4.34), (4.35), and also

$$\mathcal{A}(\mathbf{B}) = \{(\alpha_1, \alpha_2, t) : -\log\omega < \alpha_1 < \alpha_2,\ t \in [0, T(\alpha_2, \alpha_1; \mathbf{B}))\}. \tag{4.39}$$

Note that $T(\alpha_2, \alpha_1; B^{\Delta}_{\upsilon})$ coincides with $T(\alpha_2, \alpha_1)$ defined in (2.27).

Lemma 4.1.3. *For both choices of $\mathbf{B}$, there exist the corresponding families $\{Q_{\alpha_2\alpha_1}(t; \mathbf{B}) : (\alpha_1, \alpha_2, t) \in \mathcal{A}(\mathbf{B})\}$, each element of which has the following properties:*

(a) $Q_{\alpha_2\alpha_1}(t; \mathbf{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$;

(b) the map $[0, T(\alpha_2, \alpha_1; \mathbf{B})) \ni t \mapsto Q_{\alpha_2\alpha_1}(t; \mathbf{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$ *is continuous;*

(c) the operator norm of $Q_{\alpha_2\alpha_1}(t; \mathbf{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$ *satisfies*

$$\|Q_{\alpha_2\alpha_1}(t; \mathbf{B})\| \leq \frac{T(\alpha_2, \alpha_1; \mathbf{B})}{T(\alpha_2, \alpha_1; \mathbf{B}) - t},$$

(d) for each $\alpha_3 \in (\alpha_1, \alpha_2)$ *and* $t < T(\alpha_3, \alpha_1; \mathbf{B})$, *the following holds*

$$\frac{d}{dt} Q_{\alpha_2\alpha_1}(t; \mathbf{B}) = ((A^{\Delta}_{\upsilon})_{\alpha_2\alpha_3} + \mathbf{B}_{\alpha_2\alpha_3}) Q_{\alpha_3\alpha_1}(t; \mathbf{B}), \tag{4.40}$$

which yields, in turn, that

$$\begin{aligned} \frac{d}{dt} Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{\upsilon}) &= L^{\Delta}_{\alpha_2} Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{\upsilon}) \qquad (4.41) \\ \frac{d}{dt} Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{2,\upsilon}) &= ((A^{\Delta}_{\upsilon})_{\alpha_2} + (B^{\Delta}_{2,\upsilon})_{\alpha_2}) Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{2,\upsilon}), \end{aligned}$$

where $L^{\Delta}_{\alpha_2}$ *is as in (2.26), see also (4.31), and* $(B^{\Delta}_{2,\upsilon})_{\alpha_2}$ *denotes* $(B^{\Delta}_{2,\upsilon}, \mathcal{D}^{\Delta}_{\alpha_2})$, *see (2.22).*

Proof. Fix some $T < T(\alpha_2, \alpha_1; \mathbf{B})$ and then take $\alpha \in (\alpha_1, \alpha_2]$ and positive $\delta < \alpha - \alpha_1$ such that

$$T < T_\delta := \frac{\alpha - \alpha_1 - \delta}{\beta(\alpha_2; \mathbf{B})}.$$

Then take some $l \in \mathbb{N}$ and divide the interval $[\alpha_1, \alpha]$ into $2l+1$ subintervals in the following way $\alpha_1 = \alpha^0$, $\alpha = \alpha^{2l+1}$ and

$$\alpha^{2s} = \alpha_1 + \frac{s}{l+1}\delta + s\epsilon, \qquad \alpha^{2s+1} = \alpha_1 + \frac{s+1}{l+1}\delta + s\epsilon, \tag{4.42}$$

where $\epsilon = (\alpha - \alpha_1 - \delta)/l$ and $s = 0, 1, ..., l$. Now for $0 \leq t_l \leq t_{l-1} \cdots \leq t_1 \leq t_0 := t$, define

$$\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, t_2, ..., t_l; \mathbf{B}) = \Sigma_{\alpha\alpha^{2l}}(t - t_1)\mathbf{B}_{\alpha^{2l}\alpha^{2l-1}} \cdots \Sigma_{\alpha^{2s+1}\alpha^{2s}}(t_{l-s} - t_{l-s+1})$$
$$\times \mathbf{B}_{\alpha^{2s}\alpha^{2s-1}} \Sigma_{\alpha^3\alpha^2}(t_{l-1} - t_l)\mathbf{B}_{\alpha^2\alpha^1}\Sigma_{\alpha^1\alpha_1}(t_l). \tag{4.43}$$

By the very construction we have that $\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, t_2, ..., t_l; \mathbf{B}) \in \mathcal{L}(\mathcal{K}_\alpha, \mathcal{K}_{\alpha_1})$, and the map

$$(t, t_1, ..., t_l) \mapsto \Pi^{(l)}_{\alpha\alpha_1}(t, t_1, t_2, ..., t_l; \mathbf{B})$$

is continuous (Proposition 4.1.2 and the fact that each $\mathbf{B}_{\alpha^{2s}\alpha^{2s-1}}$ is bounded). Moreover, by (4.36) and (4.34) we have

$$\begin{aligned} \|\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, t_2, ..., t_l; \mathbf{B})\| &\leq \prod_{s=1}^{l} \|\mathbf{B}_{\alpha^{2s}\alpha^{2s-1}}\| \leq \prod_{s=1}^{l} \frac{\varpi(\alpha^{2s}; \mathbf{B})}{e(\alpha^{2s} - \alpha^{2s-1})} \\ &\leq \left(\frac{l\upsilon(\alpha_2; \mathbf{B})}{e(\alpha - \alpha_1 - \delta)}\right)^l \leq \left(\frac{l}{eT_\delta}\right)^l. \end{aligned} \tag{4.44}$$

By (4.37) we have that

$$\Sigma_{\alpha^{2s+1}\alpha^{2s}}(t_{l-s} - t_{l-s+1}) = \Sigma_{\alpha^{2s+1}\alpha^{2s}}(0)S^{\odot}_{\alpha^{2s}}(t_{l-s} - t_{l-s+1}).$$

Taking the derivative of both sides of the latter we obtain

$$\frac{d}{dt}\Sigma_{\alpha^{2s+1}\alpha^{2s}}(t) = (A^{\Delta}_{\upsilon})_{\alpha^{2s+1}\alpha''}\Sigma_{\alpha''\alpha^{2s}}(t) = (A^{\Delta}_{\upsilon})_{\alpha^{2s+1}}\Sigma_{\alpha^{2s+1}\alpha^{2s}}(t),$$

holing for each $\alpha'' \in (\alpha^{2s}, \alpha^{2s+1})$. Here $(A_\upsilon^\Delta)_\alpha$ stands for the unbounded operator defined in (4.25). Then we obtain from (4.43) the following

$$\begin{aligned} \frac{d}{dt}\Pi^{(l)}_{\alpha\alpha_1}(t,t_1,t_2,...,t_l;\mathbf{B}) &= (A_\upsilon^\Delta)_{\alpha\alpha'}\Pi^{(l)}_{\alpha'\alpha_1}(t,t_1,t_2,...,t_l;\mathbf{B}) \\ &= (A_\upsilon^\Delta)_{\alpha}\Pi^{(l)}_{\alpha\alpha_1}(t,t_1,t_2,...,t_l;\mathbf{B}). \end{aligned} \quad (4.45)$$

Now we set

$$Q_{\alpha\alpha_1}(t;\mathbf{B}) = \Sigma_{\alpha\alpha_1}(t)+\sum_{l=1}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{l-1}}\Pi^{(l)}_{\alpha\alpha_1}(t,t_1,t_2,...,t_l;\mathbf{B})dt_l...dt_1. \quad (4.46)$$

By (4.44) the series in (4.46) converges uniformly of compact subsets of $[0,T_\delta)$, which proves claims (a) and (b). The estimate in (c) follows directly from (4.44). Finally, (4.41) follows by (4.45), cf. (4.21). □

By solving (4.40) with the initial condition $Q_{\alpha_2\alpha_1}(t+s;\mathbf{B})|_{t=0} = Q_{\alpha_2\alpha_1}(s;\mathbf{B})$ we obtain the following 'semigroup' property of the family $\{Q_{\alpha_2\alpha_1}(t;\mathbf{B}) : (\alpha_1,\alpha_2,t) \in \mathcal{A}(\mathbf{B})\}$.

Corollary 4.1.4. *For each $\alpha \in (\alpha_1,\alpha_2)$ and $t,s>0$ such that*

$$s < T(\alpha,\alpha_1;\mathbf{B}), \quad t < T(\alpha_2,\alpha;\mathbf{B}), \quad t+s < T(\alpha_2,\alpha_1;\mathbf{B}),$$

the following holds

$$Q_{\alpha_2\alpha_1}(t+s;\mathbf{B}) = Q_{\alpha_2\alpha}(t;\mathbf{B})Q_{\alpha\alpha_1}(s;\mathbf{B}).$$

Remark 4.1.5. Since $B^\Delta_{2,\upsilon}$ is positive, by (4.43) we obtain that $Q_{\alpha_2\alpha_1}(t;B^\Delta_{2,\upsilon}) : \mathcal{K}^+_{\alpha_1} \to \mathcal{K}^+_{\alpha_2}$. This positivity will be used to make the continuation of k_t to all $t>0$. It is the only reason for us to use $Q_{\alpha_2\alpha_1}(t;B^\Delta_{2,\upsilon})$ since B^Δ_υ is not positive, and hence the positivity of $Q_{\alpha_2\alpha_1}(t;B^\Delta_\upsilon)$ cannot be secured.

Proof of Lemma 2.3.3. Set

$$Q_{\alpha_2\alpha_1}(t) = Q_{\alpha_2\alpha_1}(t;B^\Delta_\upsilon), \qquad t < T(\alpha_2,\alpha_1;B^\Delta_\upsilon) = T(\alpha_2,\alpha_1) \quad (4.47)$$

Then the solution of equation (2.26) is obtained by setting $k_t = Q_{\alpha_2\alpha_1}(t)k_0$, which definitely satisfies (2.26) by (4.41) and (4.37). To prove its uniqueness we proceed as follows (cf. the proof of Lemma

4.8 in [34]). Assume that k_t and $\hat{k}_t$ are two different solutions of (1.36) with the same initial conditions and $\hat{k}_t \in D^{\Delta}_{\alpha_2}$. Then $w_t := k_t - \hat{k}_t$ with zero initial condition is a solution in $\mathcal{K}_{\alpha_3}$ for each $\alpha_3 > \alpha_2$ and $t < T(\alpha_3, \alpha_1)$. Therefore, it takes the form

$$w_t = \int_0^t \Sigma_{\alpha_3\alpha}(t-s)(B^{\Delta}_b)_{\alpha\alpha_1} w_s ds, \quad \alpha \in (\alpha_2, \alpha_3). \tag{4.48}$$

Now for a given $n > 1$, we split interval $[\alpha_2, \alpha_3]$ into $2n+1$ subintervals $[\alpha^l, \alpha^{l+1}]$, where

$$\alpha^l = \alpha_2 - l\epsilon, \text{ where } \epsilon = \frac{\alpha_3 - \alpha_2}{2n} \text{ and } l = 0, 1, ..., 2n.$$

Then we reintegrate formula (4.48) n times and obtain

$$\begin{aligned} w_t &= \int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} \Sigma_{\alpha_3\alpha^{2n-1}}(t-t_1)(B^{\Delta}_b)_{\alpha^{2n-1}\alpha^{2n-2}} \times ... \times \\ &\quad \times \Sigma_{\alpha^2\alpha^1}(t_{n-1}-t_n)(B^{\Delta}_b)_{\alpha^1\alpha_2} w_{t_n} dt_n ... dt_1. \end{aligned}$$

Since w_t is considered as an element of $\mathcal{K}_{\alpha_3}$, we get that, see (4.35),

$$\|w_t\|_{\alpha_3} \leq \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(2t\frac{\varpi(\alpha; B^{\Delta}_{\upsilon})}{\alpha_3 - \alpha_2}\right)^n \sup_{s\in[0,t]} \|w_t\|_{\alpha_2},$$

which yields that $w_t = 0$ for

$$t < \frac{\alpha_3 - \alpha_2}{2\varpi(\alpha; B^{\Delta}_{\upsilon})}.$$

To prove that $w_t = 0$ for all t, we need to repeat this construction due times.

□

Before proceed further, we prove some corollary of Lemma 4.1.3 related to the predual evolution in $\mathcal{G}_\alpha$, see (4.23). Let S_α be the semigroup as in Lemma 4.1.1. For $\alpha' > \alpha$, let $S_{\alpha\alpha'}(t)$ be the restriction of $S_\alpha(t)$ to $\mathcal{G}_{\alpha'} \hookrightarrow \mathcal{G}_\alpha$. Along with the operators defined in (4.25) we consider the predual operators to B^{Δ}_{υ}, see (2.20) and (4.32). That is, they act

$$\begin{aligned} (B_1 G)(\eta) &= -\sum_{x\in\eta} G(\eta\setminus x)E^a(x, \eta\setminus x), \\ (B_{2,\upsilon} G)(\eta) &= 2\int_{(\mathbb{R}^d)^2} \sum_{x\in\eta} G(\eta\setminus x \cup y_1) b(x|y_1, y_2) dy_1 dy_2 + \upsilon|\eta|G(\eta). \end{aligned}$$

By means of these expressions we can define bounded operators acting from $\mathcal{G}_\alpha$ to $\mathcal{G}_{\alpha'}$ for $\alpha' < \alpha$. It turns out that the estimate is exactly as in (4.33), that is,

$$\|(B_\upsilon)_{\alpha'\alpha}\| = \frac{2\langle\beta\rangle + \upsilon + \langle a\rangle e^{\alpha'}}{e(\alpha - \alpha')}.$$

Recall that $\mathcal{A}(B_\upsilon^\Delta)$ is defined in (4.39). For $(\alpha_2, \alpha_1, t) \in \mathcal{A}(B_\upsilon^\Delta)$, let $T < T(\alpha_2, \alpha_1)$ be fixed. Then we pick $\alpha \in [\alpha_1, \alpha_2)$ and $\delta < \alpha_2 - \alpha$ such that $T < T(\alpha_2, \alpha + \delta)$. Then, for some $l \in \mathbb{N}$, set, cf. (4.42),

$$\alpha_{2s} = \alpha_2 - \frac{s}{l+1}\delta - s\epsilon, \quad \alpha^{2s+1} = \alpha_2 - \frac{s+1}{l+1}\delta - s\epsilon,$$

where $\epsilon = (\alpha_2 - \alpha - \delta)/l$. Then for $0 \le t_l \le \cdots \le t_1 \le t_0 := t$ we define, cf. (4.43),

$$\begin{aligned} \Omega^{(l)}_{\alpha\alpha_2}(t, t_1, \ldots, t_n) &= S_{\alpha\alpha^{2l}}(t - t_1)(B_\upsilon)_{\alpha^{2l}\alpha^{2l-1}} S_{\alpha^{2l-1}\alpha^{2l-2}}(t_1 - t_2) \times \\ &\times S_{\alpha^3\alpha^2}(t_{l-1} - t_l)(B_\upsilon)_{\alpha^2\alpha^1} S_{\alpha^1\alpha_2}(t_l). \end{aligned}$$

Thereafter, we set

$$H_{\alpha\alpha_2}(t) = S_{\alpha\alpha_2}(t) + \sum_{l=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} \Omega^{(l)}_{\alpha\alpha_2}(t, t_1, \ldots, t_n) dt_l dt_{l-1} \cdots dt_1. \tag{4.49}$$

Then exactly as in the case of Lemma 4.1.3 we prove the following statement.

Proposition 4.1.6. *Each member of the family of operators* $\{H_{\alpha\alpha_2}(t) : (\alpha_2, \alpha, t) \in \mathcal{A}(B_\upsilon^\Delta)\}$ *defined in (4.49) has the following properties:*

(a) $H_{\alpha\alpha_2}(t) \in \mathcal{L}(\mathcal{G}_{\alpha_2}, \mathcal{G}_\alpha)$, *the operator norm of which satisfies*

$$\|H_{\alpha\alpha_2}(t)\| \le \frac{T(\alpha_2, \alpha)}{T(\alpha_2, \alpha) - t};$$

(b) For each $k \in \mathcal{K}_\alpha$ *and* $G \in \mathcal{G}_{\alpha_2}$, *it follows that*

$$\langle\!\langle G, Q_{\alpha_2\alpha}(t)k\rangle\!\rangle = \langle\!\langle H_{\alpha\alpha_2}(t)G, k\rangle\!\rangle. \tag{4.50}$$

4.1.6 Proof of the Identification Lemma 2.3.4

In the proof of the Identification Lemma, we follow the steps given in the sketch of the proof below the Lemma itself.
Auxiliary evolutions. For $\sigma > 0$ and $x \in \mathbb{R}^d$, we set

$$\phi_\sigma(x) = \exp\left(-\sigma|x|^2\right), \quad \langle\phi_\sigma\rangle = \int_{\mathbb{R}^d} \phi_\sigma(x)dx. \tag{4.51}$$

$$b_\sigma(x|y_1, y_2) = b(x|y_1, y_2)\phi_\sigma(y_1)\phi_\sigma(y_2).$$

Then we consider

$$L^{\Delta,\sigma} = A^{\Delta,\sigma} + B^{\Delta,\sigma} = A^{\Delta,\sigma}_\upsilon + B^{\Delta,\sigma}_\upsilon, \tag{4.52}$$

which we obtain from the corresponding operators in (2.20) and (4.31), (4.32) by replacing b by b_σ given in (4.51). Since this substitution does not affect $\mathcal{D}^\Delta_\alpha$, see (2.22), we will use the latter as the domain of the corresponding unbounded operators. Then we repeat the construction as in the proof of Lemma 4.1.3 and obtain the family $\{Q^\sigma_{\alpha_2\alpha_1}(t) : (\alpha_1, \alpha_2, t) \in \mathcal{A}(B^\Delta_\upsilon)\}$ corresponding to the choice $\mathbf{B} = B^{\Delta,\sigma}_\upsilon$. Along with the evolution $t \mapsto Q^\sigma_{\alpha_2\alpha_1}(t)k_0$ we will consider two more evolutions in L^∞- and L^1-like spaces. The latter one is positive in the sense of Proposition 1.2.5 by the very construction and is related somehow with the L^∞-like evolution, which, in turn, coincides with $t \mapsto Q^\sigma_{\alpha_2\alpha_1}(t)k_0$.

L^∞-like evolution. For $u : \Gamma_0 \to \mathbb{R}$, we define the norm

$$\|u\|_{\sigma,\alpha} = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \frac{|u(\eta)|\exp(-\alpha|\eta|)}{e(\phi_\sigma;\eta)}, \tag{4.53}$$

where

$$e(\phi_\sigma;\eta) = \prod_{x\in\eta}\phi_\sigma(x) = \exp\left(-\sigma\sum_{x\in\eta}|x|^2\right),$$

cf. (1.24). Then we consider the Banach space $\mathcal{U}_{\sigma,\alpha} = \{u : \Gamma_0 \to \mathbb{R} : \|u\|_{\sigma,\alpha} < \infty\}$. Clearly,

$$\mathcal{U}_{\sigma,\alpha} \hookrightarrow \mathcal{K}_\alpha, \qquad \alpha \in \mathbb{R}. \tag{4.54}$$

The space predual to $\mathcal{U}_{\sigma,\alpha}$ is the L^1-space equipped with the norm, cf. (4.23), (4.24),

$$|G|_{\sigma,\alpha} = \int_{\Gamma_0} |G(\eta)|\exp(\alpha|\eta|)e(\phi_\sigma;\eta)\lambda(d\eta). \tag{4.55}$$

In this space, we define $A^\sigma_{1,\upsilon}$ which acts exactly as in (4.25), and A^σ_2 which acts as in (4.25) with b replaced by b_σ. Their domain is the same $\mathcal{D}_\alpha$. Then like in (4.28) by means of (1.32) and (4.55) we obtain

$$\begin{aligned}
|A^\sigma_2 G|_{\sigma,\alpha} = &\int_{\Gamma_0}\left(\sum_{x\in\eta}\int_{(\mathbb{R}^d)^2}|G(\eta\setminus x\cup\{y_1,y_2\})|b_\sigma(x|y_1,y_2)dy_1dy_2\right)\\
&\times\exp(\alpha|\eta|)e(\phi_\sigma;\eta)\lambda(d\eta)\\
=&e^\alpha\int_{\Gamma_0}\left(\int_{(\mathbb{R}^d)^3}|G(\eta\cup\{y_1,y_2\})|b_\sigma(x|y_1,y_2)\phi_\sigma(x)dxdy_1dy_2\right)\\
&\times\exp(\alpha|\eta|)e(\phi_\sigma;\eta)\lambda(d\eta)\\
\leq&e^\alpha\int_{\Gamma_0}\left(\int_{(\mathbb{R}^d)^2}|G(\eta\cup\{y_1,y_2\})|\beta(y_2-y_1)e(\phi_\sigma;\eta\cup\{y_1,y_2\})dy_1dy_2\right)\\
&\times\exp(\alpha|\eta|)\lambda(d\eta)\\
=&e^{-\alpha}\int_{\Gamma_0}E^b(\eta)|G(\eta)|e^{\alpha|\eta|}e(\phi_\sigma;\eta)\lambda(d\eta)\\
\leq&(e^{-\alpha}/\omega)\int_{\Gamma_0}e^{\alpha|\eta|}\Psi_\upsilon(\eta)|G(\eta)|e(\phi;\eta)\lambda(d\eta)\\
=&(e^{-\alpha}/\omega)|A^\sigma_{1,\upsilon}G|_{\sigma,\alpha}.
\end{aligned}$$

This allows us to prove the following analog of Lemma 4.1.1.

Proposition 4.1.7. *Let υ and ω be as in Proposition 2.1.2 and $A^\sigma_{1,\upsilon}$, A^σ_2 and $\mathcal{D}_\alpha$ be as just described. Then for each $\alpha > -\log\omega$, the operator $(A^\sigma_\upsilon, \mathcal{D}_\alpha) := (A^\sigma_{1,\upsilon} + A^\sigma_2, \mathcal{D}_\alpha)$ is the generator of a sub-stochastic semigroup $S_{\sigma,\alpha} = \{S_{\sigma,\alpha}(t)\}_{t\geq 0}$ on $\mathcal{G}_{\sigma,\alpha}$.*

Let $S^\odot_{\sigma,\alpha}$ be the sun-dual semigroup, the definition of which is pretty analogous to that of $S^\odot_\alpha$, see Proposition 4.1.2. Then, for $\alpha' < \alpha$, we define $\Sigma^\sigma_{\alpha\alpha'}(t) = S^\odot_{\sigma,\alpha}(t)|_{\mathcal{U}_{\sigma.\alpha'}}$. As in Proposition 4.1.2 we then get that the map

$$[0,+\infty)\ni t\mapsto \Sigma^\sigma_{\alpha\alpha'}(t)\in\mathcal{L}(\mathcal{U}_{\sigma,\alpha'},\mathcal{U}_{\sigma,\alpha})$$

is continuous and

$$\|\Sigma^\sigma_{\alpha,\alpha'}(t)\|\leq 1,\qquad \text{for all } t\geq 0.$$

The operators $B_\upsilon^{\Delta,\sigma} = B_1^{\Delta,\sigma} + B_{2,\upsilon}^{\Delta,\sigma}$ act as in (4.32) with b replaced by b_σ. Then we define the corresponding bounded operators and obtain, cf. (4.33),

$$\|(B_\upsilon^{\Delta,\sigma})_{\alpha\alpha'}\| \leq \frac{2\langle b\rangle + \upsilon + \langle a\rangle e^{\alpha'}}{e(\alpha-\alpha')}.$$

Thereafter, we take $\delta > 0$ as in Lemma 4.1.3 and the division as in (4.42), and then define

$$\Pi^{l,\sigma}_{\alpha\alpha'}(t,t_1,t_2,...,t_l) = \Sigma^\sigma_{\alpha\alpha^{2l}}(t-t_1)(B_\upsilon^{\Delta,\sigma})_{\alpha^{2l}\alpha^{2l-1}}\cdots\Sigma^\sigma_{\alpha^{2s+1}\alpha^{2s}}(t_{l-s}-t_{l-s+1})$$
$$\times(B_\upsilon^{\Delta,\sigma})_{\alpha^{2s}\alpha^{2s-1}}\cdots\Sigma^\sigma_{\alpha^3\alpha^2}(t_{l-1}-t_l)(B_\upsilon^{\Delta,\sigma})_{\alpha^2\alpha^1}\Sigma^\sigma_{\alpha^1\alpha'}(t_l),$$

As in the proof of Lemma 4.1.3 we obtain the family $\{U^\sigma_{\alpha_2\alpha_1}(t) : (\alpha_1,\alpha_2,t)\in\mathcal{A}(B_\upsilon^\Delta)\}$, see (4.39), with members defined by

$$U^\sigma_{\alpha_2\alpha_1}(t) = \Sigma^\sigma_{\alpha_2\alpha_1}(t) + \sum_{l=1}^\infty\int_0^t\int_0^{t_1}...\int_0^{t_{l-1}}\Pi^{l,\sigma}_{\alpha_2\alpha_1}(t,t_1,t_2,...,t_l)dt_l...dt_1,$$

where the series converges for $t < T(\alpha_2,\alpha_1)$ defined in (2.27), cf. (4.38) and (4.47). For this family, the following holds, cf. (4.41),

$$\frac{d}{dt}U^\sigma_{\alpha_2\alpha_1}(t) = L^{\Delta,\sigma}_{\alpha_2,u}U^\sigma_{\alpha_2\alpha_1}(t), \tag{4.56}$$

where the action of of $L^{\Delta,\sigma}_{\alpha_2,u}$ is as in (4.52) and the domain is

$$\mathcal{D}^{\Delta,\sigma}_{\alpha_2,u} = \{u\in\mathcal{U}_{\sigma,\alpha_2} : \Psi_\upsilon u\in\mathcal{U}_{\sigma,\alpha_2}\}\subset\mathcal{D}^\Delta_{\alpha_2}, \tag{4.57}$$

where the latter inclusion follows by (4.54) and (2.22). Then by (4.57) we have that

$$(L^{\Delta,\sigma}_{\alpha,u},\mathcal{D}^{\Delta,\sigma}_{\alpha,u})\subset(L^{\Delta,\sigma}_\alpha,\mathcal{D}^\Delta_\alpha). \tag{4.58}$$

Now by (4.56) we prove the following statement.

Proposition 4.1.8. *For each $\alpha_2 > \alpha_1 > -\log\omega$, the problem*

$$\dot{u}_t = L^{\Delta,\sigma}_{\alpha_2,u}u_t, \qquad u_t|_{t=0} = u_0\in\mathcal{U}_{\sigma,\alpha_1} \tag{4.59}$$

has a unique solution $u_t\in\mathcal{U}_{\sigma,\alpha_2}$ on the time interval $[0,T(\alpha_2,\alpha_1))$. This solution is given by $u_t = U^\sigma_{\alpha_2\alpha_1}(t)u_0$.

Corollary 4.1.9. *Let $\alpha_2 > \alpha_1 > -\log\omega$ be as in Proposition 4.1.8 and $Q^\sigma_{\alpha_2\alpha_1}(t)$ be as described at the beginning of this subsection. Then for each $t < T(\alpha_2, \alpha_1)$ and $u_0 \in \mathcal{U}_{\sigma,\alpha_1} \subset \mathcal{K}_{\alpha_1}$, it follows that*

$$U^\sigma_{\alpha_2\alpha_1}(t)u_0 = Q^\sigma_{\alpha_2\alpha_1}(t)u_0. \tag{4.60}$$

Proof. By (4.58) we get that the solution of (4.59) is also the unique solution of the following "σ-analog" of (2.26)

$$\dot{u}_t = L^{\Delta,\sigma}_{\alpha_2} u_t, \quad u_t|_{t=0} = u_0,$$

and hence is given by the right-hand side of (4.60). Then the equality in (4.60) follows by the uniqueness just mentioned. □

L^1-like evolution. Now we take $L^{\Delta,\sigma}$ as given in (4.52) and define the corresponding operator $L^{\Delta,\sigma}_\vartheta$ in $\mathcal{G}_\vartheta$, $\vartheta \in \mathbb{R}$, introduced in (4.23), (4.24), with domain $\mathcal{D}_\vartheta$ given in (4.25). By (4.52) and (2.20) we have that $A^\Delta_1 : \mathcal{D}_\vartheta \to \mathcal{G}_\vartheta$. Next, for $q \in \mathcal{D}_\vartheta$, we have

$$\begin{aligned}
&|A^{\Delta,\sigma}_2 q|_\vartheta \\
&\quad \leq \int_{\Gamma_0} e^{\vartheta|\eta|} \left(\int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} |q(\eta\cup x \setminus \{y_1, y_2\})| b_\sigma(x|y_1.y_2)dx \right) \lambda(d\eta) \\
&\quad \leq \int_{\Gamma_0} e^{\vartheta|\eta|+2\vartheta} \int_{\mathbb{R}^d} |q(\eta\cup x)| \left(\int_{(\mathbb{R}^d)^2} b(x|y_1, y_2)dy_1 dy_2 \right) dx\lambda(d\eta) \\
&\quad = \langle b\rangle e^\vartheta \int_{\Gamma_0} |\eta| e^{\vartheta|\eta|} |q(\eta)|\lambda(d\eta) \\
&\quad \leq e^\vartheta \int_{\Gamma_0} \Psi(\eta) e^{\vartheta|\eta|} |q(\eta)|\lambda(d\eta),
\end{aligned} \tag{4.61}$$

see item (iii) of Assumption 1 and (2.15). Hence, $A^{\Delta,\sigma}_2 : \mathcal{D}_\vartheta \to \mathcal{G}_\vartheta$. Next, for the same q, we have

$$\begin{aligned}
|B^\Delta_1 q|_\vartheta &\leq \int_{\Gamma_0} e^{\vartheta|\eta|} \left(\int_{\mathbb{R}^d} |q(\eta\cup x)| E^a(x,\eta)dx \right) \lambda(d\eta) \qquad (4.62) \\
&= e^{-\vartheta} \int_{\Gamma_0} e^{\vartheta|\eta|} E^a(\eta)|q(\eta)|\lambda(d\eta) \leq e^{-\vartheta} \int_{\Gamma_0} \Psi(\eta) e^{\vartheta|\eta|} |q(\eta)|\lambda(d\eta).
\end{aligned}$$

Hence, $B_1^\Delta : \mathcal{D}_\vartheta \to \mathcal{G}_\vartheta$. Finally,

$$\begin{aligned}|B_2^{\Delta,\sigma} q|_\vartheta &\leq 2\int_{\Gamma_0} e^{\vartheta|\eta|}\left(\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta}|q(\eta\cup x\setminus y_1)|b_\sigma(x|y_1,y_2)dy_2dx\right)\lambda(d\eta)\\ &\leq 2\int_{\Gamma_0} e^{\vartheta|\eta|+\vartheta}\left(\int_{(\mathbb{R}^d)^3}|q(\eta\cup x)|b(x|y_1,y_2)dxdy_1dy_2\right)\lambda(d\eta)\\ &= 2\langle b\rangle\int_{\Gamma_0} e^{\vartheta|\eta|}|\eta||q(\eta)|\lambda(d\eta) \leq \int_{\Gamma_0}\Psi(\eta)e^{\vartheta|\eta|}|q(\eta)|\lambda(d\eta).\end{aligned} \tag{4.63}$$

Then by (4.61), (4.62) and (4.63) we conclude that, for an arbitrary $\vartheta \in \mathbb{R}$, $L^{\Delta,\sigma} = A_1^\Delta + A_2^{\Delta,\sigma} + B_1^\Delta + B_2^{\Delta,\sigma}$ maps $\mathcal{D}_\vartheta$ to $\mathcal{G}_\vartheta$ and hence can be used to define the corresponding unbounded operator $(L_\vartheta^{\Delta,\sigma}, \mathcal{D}_\vartheta)$. Let us then consider the corresponding Cauchy problem

$$\dot{q}_t = L_\vartheta^{\Delta,\sigma} q_t, \qquad q_t|_{t=0} = q_0 \in \mathcal{D}_\vartheta. \tag{4.64}$$

Recall that $\mathcal{G}_{\vartheta'} \subset \mathcal{D}_\vartheta$ for each $\vartheta' > \vartheta$.

Lemma 4.1.10. *For a given $\vartheta > 0$ and $\vartheta' > \vartheta$, assume that the problem in (4.64) with $q_0 \in \mathcal{G}_{\vartheta'}$ has a solution $q_t \in \mathcal{G}_\vartheta$ on a time interval $[0, \tau_q)$. Then this solution is unique.*

Proof. Set

$$w_t(\eta) = (-1)^{|\eta|} q_t(\eta).$$

Then $|w_t|_\vartheta = |q_t|_\vartheta$ and q_t solves (4.64) if and only if w_t solves the following equation

$$\dot{w}_t = \left(A_1^\Delta - A_2^{\Delta,\sigma} - B_1^\Delta + B_2^{\Delta,\sigma}\right) w_t. \tag{4.65}$$

By Proposition 1.5.1 we prove that $(A_1^\Delta - B_1^\Delta, \mathcal{D}_\vartheta)$ generates a sub-stochastic semigroup on $\mathcal{G}_\vartheta$. Indeed, $(A_1^\Delta, \mathcal{D}_\vartheta)$ generates a sub-stochastic semigroup defined in (4.27) with $\upsilon = 0$, and $-B_1^\Delta$ is positive and defined on $\mathcal{D}_\vartheta$, see (4.62). Also by (4.62), for $w \in \mathcal{G}_\vartheta^+$ and $r \in (0,1)$, we get

$$\int_{\Gamma_0} e^{\vartheta|\eta|} \left(\left(A_1^\Delta - r^{-1}B_1^\Delta\right) w\right)(\eta)\lambda(d\eta) = -\int_{\Gamma_0} e^{\vartheta|\eta|}\Psi(\eta)w(\eta)\lambda(d\eta)$$

$$+ r^{-1}\int_{\Gamma_0} e^{\vartheta|\eta|}\left(\int_{\mathbb{R}^d} w(\eta\cup x)E^a(x,\eta)dx\right)\lambda(d\eta)$$

$$= -\int_{\Gamma_0} e^{\vartheta|\eta|}\Psi(\eta)w(\eta)\lambda(d\eta) + r^{-1}e^{-\vartheta}\int_{\Gamma_0} e^{\vartheta|\eta|}E^a(\eta)w(\eta)\lambda(d\eta)$$

$$\leq -\left(1 - r^{-1}e^{-\vartheta}\right)\int_{\Gamma_0}\Psi(\eta)e^{\vartheta|\eta|}w(\eta)\lambda(d\eta) \leq 0,$$

where the latter inequality holds for $r \in (e^{-\vartheta}, 1)$. Therefore, $(A_1^\Delta - B_1^\Delta, \mathcal{D}_\vartheta) = (A_1^\Delta - rr^{-1}B_1^\Delta, \mathcal{D}_\vartheta)$ generates a sub-stochastic semigroup $V_\vartheta = \{V_\vartheta(t)\}_{t\geq 0}$ on $\mathcal{G}_\vartheta$. For each $\vartheta'' \in (0,\vartheta)$, we have that $\mathcal{G}_\vartheta \hookrightarrow \mathcal{G}_{\vartheta''}$. By the estimates in (4.61) and (4.63), similarly as in (4.33) we obtain that

$$|(A_2^{\Delta,\sigma} w|_{\vartheta''} \leq \frac{\langle b\rangle}{e(\vartheta-\vartheta'')}|w|_\vartheta,$$

$$|(B_2^{\Delta,\sigma} w|_{\vartheta''} \leq \frac{2\langle b\rangle}{e(\vartheta-\vartheta'')}|w|_\vartheta,$$

which we then use to define a bounded operator $C^{\Delta,\sigma}_{\vartheta''\vartheta} : \mathcal{G}_\vartheta \to \mathcal{G}_{\vartheta''}$. It acts as $-A_2^{\Delta,\sigma} + B_2^{\Delta,\sigma}$ and its norm satisfies

$$\|C^{\Delta,\sigma}_{\vartheta''\vartheta}\| \leq \frac{3\langle b\rangle}{e(\vartheta-\vartheta'')}. \tag{4.66}$$

Assume now that (4.65) has two solutions corresponding to the same initial condition w_0. Let v_t be their difference. Then it solves (4.65) with the zero initial condition and hence satisfies

$$v_t = \int_0^t V_{\vartheta''}(t-s)C^{\Delta,\sigma}_{\vartheta''\vartheta}v_s ds \tag{4.67}$$

where v_t in the left-hand side is considered as an element of $\mathcal{G}_{\vartheta''}$ and $t > 0$ will be chosen later. Now for a given $n \in \mathbb{N}$, we set $\epsilon = (\vartheta - \vartheta'')/n$

and $\vartheta^l = \vartheta - l\epsilon$, $l = 0, \ldots, n$. Next, we iterate (4.67) due times and get

$$v_t = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} V_{\vartheta''}(t - t_1) C^{\Delta,\sigma}_{\vartheta''\vartheta^{n-1}} V_{\vartheta^{n-1}}(t_1 - t_2) C^{\Delta,\sigma}_{\vartheta^{n-1}\vartheta^{n-2}} \times \cdots \times V_{\vartheta^1}(t_{n-1} - t_n) C^{\Delta,\sigma}_{\vartheta^{n-1}\vartheta} v_{t_n} dt_n \cdots dt_1.$$

Then we take into account that V_ϑ is sub-stochastic, $C^{\Delta,\sigma}_{\vartheta^l\vartheta^{l-1}}$ are positive and satisfy (4.66), and thus obtain from the latter that v_t satisfies

$$|v_t|_{\vartheta''} \leq \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(\frac{3t\langle b\rangle}{\vartheta - \vartheta''}\right)^n \sup_{s\in[0,t]} |v_s|_\vartheta.$$

Then, since n is an arbitrary positive integer, for all $t < (\vartheta - \vartheta'')/3\langle b\rangle$ it follows that $v_t = 0$. To prove that $v_t = 0$ for all t of interest one has to repeat the above procedure appropriate number of times. □

Let us now take $u \in \mathcal{U}_{\sigma,\alpha}$ with some $\alpha \in \mathbb{R}$, for which by (4.53) we have

$$|u(\eta)| \leq \|u\|_{\sigma,\alpha} e^{\alpha|\eta|} e(\phi_\sigma, \eta).$$

Then the norm of this u in $\mathcal{G}_\vartheta$ can be estimated as follows, see (4.51),

$$\begin{aligned} |u|_\vartheta &\leq \|u\|_{\sigma,\alpha} \int_{\Gamma_0} \exp\left((\alpha + \vartheta)|\eta|\right) e(\phi_\sigma, \eta)\lambda(d\eta) \\ &= \|u\|_{\sigma,\alpha} \exp\left((\alpha + \vartheta)\langle\phi\rangle\right). \end{aligned} \tag{4.68}$$

This means that $\mathcal{U}_{\sigma,\alpha} \hookrightarrow \mathcal{G}_\vartheta$ for each pair of real α and ϑ. Moreover, for the operators discussed above this implies, cf. (4.58),

$$(L^{\Delta,\sigma}_{\alpha,u}, \mathcal{D}^{\Delta,\sigma}_{\alpha,u}) \subset (L^{\Delta,\sigma}_{\vartheta}, \mathcal{D}_\vartheta). \tag{4.69}$$

Corollary 4.1.11. *Let α_1 and α_2 be as in Proposition 4.1.8. Then for each $q_0 \in \mathcal{U}_{\sigma,\alpha_1}$, the problem in (4.64) has a unique solution $q_t \in \mathcal{U}_{\sigma,\alpha_2}$, $t < T(\alpha_2, \alpha_1)$, which coincides with the unique solution of (4.59).*

Proof. By (4.69) we have that the unique solution of (4.59) u_t solves also (4.64), and this is a unique solution in view of Lemma 4.1.10. □

Local approximations. Our aim now is to prove that, cf. Proposition 1.2.5, the following holds

$$\langle\!\langle G, Q^{\sigma}_{\alpha_2\alpha_1}(t)k_0\rangle\!\rangle \geq 0, \qquad G \in B^{\star}_{\rm bs}(\Gamma_0), \tag{4.70}$$

for some $t > 0$. By Corollaries 4.1.9 and 4.1.11 to this end it is enough to prove (4.70) with $Q^{\sigma}_{\alpha_2\alpha_1}(t)k_0$ replaced by q_t. To get the latter we use local approximations of this q_t.

Let $\mu_0 \in \mathcal{P}_{\rm exp}(\Gamma)$ be the initial state of our model. For a compact Λ, let $\mu_0^{\Lambda} \in \mathcal{P}(\Gamma_\Lambda)$ be its projection to Γ_Λ defined in (1.16). Finally, let R_0^{Λ} be its Radon-Nikodym derivative, see (1.25). For $N \in \mathbb{N}$ and $\eta \in \Gamma_0$, we set

$$R_0^{\Lambda,N}(\eta) = \begin{cases} R_0^{\Lambda}(\eta), & \text{if } \ \eta \in \Gamma_\Lambda \ \text{ and } \ |\eta| \leq N; \\ 0, & \text{otherwise.} \end{cases} \tag{4.71}$$

Let us stress here that R_0^{Λ} whereas $R_0^{\Lambda,N}$ is defined on Γ_0. Until the end of this subsection, Λ and N are fixed.

Having in mind (1.26) we introduce the following function $q_0 : \Gamma_0 \to \mathbb{R}_+$

$$q_0^{\Lambda,N}(\eta) = \int_{\Gamma_0} R_0^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi). \tag{4.72}$$

For $G \in B^{\star}_{\rm bs}(\Gamma_0)$, by (1.28), (1.32) and (4.72) we have

$$\langle\!\langle G, q_0^{\Lambda,N}\rangle\!\rangle = \langle\!\langle KG, R_0^{\Lambda,N}\rangle\!\rangle \geq 0. \tag{4.73}$$

By (4.71) it follows that $R_0^{\Lambda,N} \in \mathcal{R}^+$ and $\|R_0^{\Lambda,N}\|_{\mathcal{R}} \leq 1$. Moreover, for each $\kappa > 0$, we have, see (1.22),

$$\|R_0^{\Lambda,N}\|_{\mathcal{R}_{\chi^\kappa}} = \int_{\Gamma_\Lambda} e^{\kappa|\eta|} R_0^{\Lambda,N}(\eta)\lambda(d\eta) \leq e^{\kappa N}\|R_0^{\Lambda,N}\|_{\mathcal{R}} \leq e^{\kappa N}. \tag{4.74}$$

Let $S^{\sigma}_{\mathcal{R}}$ be the stochastic semigroup on $\mathcal{R}$ constructed in the proof of Theorem 2.2.2 with b replaced by b_σ. Recall that the solution of (4.10) is obtained in the form $R_t = S_{\mathcal{R}}(t)R_0$. For $t > 0$ and σ as in (4.70), we set

$$\begin{aligned} R_t^{\Lambda,N} &= S^{\sigma}_{\mathcal{R}}(t)R_0^{\Lambda,N}, \\ q_t^{\Lambda,N}(\eta) &= \int_{\Gamma_0} R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi), \quad \eta \in \Gamma_0. \end{aligned} \tag{4.75}$$

Proposition 4.1.12. *For each $\vartheta \in \mathbb{R}$ and $t < \tau_\vartheta := [e\langle b\rangle(1+e^\vartheta)]^{-1}$, it follows that $q_t^{\Lambda,N} \in \mathcal{G}_\vartheta^+$. Moreover,*

$$\langle\!\langle G, q_t^{\Lambda,N}\rangle\!\rangle \geq 0 \tag{4.76}$$

holding for each $G \in B_{\rm bs}^\star(\Gamma_0)$ and all $t > 0$.

Proof. Since $S_{\mathcal{R}}^\sigma$ is stochastic and $R_0^{\Lambda,N}$ is as in (4.71), then $R_t^{\Lambda,N} \in \mathcal{R}^+$ for all $t > 0$. Hence, $q_t^{\Lambda,N}(\eta) \geq 0$ for all those $t > 0$ for which the integral in the second line in (4.75) makes sense. By (2.19) we have that $T(\kappa, \kappa')$, as a function of κ, attains its maximum value $T_{\kappa'} = e^{-\kappa'}/e\langle b\rangle$ at $\kappa = \kappa' + 1$. By (4.74) we have that $R_0^{\Lambda,N} \in \mathcal{R}_{\chi^\kappa}$ for any $\kappa > 0$. Then by Proposition 1.5.2 it follows that, for each $\kappa > 0$, $R_0^{\Lambda,N} \in \mathcal{R}_{\chi^\kappa}$ for $t < T_\kappa$. Taking all these fact into account we then get

$$\begin{aligned} |q_t^{\Lambda,N}|_\vartheta &= \int_{\Gamma_0} e^{\vartheta|\eta|} q_t^{\Lambda,N}(\eta)\lambda(d\eta) \\ &= \int_{\Gamma_0}\int_{\Gamma_0} e^{\vartheta|\eta|} R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\eta)\lambda(d\xi) \\ &= \int_{\Gamma_0} \left(1+e^\vartheta\right)^{|\eta|} R_t^{\Lambda,N}(\eta)\lambda(d\eta) = \|R_t^{\Lambda,N}\|_{\mathcal{R}_{\chi^\kappa}} \end{aligned} \tag{4.77}$$

with $\kappa = \log(1+e^\vartheta)$. For these κ and ϑ, we have that $T_\kappa = \tau_\vartheta$. Then by (4.77) we have $q_t^{\Lambda,N} \in \mathcal{G}_\vartheta$ for $t < \tau_\vartheta$. The existence of the integral and the validity of the inequality in (4.76) follows analogously to that in (4.73). □

Corollary 4.1.13. *For each $\alpha \in \mathbb{R}$, it follows that $q_0^{\Lambda,N} \in \mathcal{U}_{\sigma,\alpha}^+$.*

Proof. Set $I_N(\eta) = 1$ whenever $|\eta| \leq N$ and $I_N(\eta) = 0$ otherwise. By (4.71), (4.72) and (1.26) we have that

$$q_0^{\Lambda,N}(\eta) \leq I_N(\eta)\mathbb{1}_{\Gamma_\Lambda}(\eta)\int_{\Gamma_\Lambda} R_0(\eta\cup\xi)\lambda(d\xi) \leq k_0(\eta)I_N(\eta)\mathbb{1}_{\Gamma_\Lambda}(\eta).$$

Since $k_0 = k_{\mu_0}$ for some $\mu_0 \in \mathcal{P}_{\rm exp}(\Gamma)$, by Definition 1.2.3 and then by (1.21) we have that $k_0(\eta) \leq \|k\|_\alpha \exp(\alpha_0|\eta|)$ for some $\alpha_0 \in \mathbb{R}$. We apply this in the latter estimate to check by means of (4.53) that $\|q_0^{\Lambda,N}\|_{\sigma,\alpha} < \infty$. The stated positivity i immediate. □

By (4.68) and Corollary 4.1.13 we obtain that $q_0^{\Lambda,N} \in \mathcal{G}_\vartheta^+$ for each $\vartheta \in \mathbb{R}$. Now we relate $q_t^{\Lambda,N}$ with solutions of (4.64).

Lemma 4.1.14. *For each $\vartheta \in \mathbb{R}$, the map $[0, \tau_\vartheta) \ni t \mapsto q_t^{\Lambda,N} \in \mathcal{G}_\vartheta$ is continuous and continuously differentiable on $(0, \tau_\vartheta)$. Moreover, $q_t^{\Lambda,N} \in \mathcal{D}_\vartheta$, see (4.25), and solves the problem in (4.64) on the time interval $[0, \tau_\vartheta)$ with $q_0^{\Lambda,N}$ as the initial condition.*

Proof. Fix an arbitrary $\vartheta \in \mathbb{R}$. The stated continuity of $t \mapsto q_t^{\Lambda,N}$ follows by (4.75). Let us prove that $q_t^{\Lambda,N}$ be differentiable in $\mathcal{G}_\vartheta$ on $(0, \tau_\vartheta)$ and the following holds

$$\dot{q}_t^{\Lambda,N}(\eta) = \int_{\Gamma_0} R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi). \tag{4.78}$$

For small enough τ, we have

$$\frac{1}{\tau}\left(q_{t+\tau}^{\Lambda,N}(\eta) - q_t^{\Lambda,N}(\eta)\right) - \int_{\Gamma_0} \dot{R}_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi) \tag{4.79}$$

$$= \int_{\Gamma_0}\left[\frac{1}{\tau}\left(R_{t+\tau}^{\Lambda,N}(\eta \cup \xi) - R_t^{\Lambda,N}(\eta \cup \xi)\right) - \dot{R}_t^{\Lambda,N}(\eta \cup \xi)\right]\lambda(d\xi).$$

Then by (1.32) we get

$$|\mathrm{LHS}(4.79)|_\theta \leq \int_{\Gamma_0}\left(1 + \mathrm{e}^\vartheta\right)^{|\eta|}\left|\frac{1}{\tau}\left(\mathrm{R}_{t+\tau}^{\Lambda,\mathrm{N}}(\eta) - \mathrm{R}_{\mathrm{t}}^{\Lambda,\mathrm{N}}(\eta)\right) - \dot{\mathrm{R}}_{\mathrm{t}}^{\Lambda,\mathrm{N}}(\eta)\right|\lambda(\mathrm{d}\eta),$$

that proves (4.78), cf. (4.77). The continuity of $t \in \dot{q}_t^{\Lambda,N}$ follows by (4.78) and the fact that $R_t^{\Lambda,N} = S_{\mathcal{R}}^\sigma(t)R_0^{\Lambda,N}$, which also yields that

$$\dot{q}_t^{\Lambda,N}(\eta) = \int_{\Gamma_0}\left(L_\vartheta^{\dagger,\sigma}R_t^{\Lambda,N}\right)(\eta \cup \xi)\lambda(d\xi), \tag{4.80}$$

where $L_\vartheta^{\dagger,\sigma}$ is the trace of $L^{\dagger,\sigma}$ (the generator of $S_{\mathcal{R}}^\sigma$) in $\mathcal{R}_{\chi^\kappa}$ with $\kappa = \log(1 + e^\vartheta)$. By (4.25) it follows that $\Psi_\upsilon(\eta) \leq C_\varepsilon e^{\varepsilon|\eta|}$ holding for arbitrary $\varepsilon > 0$ and the corresponding $C_\varepsilon > 0$. For each $t < T_\kappa = \tau_\vartheta$, one can pick $\kappa' > \kappa$ such that $R_t^{\Lambda,N} \in \mathcal{R}_{\chi^{\kappa'}}$. For these t and κ', we thus pick $\varepsilon > 0$ such that $1 + e^{\vartheta+\varepsilon} = e^{\kappa'}$, and then obtain, cf. (1.30),

$$|\Psi_\upsilon q_t^{\Lambda,N}|_\vartheta \leq C_\varepsilon \|R_t^{\Lambda,N}\|_{\mathcal{R}_{\chi^{\kappa'}}}. \tag{4.81}$$

Hence, $q_t^{\Lambda,N} \in \mathcal{D}_\vartheta$ for this t. Let us now prove that $q_t^{\Lambda,N}$ solves (4.64). In view of (4.80), (4.3) and (4.81), to this end it is enough to prove that

$$\left(L^{\Delta}q_t^{\Lambda,N}\right)(\eta) = -\int_{\Gamma_0}\Psi(\eta\cup\xi)R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi) \tag{4.82}$$
$$+\int_{\mathbb{R}^d}\int_{\Gamma_0}(m(x)+E^a(x,\eta\cup\xi))R_t^{\Lambda,N}(\eta\cup\xi\cup x)\lambda(d\xi)dx$$
$$+\int_{\mathbb{R}^d}\int_{\Gamma_0}\sum_{y_1\in\eta\cup\xi}\ \sum_{y_1\in\eta\cup\xi\setminus y_1} b(x|y_1,y_2)\times$$
$$\times R_t^{\Lambda,N}(\eta\cup\xi\cup x\setminus\{y_1,y_2\})\lambda(d\xi)dx$$

holding point-wise in $\eta\in\Gamma_0$. By (2.15) and (2.2) we get

$$\Psi(\eta\cup\xi) = \Psi(\eta)+\Psi(\xi)+2\sum_{x\in\eta}\sum_{x\in\xi}a(x-y). \tag{4.83}$$

Let $I_1(\eta)$ denote the first summand in the right-hand side of (4.82). By (1.32) and (4.83) we then write it as follows

$$I_1(\eta) \;=\; -\Psi(\eta)q_t^{\Lambda,N} - 2\int_{\mathbb{R}^d}E^a(x\eta)q_t^{\Lambda,N}(\eta\cup x)dx \tag{4.84}$$
$$-\int_{\Gamma_0}\Psi(\xi)R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi).$$

To calculate the latter summand in (4.84) we again use (2.15) and (1.32) to obtain the following:

$$\int_{\Gamma_0}\left(\sum_{x\in\xi}m(x)\right)R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi) \tag{4.85}$$
$$= \int_{\Gamma_0}\int_{\mathbb{R}^d}m(x)R_t^{\Lambda,N}(\eta\cup\xi\cup x)\lambda(d\xi)dx$$
$$= \int_{\mathbb{R}^d}m(x)q_t^{\Lambda,N}(\eta\cup x)dx.$$

$$\int_{\Gamma_0}\left(\sum_{x\in\xi}\sum_{y\in\xi\setminus x}a(x-y)\right)R_t^{\Lambda,N}(\eta\cup\xi)\lambda(d\xi) \tag{4.86}$$
$$= \int_{\Gamma_0}\int_{(\mathbb{R}^d)^2}a(x-y)R_t^{\Lambda,N}(\eta\cup\xi\cup\{x,y\})\lambda(d\xi)dxdy$$
$$= \int_{(\mathbb{R}^d)^2}a(x-y)q_t^{\Lambda,N}(\eta\cup\{x,y\}).dxdy$$

$$\int_{\Gamma_0} \left(\langle b \rangle \sum_{x\in\xi} 1 \right) R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi) = \langle b \rangle \int_{\mathbb{R}^d} q_t^{\Lambda,N}(\eta \cup x)dx. \qquad (4.87)$$

In a similar way, we get the second I_2 (resp. the third I_3) summands of the right-hand side of (4.82) as follows

$$\begin{aligned} I_2(\eta) &= \int_{\mathbb{R}^d} (m(x) + E^a(x,\eta))q_t^{\Lambda,N}(\eta \cup x)dx \qquad (4.88) \\ &\quad + \int_{(\mathbb{R}^d)^2} a(x-y)q_t^{\Lambda,N}(\eta \cup \{x,y\})dxdy, \end{aligned}$$

$$\begin{aligned} I_3(\eta) &= \int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} b(x|y_1y_2)q_t^{\Lambda,N}(\eta \cup x \setminus \{y_1,y_2\})dx \qquad (4.89) \\ &\quad +2\int_{(\mathbb{R}^d)^2} \sum_{y_1\in\eta} b(x|y_1y_2)q_t^{\Lambda,N}(\eta \cup x \setminus y_1)dxdy_2 \\ &\quad +\langle b \rangle \int_{\mathbb{R}^d} q_t^{\Lambda,N}(\eta \cup x)dx. \end{aligned}$$

Now we plug (4.85), (4.86) and (4.87) into (4.84), and then use it together with (4.88) and (4.89) in the right-hand side of (4.82) to get its equality with the left-hand side, see (2.20). This completes the proof.

□

Corollary 4.1.15. *Let $\alpha_1 > -\log\omega$ and $\alpha_2 > \alpha_1$ be chosen. Then $k_t^{\Lambda,N} = Q^\sigma_{\alpha_2\alpha_1}(t)q_0^{\Lambda,N}$ has the property*

$$\langle\!\langle G, k_t^{\Lambda,N} \rangle\!\rangle \geq 0, \qquad (4.90)$$

holding for all $G \in B^\star_{\mathrm{bs}}(\Gamma_0)$ and $t < T(\alpha_2,\alpha_1)$.

Proof. The proof of (4.90) will be done by showing that $k_t^{\Lambda,N} = q_t^{\Lambda,N}$, for $t < T(\alpha_2,\alpha_1)$ and then by employing (4.76), which holds for all $t > 0$.

By Corollary 4.1.13 it follows that $q_0^{\Lambda,N} \in \mathcal{U}_{\sigma,\alpha_1}$, and hence $u_t = U^\sigma_{\alpha_2\alpha_1}(t)q_0^{\Lambda,N}$ is a unique solution of (4.59), see Proposition 4.1.8. By Lemma 4.1.14 $q_t^{\Lambda,N}$ solves (4.64) in on $[0,\tau_\vartheta)$, which by Corollary 4.1.11 yields $u_t = q_t^{\Lambda,N}$ for $t < \min\{\tau_\vartheta; T(\alpha_2,\alpha_1)\}$. If $\tau_\vartheta < T(\alpha_2,\alpha_1$, we can continue $q_t^{\Lambda,N}$ beyond τ_ϑ by means of the following arguments. Since $u_t = q_t^{\Lambda,N}$ lies in $\mathcal{U}_{\sigma,\alpha_2}$ for all $t < \min\{\tau_\vartheta; T(\alpha_2,\alpha_1)\}$ by (4.68) this yields that $q_t^{\Lambda,N}$ lies in the initial space $\mathcal{G}_{\vartheta'}$ and hence can further by continued. This yields that $u_t = q_t^{\Lambda,N}$ for all $t < T(\alpha_2,\alpha_1)$. Now we make use of (4.60) and obtain that $q_t^{\Lambda,N} = u_t = k_t^{\Lambda,N}$, that completes the proof. □

Taking the limits. The aim of this part is to prove that (4.90) holds when the approximation is removed. Recall that $k_t^{\Lambda,N}$ in (4.90) depends on $\sigma > 0$, Λ and N. We first take the limits $\Lambda \to \mathbb{R}^d$ and $N \to +\infty$. Below, by an exhausting sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ we mean a sequence of compact Λ_n such that: (a) $\Lambda_n \subset \Lambda_{n+1}$ for all n; (b) for each $x \in \mathbb{R}^d$, there exits n such that $x \in \Lambda_n$.

Proposition 4.1.16. *Let $\alpha_1 > -\log\omega$ and $\alpha_2 > \alpha_1$ be fixed and $k_0 \in \mathcal{K}_{\alpha_1}$ be the correlation function of the initial state $\mu_0 \in \mathcal{P}_{\exp}(\Gamma)$ the local density of which was used in (4.71). For these α_1, α_2 and $t < T(\alpha_2,\alpha_1)$, let $Q^\sigma_{\alpha_2\alpha_1}(t)$ be the same as in (4.70). Then, for each $G \in B_{\mathrm{bs}}(\Gamma_0)$ and any $t < T(\alpha_2,\alpha_1)$, the following holds*

$$\lim_{n\to+\infty}\lim_{l\to+\infty}\langle\!\langle G, k_t^{\Lambda_n,N_l}\rangle\!\rangle = \langle\!\langle G, Q^\sigma_{\alpha_2\alpha_1}(t)k_0\rangle\!\rangle,$$

for arbitrary exhausting $\{\Lambda_n\}_{n\in\mathbb{N}}$ and increasing $\{N_l\}_{l\in\mathbb{N}}$ sequences of sets and positive integers, respectively.

The proof of this statement can be performed by the literal repetition of the proof given in Appendix of [11].

Recall that, for $\alpha_2 > \alpha_1$, $T(\alpha_2,\alpha_1)$ was defined in (2.27). For these α_1 and α_2 we set

$$\alpha = \frac{1}{3}\alpha_2 + \frac{2}{3}\alpha_1, \qquad \alpha' = \frac{2}{3}\alpha_2 + \frac{1}{4}\alpha_1. \tag{4.91}$$

Clearly,

$$\tau(\alpha_2,\alpha_1) := \frac{1}{3}T(\alpha_2,\alpha_1) < \min\{T(\alpha_2,\alpha'); T(\alpha,\alpha_1)\}. \tag{4.92}$$

Lemma 4.1.17. *Let α_1, α_2 and k_0 be as in Proposition 4.1.16, and let k_t be the solution of (2.26). Then for each $G \in B_{\mathrm{bs}}(\Gamma_0)$ and $t \in [0,\tau(\alpha_2,\alpha_1)]$, the following holds*

$$\lim_{\sigma\to0^+}\langle\!\langle G, Q^\sigma_{\alpha_2\alpha_1}(t)k_0\rangle\!\rangle = \langle\!\langle G, k_t\rangle\!\rangle. \tag{4.93}$$

Proof. We recall that the solution of (1.36)is in the form $k_t = Q_{\alpha_2\alpha_1}(t)k_0$ with $Q_{\alpha_2\alpha_1}(t)$ given in (4.47) and $t \leq T(\alpha_2, \alpha_1)$, see Lemma 2.3.3. For α and α' as in (4.92) and $t \leq \tau(\alpha_2, \alpha_1)$, write

$$\begin{aligned} Q_{\alpha_2\alpha_1}(t)k_0 &= Q^{\sigma}_{\alpha_2\alpha_1}(t)k_0 + \Upsilon_1(t,\sigma) + \Upsilon_2(t,\sigma), \qquad (4.94) \\ \Upsilon_1(t,\sigma) &= \int_0^t Q_{\alpha_2\alpha'}(t-s)\left[(A_2^{\Delta})_{\alpha'\alpha} - (A_2^{\Delta.\sigma})_{\alpha'\alpha}\right] Q^{\sigma}_{\alpha\alpha_1}(s)k_0 ds, , \\ \Upsilon_2(t,\sigma) &= \int_0^t Q_{\alpha_2\alpha'}(t-s)\left[(B_2^{\Delta})_{\alpha'\alpha} - (B_2^{\Delta,\sigma})_{\alpha'\alpha}\right] Q^{\sigma}_{\alpha\alpha_1}(s)k_0 ds, \end{aligned}$$

Recall that the norms of the bounded operators $(A_2^{\Delta})_{\alpha'\alpha}$. $(B_2^{\Delta})_{\alpha'\alpha}$, $(A_2^{\Delta,\sigma})_{\alpha'\alpha}$, $(B_2^{\Delta.\sigma})_{\alpha'\alpha}$ can be estimated as in (4.34). The validity of (4.94) can be checked by taking the t-derivatives from both sides and then by using e.g., (4.40). For G as in (4.93), we then have

$$\langle\!\langle G, Q_{\alpha_2\alpha_1}(t)k_0\rangle\!\rangle - \langle\!\langle G, Q^{\sigma}_{\alpha_2\alpha_1}(t)k_0\rangle\!\rangle = \langle\!\langle G, \Upsilon_1(t,\sigma)\rangle\!\rangle + \langle\!\langle G, \Upsilon_2(t,\sigma)\rangle\!\rangle. \qquad (4.95)$$

By (4.50) and (4.94) it follows that

$$\begin{aligned} &\langle\!\langle G, \Upsilon_1(t,\sigma)\rangle\!\rangle \\ &= \int_0^t \langle\!\langle G, Q_{\alpha_2\alpha'}(t-s)\left[(A_2^{\Delta})_{\alpha'\alpha} - (A_2^{\Delta.\sigma})_{\alpha'\alpha}\right] Q^{\sigma}_{\alpha\alpha_1}(s)k_0\rangle\!\rangle ds \\ &= \int_0^t \langle\!\langle H_{\alpha'\alpha_2}(t-s)G, v_s^{\sigma}\rangle\!\rangle ds = \int_0^t \langle\!\langle G_{t-s}, v_s^{\sigma}\rangle\!\rangle ds, \qquad (4.96) \end{aligned}$$

where

$$\begin{aligned} v_s^{\sigma} &= \left[(A_2^{\Delta})_{\alpha'\alpha} - (A_2^{\Delta.\sigma})_{\alpha'\alpha}\right] k_s^{\sigma} \qquad (4.97) \\ &:= \left[(A_2^{\Delta})_{\alpha'\alpha} - (A_2^{\Delta.\sigma})_{\alpha'\alpha}\right] Q^{\sigma}_{\alpha\alpha_1}(s)k_0 \in \mathcal{K}_{\alpha'}, \end{aligned}$$

and

$$G_{t-s} = H_{\alpha'\alpha_2}(t-s)G \in \mathcal{G}_{\alpha'}, \qquad (4.98)$$

since obviously $G \in \mathcal{G}_{\alpha_2}$. We apply (2.20) in (4.97) and transform

(4.96) into the following expression

$$\int_0^t \langle\!\langle G_{t-s}, v_s^\sigma \rangle\!\rangle ds = \int_{\Gamma_0} G_{t-s}(\eta)\Big(\int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} k_s^\sigma(\eta\cup x\setminus\{y_1,y_2\}) \times [1-\phi_\sigma(y_1)\phi_\sigma(y_2)]\, b(x|y_1,y_2)dx\Big)\lambda(d\eta) \quad (4.99)$$

$$= \int_{\Gamma_0}\Big(\int_{(\mathbb{R}^d)^3} G_{t-s}(\eta\cup\{y_1,y_2\})k_s^\sigma(\eta\cup x) \times [1-\phi_\sigma(y_1)\phi_\sigma(y_2)]\, b(x|y_1,y_2)dxdy_1dy_2\Big)\lambda(d\eta)$$

Since $k^\sigma = Q^\sigma_{\alpha\alpha_1}(s)k_0$ is in $\mathcal{K}_\alpha$ we have that

$$|k_s^\sigma(\eta\cup x)| \le \|k_s^\sigma\|_\alpha e^{\alpha|\eta|+\alpha} \le e^{\alpha|\eta|+\alpha}\frac{T(\alpha,\alpha_1)\|k_0\|_{\alpha_1}}{T(\alpha,\alpha_1)-\tau(\alpha_2,\alpha_1)}, \quad (4.100)$$

where α is as in (4.91) and $s\le t\le \tau(\alpha_2,\alpha_1)$. Now for $s\le t$, we set

$$g_s(y_1,y_2) = \int_{\Gamma_0} e^{\alpha|\eta|}|G_s(\eta\cup\{y_1,y_2\})|\lambda(d\eta). \quad (4.101)$$

Let us show that $g_s\in L^1((\mathbb{R}^d)^2)$. By (4.98) we have

$$\int_{(\mathbb{R}^d)^2} g_s(y_1,y_2)dy_1dy_2 \quad (4.102)$$

$$= e^{-2\alpha}\int_{\Gamma_0}|\eta|(|\eta|-1)e^{-(\alpha'-\alpha)|\eta|}|G_s(\eta)|e^{\alpha'|\eta|}\lambda(d\eta)$$

$$\le \frac{4e^{-2\alpha-2}}{(\alpha'-\alpha)^2}|G_s|_{\alpha'}$$

$$\le \frac{4e^{-2\alpha-2}T(\alpha_2,\alpha')|G|_{\alpha_2}}{(\alpha'-\alpha)^2[T(\alpha_2,\alpha')-\tau(\alpha_2,\alpha_1)]}$$

Turn now to (4.99). By means of item (iv) of Assumption 1 and by (4.100) and (4.101) we get

$$\int_0^t |\langle\!\langle G_{t-s}, v_s^\sigma\rangle\!\rangle|\, ds \le \beta^* C(\alpha_2,\alpha_1)\|k_0\|_{\alpha_1} \times \int_0^t\int_{(\mathbb{R}^d)^2} g_s(y_1,y_2)\,[1-\phi_\sigma(y_1)\phi_\sigma(y_2)]\, dsdy_1dy_2,$$

where we have taken into account that α and α' are expressed through α_2 and α_1, see (4.91). Then the function under the latter integral is bounded from above by $g_s(y_1, y_2)$ which by (4.102) is integrable on $[0,t] \times (\mathbb{R}^d)^2$. Since this function converges point-wise to 0 as $\sigma \to 0^+$, by Lebesgue's dominated convergence theorem we get that

$$\langle\!\langle G, \Upsilon_1(t,\sigma)\rangle\!\rangle \to 0, \qquad \text{as } \ \sigma \to 0^+.$$

The proof that the second summand in the right-hand side of (4.95) vanishes in the limit $\sigma \to 0^+$ is pretty analogous. $\square$

Proof of Lemma 2.3.4. By (1.44) and Proposition 1.2.5 we have that each $k_0 \in \mathcal{K}^\star_{\alpha_1}$ is the correlation function of some $\mu_0 \in \mathcal{P}_{\exp}(\Gamma_0)$. By (2.20) we readily conclude that

$$\dot{k}_t(\varnothing) = (L^\Delta_{\alpha_2} k_t)(\varnothing) = 0.$$

Hence, $k_t(\varnothing) = k_0(\varnothing) = 1$. At the same time, for $t \leq \tau(\alpha_2, \alpha_1)$ given in (4.92), we have that

$$\langle\!\langle G, k_t\rangle\!\rangle = \lim_{\sigma\to 0^+} \lim_{n\to+\infty} \lim_{l\to+\infty} \langle\!\langle G, k_t^{\Lambda_n, N_l}\rangle\!\rangle,$$

that follows by Lemma 4.1.17 and Proposition 4.1.16. Then $\langle\!\langle G, k_t\rangle\!\rangle \geq 0$ by (4.90) which completes the proof. $\square$

4.1.7 Proofs of Theorem 2.3.5 and Corollary 2.3.6

The main peculiarity of the solution obtained in Lemma 2.3.3 is that it resides in an ascending scale of Banach spaces and, in general, may abandon these spaces in finite time. Indeed, for a fixed $\alpha_1 > -\log\omega$, see, e.g., Lemma 2.3.4, the time bound $T(\alpha, \alpha_1)$ defined in (2.27) is a bounded function of $\alpha > \alpha_1$. To overcome this difficulty we compare k_t with some auxiliary functions.

Lemma 4.1.18. *Let α_2, α_1 and $\tau(\alpha_2, \alpha_1)$ be as in Lemma 2.3.4. Then for each $t \in [0, \tau(\alpha_2, \alpha_1)]$ and arbitrary $k_0 \in \mathcal{K}^\star_{\alpha_1}$, the following holds*

$$0 \leq (Q_{\alpha_2\alpha_1}(t; B^\Delta_\upsilon)k_0)(\eta) \leq (Q_{\alpha_2\alpha_1}(t; B^\Delta_{2,\upsilon})k_0)(\eta), \qquad \eta \in \Gamma_0. \quad (4.103)$$

Proof. The left-hand side inequality follows by Lemma 2.3.4 and (1.45). By the second line in (4.41) we conclude that $w_t = Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{2,\upsilon})k_0$ is the unique solution of the equation

$$\dot{w}_t = ((A^{\Delta}_{\upsilon})_{\alpha_2} + (B^{\Delta}_{2,\upsilon})_{\alpha_2})w_t, \qquad w_t|_{t=0} = k_0,$$

on the time interval $[0, T(\alpha_2, \alpha_1; B^{\Delta}_{2,\upsilon})) \supset [0, T(\alpha_2, \alpha_1; B^{\Delta}_{\upsilon}))$ since $T(\alpha_2, \alpha_1; B^{\Delta}_{\upsilon}) \leq T(\alpha_2, \alpha_1; B^{\Delta}_{2,\upsilon})$. Then we have that $w_t - k_t \in \mathcal{K}_{\alpha_2}$ for all $t < \tau(\alpha_2, \alpha_1)$. Now we choose $\alpha', \alpha \in [\alpha_1, \alpha_2]$ according to (4.91) so that (4.92) holds, and then write

$$\begin{aligned} w_t - k_t &= (Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{2,\upsilon})k_0)(\eta) - (Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{\upsilon})k_0)(\eta) \\ &= \int_0^t Q_{\alpha_2\alpha'}(t-s; B^{\Delta}_{2,\upsilon})(-B^{\Delta}_1)_{\alpha'\alpha}k_s ds, \qquad t < \tau(\alpha_2, \alpha_1), \end{aligned} \tag{4.104}$$

where the operator $(-B^{\Delta}_1)_{\alpha'\alpha}$ is positive with respect to the cone $\mathcal{K}^+_{\alpha}$ defined in (1.45). In the integral in (4.104), for all $s \in [0, \tau(\alpha_2, \alpha_1)]$, we have that $k_s \in \mathcal{K}_{\alpha}$ and $Q_{\alpha_2\alpha'}(t-s; B^{\Delta}_{2,\upsilon}) \in \mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_{\alpha_2})$ is positive. We also have that $k_s \in \mathcal{K}^{\star}_{\alpha} \subset \mathcal{K}^+_{\alpha}$ (by Lemma 2.3.4). Therefore $w_t - k_t \in \mathcal{K}^+_{\alpha_2}$ for $t \leq \tau(\alpha_2, \alpha_1)$, which yields (4.103). $\square$

The next step is to compare k_t with

$$r_t(\eta) = \|k_0\|_{\alpha_1} \exp\left((\alpha_1 + ct)|\eta|\right), \tag{4.105}$$

where α_1 is as in Lemma 4.1.18 and

$$c = \langle b \rangle + \upsilon - m_*, \quad m_* = \inf_{x\in\mathbb{R}^d} m(x). \tag{4.106}$$

Let us show that $r_t \in \mathcal{K}_{\alpha}$ for $t \leq \tau(\alpha_2, \alpha_1)$, where α is given in (4.91). In view of (1.42), this is the case if the following holds

$$\alpha_1 + c\tau(\alpha_2, \alpha_1) \leq \frac{1}{3}\alpha_2 + \frac{2}{3}\alpha_1, \tag{4.107}$$

which amounts to $c \leq \langle b \rangle + \upsilon + \langle a \rangle e^{\alpha_2}$, see (4.92) and (2.27). The latter obviously holds by (4.106).

Lemma 4.1.19. *Let α_1, α_2 and $k_t = Q_{\alpha_2\alpha_1}k_0$ be as in Lemma 4.1.18, and r_t be as in* (4.105) *and* (4.106). *Then $k_t(\eta) \leq r_t(\eta)$ for all $t \leq \tau(\alpha_2, \alpha_1)$ and $\eta \in \Gamma_0$.*

Proof. The idea is to show that $w_t(\eta) \leq r_t(\eta)$ and then apply the estimate obtained in Lemma 4.1.18. Set $\tilde{w}_t = Q_{\alpha_2\alpha_1}(t; B^{\Delta}_{2,\upsilon})r_0$. Since $k_0 \in \mathcal{K}_{\alpha_1}$, we have that $k_0 \leq r_0$. Then by the positivity discussed in Remark 4.1.5 we obtain $w_t \leq \tilde{w}_t$, and hence $k_t \leq \tilde{w}_t$, holding for all $t \leq \tau(\alpha_2, \alpha_1)$. Thus, it remains to prove that $w_t(\eta) \leq r_t(\eta)$. To this end we write, cf. (4.104),

$$\tilde{w}_t - r_t = \int_0^t Q_{\alpha_2\alpha'}(t-s; B^{\Delta}_{2,\upsilon}) D_{\alpha'\alpha} r_s ds, \tag{4.108}$$

where α' and α are as in (4.91) and the bounded operator $D_{\alpha'\alpha}$ acts as follows: $D = A^{\Delta}_{\upsilon} + B^{\Delta}_{2,\upsilon} - J_{c_1}$, where $(J_{c_1}k)(\eta) = c_1|\eta|k(\eta)$. The validity of (4.108) can be established by taking the t-derivative of both sides and then taking into account (4.105) and (4.41). Note that r_s in (4.108) lies in $\mathcal{K}_\alpha$, as it was shown above. By means of (2.20) the action of D on r_s can be calculated explicitly yielding

$$\begin{aligned}
&(Dr_t)(\eta) \qquad (4.109)\\
&= -\Psi_\upsilon(\eta)r_t(\eta) + \int_{\mathbb{R}^d} \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus\eta_1} r_t(\eta \cup x \setminus \{y_1, y_2\}) b(x|y_1, y_2) dx \\
&\quad + \upsilon|\eta| r_t(\eta) + 2\int_{(\mathbb{R}^d)^2} \sum_{y_1\in\eta} r_t(\eta \cup x \setminus y_1) b(x|y_1, y_2) dx dy_2 - c_1|\eta| r_t(\eta) \\
&= \left(-M(\eta) - E^a(\eta) - \langle b\rangle|\eta| + e^{-c_0-c_1 t}E^b(\eta) + 2\langle\beta\rangle|\eta| - c_1|\eta|\right) r_t(\eta).
\end{aligned}$$

Since $\alpha_1 > -\log\omega$, by Proposition 2.1.2 we have that

$$-E^a(\eta) + e^{-\alpha_1 - ct}E^b(\eta) \leq \upsilon|\eta|,$$

by which we obtain from (4.109) the following estimate $(Dr_t)(\eta) \leq 0$. We apply this in (4.108) and obtain $\tilde{w}_t \leq r_t$ which completes the proof. □

Remark 4.1.20. By (4.106) we obtain that $c \leq 0$ (and hence $k_t \in \mathcal{K}_{\alpha_1}$) whenever

$$m_* \geq \langle b\rangle + \upsilon.$$

In the short dispersal case, see Remark 2.1.1, one can take $\upsilon = 0$. In the long dispersal case, by Proposition 2.1.3 one can make υ as small as one wants by taking small enough ω and hence big enough α_1. Then, the evolution of k_t leaves the initial space invariant if the following holds

$$m_* > \langle b \rangle. \tag{4.110}$$

In the short dispersal case, one can allow equality in (4.110).

Continuation. The choice of the initial space should satisfy the condition $\alpha_1 > -\log \omega$. At the same time, the parameter $\alpha_2 > \alpha_1$ can be taken arbitrarily. In view of the dependence of $T(\alpha_2, \alpha_1)$ on α_2, see (2.27), the function $\alpha_2 \mapsto T(\alpha_2, \alpha_1)$ attains maximum at $\alpha_2 = \alpha_1 + \delta(\alpha_1)$, where

$$\delta(\alpha) = 1 + W\left(\frac{2\langle b \rangle + \upsilon}{\langle a \rangle} e^{-\alpha - 1}\right), \tag{4.111}$$

Here W is Lambert's function, see [19]. Then we have

$$T_{\max}(\alpha_1) = \max_{\alpha_2 > \alpha_1} T(\alpha_2, \alpha_1) = \exp\left(-\alpha_1 - \delta(\alpha_1)\right) / \langle a \rangle. \tag{4.112}$$

Proof of Theorem 2.3.5. Fix υ and then find small ω (see Proposition 2.1.3) such that the inequality in Proposition 2.1.2 holds true. Thereafter takes $\alpha_0 > -\log \omega$ such that $k_{\mu_0} \in \mathcal{K}_{\alpha_0}$. Then take c as given in (4.106) with this υ. Next, set $T_1 = T_{\max}(\alpha_0)/3$, see (4.112), and also $\alpha_1^* = \alpha_0 + cT_1$, $\alpha_1 = \alpha_0 + \delta(\alpha_0)$, see (4.111). Clearly, $\alpha_1^* < \alpha_1$ that can be checked similarly as in (4.107). By Lemma 2.3.4 it follows that, for $t \leq T_1$, $k_t = Q_{\alpha_1 \alpha_0}(t) k_{\mu_0}$ lies in $\mathcal{K}^\star_{\alpha_1}$, whereas by Lemma 4.1.19 we have that $k_t \in \mathcal{K}^\star_{\alpha_t}$ with $\alpha_t = \alpha_0 + ct \leq \alpha_1^*$. Clearly, for $T \leq T_1$, the map $[0, T) \ni t \mapsto k_t \in \mathcal{K}_{\alpha_T}$ is continuous and continuously differentiable, and both claims *(i)* and *(j)* are satisfied (by construction) $\dot{k}_t = L^\Delta_{\alpha_1} k_t = L^\Delta_{\alpha_T} k_t$, see (2.25). Now for $n \geq 2$, we set

$$\begin{aligned} T_n &= T_{\max}(\alpha_{n-1}^*)/3, \quad \alpha_n^* = \alpha_{n-1}^* + cT_n, \\ \alpha_n &= \alpha_{n-1}^* + \delta(\alpha_{n-1}^*). \end{aligned} \tag{4.113}$$

As for $n = 1$, we have that $\alpha_n^* < \alpha_n$ and $T_n < T(\alpha_n, \alpha_{n-1}^*)$ holding for all $n \geq 2$. Thereafter, set

$$k_t^{(n)} = Q_{\alpha_n \alpha_{n-1}^*}(t) k_{T_{n-1}}^{(n-1)}, \qquad t \in [0, T(\alpha_n, \alpha_{n-1}^*)),$$

where $k_t^{(1)} = Q_{\alpha_1\alpha_0}(t)k_{\mu_0}$. Then for each $T < T_n$ both maps $[0,T) \ni t \mapsto k_t^{(n)} \in \mathcal{K}_{\tilde{\alpha}_{n-1}(T)}$ and $[0,T) \ni t \mapsto L^{\Delta}_{\tilde{\alpha}_{n-1}(T)}k_t^{(n)} \in \mathcal{K}_{\tilde{\alpha}_{n-1}(T)}$ are continuous, where $\tilde{\alpha}_{n-1}(T) := \alpha^*_{n-1} + cT$. The continuity of the latter map follows by the fact that $k_t^{(n)} : \mathcal{K}_{\tilde{\alpha}_{n-1}(t)} \hookrightarrow \mathcal{K}_{\tilde{\alpha}_{n-1}(T)}$ and that $L^{\Delta}_{\tilde{\alpha}_{n-1}(T)}|_{\mathcal{K}_{\tilde{\alpha}_{n-1}(t)}} = L^{\Delta}_{\tilde{\alpha}_{n-1}(T)\tilde{\alpha}_{n-1}(t)}$, see (2.25). Moreover, $k_0^{(n)} = k_{T_{n-1}}^{(n-1)}$ and $L^{\Delta}_{\alpha^*_{n-1}+\varepsilon}k_0^{(n)} = L^{\Delta}_{\alpha^*_{n-1}+\varepsilon}k_{T_{n-1}}^{(n-1)}$ holding for each $\varepsilon > 0$. Then the map in question $t \mapsto k_t$ is

$$k_{t+T_1+\ldots+T_{n-1}} = k_t^{(n)}, \qquad t \in [0, T_n],$$

provides that the series $\sum_{n\geq 1} T_n$ is divergent. By (4.112) we have

$$\sum_{n\geq 1} T_n = \frac{1}{3\langle a\rangle} \sum_{n\geq 1} \exp\left(-\alpha^*_{n-1} - \delta(\alpha^*_{n-1})\right). \tag{4.114}$$

For the convergence of the series in the right-hand side it is necessary that $\alpha^*_{n-1} + \delta(\alpha^*_{n-1}) \to +\infty$, and hence $\alpha^*_{n-1} \to +\infty$ as $n \to +\infty$, since $\delta(\alpha)$ is decreasing. By (4.113) we have $\alpha^*_n = \alpha_0 + c(T_1 + \cdots + T_n)$. Then the convergence of $\sum_{n\geq 1} T_n$ would imply that $\alpha^*_n \leq \alpha^*$ for some number $\alpha^* > 0$ that contradicts the convergence of the right-hand side of (4.114). $\square$

Proof of Corollary 2.3.6. For a compact Λ, let us show that $\mu_t^\Lambda \in \mathcal{D}$, that is, $R^\Lambda_{\mu_t} \in \mathcal{D}^\dagger$, see (4.4). For $k_t = k_{\mu_t}$ described in Theorem 2.3.5, by (1.26) we have

$$R^\Lambda_{\mu_t}(\eta) = \int_{\Gamma_\Lambda} (-1)^{|\xi|} k_t(\eta \cup \xi)\lambda(d\xi).$$

Let $\alpha > \alpha_0$ be such that $k_t \in \mathcal{K}_\alpha$. Then using (1.42), (1.22), (2.15) and (1.30) we calculate

$$\begin{aligned}
&\int_{\Gamma_\Lambda} \Psi(\eta) R^\Lambda_{\mu_t}(\eta)\lambda(d\eta) \\
&\qquad = \int_{\Gamma_\Lambda} \Psi(\eta) \int_{\Gamma_\Lambda} (-1)^{|\xi|} k_t(\eta \cup \xi)\lambda(d\xi)\lambda(d\eta) \\
&\qquad \leq \int_{\Gamma_\Lambda} \Psi(\eta)\|k\|_\alpha e^{\alpha|\eta|}\lambda(d\eta) \int_{\Gamma_\Lambda} e^{\alpha|\xi|}\lambda(d\xi)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|k\|_\alpha (m^* + a^* + \langle b\rangle) \int_{\Gamma_\Lambda} |\eta|^2 e^{\alpha|\eta|}\lambda(d\eta)\exp(|\Lambda|e^\alpha) \\
&= \|k\|_\alpha (m^* + a^* + \langle b\rangle)|\Lambda|e^\alpha(2+|\Lambda|e^\alpha)\exp(2|\Lambda|e^\alpha),
\end{aligned}
$$

where $|\Lambda|$ is the Euclidean volume of Λ. That yields $\mu_t^\Lambda \in \mathcal{D}$. The vality of (2.28) follows by (1.24). □

4.1.8 Proof of Theorem 2.4.2

We rescale the interaction in (1.36), see (2.20), by multiplying a by ε and obtain the evolution equation as (1.36) with L^Δ replaced by

$$
\begin{aligned}
(L_\varepsilon^\Delta k)(\eta) \;=\; & -k(\eta)\sum_{x\in\eta}(m(x) + \varepsilon E^a(x,\eta\setminus x)) \\
& - k(\eta)\sum_{x\in\eta}\int_{(\mathbb{R}^d)^2} b(x|y_1,y_2)dy_1dy_2 \\
& + \int_{\mathbb{R}^d}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1} k(\eta\cup x\setminus\{y_1,y_2\})b(x|y_1,y_2)dx \\
& - \int_{\mathbb{R}^d}\varepsilon E^a(y,\eta)k(\eta\cup y)dy \\
& + 2\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta} k(\eta\cup x\setminus y_1)b(x|y_1,y_2)dy_2dx.
\end{aligned}
$$

Note that $\varepsilon = 1$ corresponds to the initial microscopic system and by taking $\varepsilon \to 0$ we obtain the mesoscopic description of the system. Now we introduce the rescaled correlation functions

$$
k_{t,ren}^{(\varepsilon)}(\eta) = \varepsilon^{|\eta|}k_t^{(\varepsilon)}(\eta),
$$

which is the solution of the following Cauchy problem

$$
\frac{d}{dt}k_t^{(\varepsilon)} = L_{\varepsilon,ren}^\Delta k_t^{(\varepsilon)}, \qquad k_t^{(\varepsilon)}|_{t=0} = k_0.
$$

'Operator' $L_\varepsilon^{\Delta,ren}$ is obtained from L_ε^Δ by the formula

$$
(L_{\varepsilon,ren}^\Delta k)(\eta) = \varepsilon^{|\eta|}L_\varepsilon^\Delta(\varepsilon^{-|\eta|}k(\eta)),
$$

and hence, it has the form

$$
\begin{aligned}
(L^{\Delta}_{\varepsilon,ren}k)(\eta) &= -k(\eta)\sum_{x\in\eta}\big(m(x)+\varepsilon E^{-}(x,\eta\setminus x)\big)\\
&\quad - k(\eta)\sum_{x\in\eta}\int_{(\mathbb{R}^d)^2} b(x|y_1,y_2)dy_1dy_2\\
&\quad + \varepsilon^{|\eta|}\int_{\mathbb{R}^d}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1}\varepsilon^{-|\eta\cup x\setminus y_1\setminus y_2|}k(\eta\cup x\setminus y_1\setminus y_2)b(x|y_1,y_2)dx\\
&\quad - \varepsilon^{|\eta|}\int_{\mathbb{R}^d}\varepsilon E^{-}(y,\eta)\varepsilon^{-|\eta\cup y|}k(\eta\cup y)dy\\
&\quad + 2\varepsilon^{|\eta|}\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta}\varepsilon^{-|\eta\cup x\setminus y_1|}k(\eta\cup x\setminus y_1)b(x|y_1,y_2)dy_2dx
\end{aligned}
$$

After direct calculation we obtain

$$
\begin{aligned}
(L^{\Delta}_{\varepsilon,ren}k)(\eta) &= -k(\eta)\sum_{x\in\eta}\big(m(x)+\varepsilon E^{-}(x,\eta\setminus x)+\langle b\rangle\big)\\
&\quad + \varepsilon\int_{\mathbb{R}^d}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1}k(\eta\cup x\setminus y_1\setminus y_2)b(x|y_1,y_2)dx\\
&\quad - \int_{\mathbb{R}^d}E^{-}(y,\eta)k(\eta\cup y)dy\\
&\quad + 2\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta}k(\eta\cup x\setminus y_1)b(x|y_1,y_2)dy_2dx.
\end{aligned}
$$

Note that $L^{\Delta}_{\varepsilon,ren}$ has the following structure

$$L^{\Delta}_{\varepsilon,ren} = V + \varepsilon C,$$

where

$$
\begin{aligned}
(Vk)(\eta) &= -k(\eta)\sum_{x\in\eta}(m(x)+\langle b\rangle)\\
&\quad - \int_{\mathbb{R}^d}E^{-}(y,\eta)k(\eta\cup y)dy\\
&\quad + 2\int_{(\mathbb{R}^d)^2}\sum_{y_1\in\eta}k(\eta\cup x\setminus y_1)b(x|y_1,y_2)dy_2dx,
\end{aligned}
$$

$$
\begin{aligned}
(Ck)(\eta) &= -k(\eta)\sum_{x\in\eta}E^{-}(x,\eta\setminus x) \\
&\quad +\int_{\mathbb{R}^d}\sum_{y_1\in\eta}\sum_{y_2\in\eta\setminus y_1}k(\eta\cup x\setminus y_1\setminus y_2)b(x|y_1,y_2)dx.
\end{aligned}
$$

For $\varepsilon\to 0$ we consider problem

$$
\frac{d}{dt}r_t = Vr_t, \qquad r_t|_{t=0}=r_0. \tag{4.115}
$$

The main property of the evolution $r_0\mapsto r_t$ is 'preserving chaos'. That is, if r_0 is the correlation function of the Poisson measure π_{ϱ_0}, i.e.

$$
r_0(\eta)=\prod_{x\in\eta}\varrho_0(x),
$$

then, for all $t>0$ for which we can solve (4.115), the solution of (4.115) has the form

$$
r_t(\eta)=\prod_{x\in\eta}\varrho_t(x).
$$

Then ϱ_t is a solution of a *kinetic equation*.
Now let us consider equation (1.36) as an infinite chain of linked equation in term of the components $k_t^{(n)}$. The first three evolution equations have the forms

$$
\frac{d}{dt}k_t^0(\emptyset)=0,
$$

$$
\begin{aligned}
\frac{d}{dt}k_t^{(1)}(x) &= -k_t^{(1)}(x)(m(x)+\langle b\rangle)-\int_{\mathbb{R}^d}a(x-y)k_t^{(2)}(x,y)dy \\
&\quad +2\int_{(\mathbb{R}^d)^2}k_t^{(1)}(y)b(y|x,y_2)dy_2dy.
\end{aligned}
$$

By set $k_t^{(2)}(x,y)=k_t^{(1)}(x)k_t^{(1)}(y)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}k_t^{(1)}(x) &= -k_t^{(1)}(x)\left((m(x)+\langle b\rangle)-\int_{\mathbb{R}^d}a(x-y)k_t^{(1)}(y)dy\right) \\
&\quad +2\int_{(\mathbb{R}^d)^2}k_t^{(1)}(y)b(y|x,y_2)dy_2dy.
\end{aligned} \tag{4.116}
$$

In view of (4.116), we yield that the *kinetic equation* has the form

$$\begin{aligned}\frac{d}{dt}\varrho_t(x) &= -\varrho_t(x)\left((m(x)+\langle b\rangle)-\int_{\mathbb{R}^d} a(x-y)\varrho_t(y)dy\right)\\ &\quad +2\int_{(\mathbb{R}^d)^2}\varrho_t(y)b(y|x,y_2)dy_2dy.\end{aligned}$$

Let $\mathbf{B}$ be either β or a. For both choice of $\mathbf{B}$ let set

$$(\mathbf{B}*\varrho_t)(x)=\int_{\mathbb{R}^d}\mathbf{B}(x-y)\varrho_t(y)dy.$$

Then

$$2\int_{(\mathbb{R}^d)^2}\varrho_t(y)b(y|x,y_2)dy_2dy=\langle b\rangle\varrho_t(x)+(b*\varrho_t)(x),$$

which give us problem (2.29).

4.2 Proofs regarding the free-branching model

4.2.1 Proof of Proposition 3.2.1

We begin by showing that $K : \mathcal{C}^T_\psi(\phi) \to \mathcal{C}^T_\psi(\phi)$ for each $T > 0$. Clearly, $x \mapsto (K\varphi)_t(x)$ is continuous and $(K\varphi)_0 = \phi$ whenever $\varphi \in \mathcal{C}^T_\psi(\phi)$. The continuity of $t \mapsto \Phi\varphi_t$ follows by the estimate, see (3.4),

$$
\begin{aligned}
|(\Phi\varphi_s)(x) - (\Phi\varphi_u)(x)| &\leq \int_{\Gamma_0} \left| \prod_{y\in\xi} \varphi_s(y) - \prod_{y\in\xi} \varphi_u(y) \right| b_x(d\xi) \\
&\leq \sup_{y\in X} |\varphi_s(y) - \varphi_u(y)| \int_{\Gamma_0} |\xi| b_x(d\xi) \leq n_* \sup_{y\in X} |\varphi_s(y) - \varphi_u(y)|.
\end{aligned} \tag{4.117}
$$

This also yields the continuity of $t \mapsto (K\varphi)_t$. In obtaining (4.117) we have used the following evident estimate

$$
|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \leq n \max_i |a_i - b_i|, \qquad a_i, b_i \in [0,1].
$$

Furthermore,

$$
0 < (K\varphi)_t(x) \leq \phi(x) e^{-t} + (1 - e^{-t}) = 1 - (1 - \phi(x)) e^{-t} \leq 1
$$

which yields

$$
1 - (K\varphi)_t(x) \geq e^{-t}\theta(x) \geq e^{-t} c_\phi \psi(x) = c_\phi(t)\psi(x), \tag{4.118}
$$

and hence the validity of the lower estimate as in (3.6). To get the upper bound, we write, see (3.2) and (3.4),

$$
\begin{aligned}
(\Phi\varphi_s)(x) &= b_x(\Gamma^0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} \phi(y_1) \cdots \phi(y_n) b_x^{(n)}(dy_1, \ldots dy_n) \\
&\geq b_x(\Gamma^0) = \delta(x) \geq 1 - \psi(x),
\end{aligned}
$$

where we used also item (iii) of Assumptions 2. By means of this estimate applied in (3.11) we then get

$$
\begin{aligned}
(K\varphi)_t(x) &\geq \phi(x) e^{-t} + (1 - e^{-t}) \delta(x) \\
&\geq (1 - \psi(x)) + e^{-t}(\phi(x) - \delta(x)) \geq 1 - \psi(x),
\end{aligned}
$$

as $\phi \in C_\psi(X)$. Thus, $K : \mathcal{C}_\psi^T(\phi) \to \mathcal{C}_\psi^T(\phi)$. Let us show that it is a contraction. To this end, similarly as in (4.117) we obtain, see also (3.10),

$$\|K\varphi - K\tilde{\varphi}\|_T \leq n_*(1 - e^{-T})\|\varphi - \tilde{\varphi}\|_T,$$

holding for each $\varphi, \tilde{\varphi} \in \mathcal{C}_\psi^T(\phi)$. Then the proof follows by Banach's contraction principle.

4.2.2 Proof of Lemma 3.2.2

We begin by fixing $T > 0$ such that the contraction condition $n_*(1 - e^{-T}) < 1$ is satisfied. Then integrating in (3.12) we arrive at the following integral equation

$$\phi_t(x) = \phi(x)e^{-t} + \int_0^t e^{-(t-s)}(\Phi\phi_s)(x)ds, \tag{4.119}$$

the set of solutions of which on $[0, T]$ coincides with the set of fixed points of $K : \mathcal{C}^T(\phi) \to \mathcal{C}^T(\phi)$ established in Proposition 3.2.1. The continuous differentiability of $t \mapsto \phi_t \in C_{\rm b}(X)$ follows by continuity $s \mapsto \Phi\psi_s$, which in turn follows by (4.117). Thus, each solution of (4.119) solves also (3.12), which yields the existence of the solution in question on the time interval $[0, T]$. For $n_* \leq 1$, the contraction condition is satisfied with any $T > 0$; hence, the aforementioned solution is global in time. For $n_* > 1$, we proceed as follows. For $t_1 + t_2 \leq T$, we rewrite (4.119) as follows

$$\begin{aligned}\phi_{t_1+t_2}(x) = e^{-t_2}&\left(\phi(x)e^{-t_1} + \int_0^{t_1} e^{-(t_1-s)}(\Phi\phi_s)(x)ds\right) \\ &+ \int_{t_1}^{t_1+t_2} e^{-(t_2+t_1-s)}(\Phi\phi_s)(x)ds \\ &= \phi_{t_1}(x)e^{-t_2} + \int_0^{t_2} e^{-(t_2-s)}(\Phi\phi_{t_1+s})(x)ds\end{aligned} \tag{4.120}$$

Since the contraction condition is independent of the initial condition in (3.12), by (4.120) one can continue the solution obtained above to any $t > 0$. Indeed, let ϕ_t be the solution on $[0, T]$. Let also $\phi_t^1 \in \mathcal{C}_\psi^T(\phi^1)$ be the solution of (3.12) on the same $[0, T]$ with the initial condition $\phi_t^1 := \phi_{T/2}$. By the uniqueness established in Lemma 3.2.2 it follows that these two solutions satisfy $\phi_{t+T/2} = \phi^1_t$ for $t \in [0, T/2]$. Hence, the function $\phi_t \mathbb{I}_{[0,T/2]}(t) + \phi^1_{t-T/2}\mathbb{I}_{[T/2,3T/2]}(t) = \phi_t\mathbb{I}_{[0,T]}(t) + \phi^1_{t-T}\mathbb{I}_{[T,3T/2]}(t)$ is the unique solution of (4.119) (hence of (3.12)) on $[0, 3T/2]$. The further continuation goes in analogous way.

For $n_* < 1$, we define $\vartheta_s = e^s\|1-\phi_x\| = e^s \sup_{x\in X}(1-\phi_s(x))$. Similarly as in (4.117), Then by (4.119) we then get

$$\vartheta_t \leq \vartheta_0 + n_* \int_0^t \vartheta_s ds.$$

which by Grönwall's inequality yields,

$$\|1-\phi_t\| \leq \|1-\phi_0\|e^{-(1-n_*)t},$$

that yields the convergence in question. Note that ϕ_∞ does not belong to $C_\psi(X)$ as it fails to obey the upper bound $\phi(x) \leq 1 - c_\phi\psi(x)$ with $c_\phi > 0$, see (3.6). However, it belongs to the closure of this set, and is a stationary solution of (3.12).

4.2.3 Proof of Lemma 3.2.4

Now by (3.12) we have

$$\begin{aligned}|\phi_{t+u}(x) - \phi_t(x)| &\leq \int_0^u |\phi_{t+s}(x) - (\Phi\phi_{t+s})(x)|ds \qquad (4.121)\\ &= \int_0^u |\theta_{t+s}(x) - (1-(\Phi\phi_{t+s})(x))|ds \leq 2\psi(x)u,\end{aligned}$$

where we have used (3.13) and (3.8). To prove (b), we denote

$$\begin{aligned}h^+(x) &= \max\{g_{t+u}(x)\psi(x); g_t(x)\psi(x)\},\\ h^-(x) &= \min\{g_{t+u}(x)\psi(x); g_t(x)\psi(x)\}.\end{aligned}$$

Then, cf. (1.13),

$$\begin{aligned}|\phi_{t+u}(x) - \phi_t(x)| &= e^{-h^+(x)}\left[e^{h^+(x)-h^-(x)} - 1\right]\\ &\geq e^{-h^-(x)}|g_{t+u}(x) - g_t(x)|\psi(x)\\ &\geq \max\{\phi_{t+u}(x); \phi_t(x)\}|g_{t+u}(x) - g_t(x)|\psi(x),\end{aligned}$$

which yields case (b) of (3.14) by (4.121) and (3.7). Next, similarly as in (4.117) we get

$$\begin{aligned} |(\Phi\phi_{t+u})(x) - (\Phi\phi_t)(x)| &\leq \int_{\Gamma_0} \left(\sum_{y \in \xi} |\phi_{t+u}(y) - \phi_t(y)| \right) b_x(d\xi) \\ &\leq 2u \int_X \psi(y) \beta_x^{(1)}(dy) \leq 2un_* m \psi(x), \end{aligned}$$

where we used (4.121), (3.2) and (3.5), see also item (i) of Assumption 2.

4.2.4 Proof of Proposition 3.2.5

By (3.1), and then by (3.7), (3.8) and (3.6), we have

$$\begin{aligned} |LF^\phi(\gamma)| &\leq \sum_{x \in \gamma} F^\phi(\gamma \setminus x) \, |(\Phi\phi)(x) - \phi(x)| \\ &\leq (F^\phi(\gamma)/\delta_*) \sum_{x \in \gamma} \Big(|1 - (\Phi\phi)(x)| + |1 - \phi(x)| \Big) \\ &\leq 2\Psi(\gamma) F^\phi(\gamma)/\delta_* \leq 2F^\phi(\gamma) e^{c_\phi \Psi(\gamma)}/(e\delta_* c_\phi) \leq 2/(e\delta_* c_\phi), \end{aligned} \tag{4.122}$$

where Ψ is as in (1.5). To get the latter two estimates in (4.122), we proceeded as follows. The first one is obtained with the help of the estimate $\alpha \leq e^{\alpha - 1}$, $\alpha > 0$. Afterwards, we did

$$F^\phi(\gamma) \exp(c_\phi \Psi(\gamma)) = \prod_{x \in \gamma} (1 - \theta(x)) e^{c_\phi \psi(x)} \leq \prod_{x \in \gamma} (1 - c_\phi \psi(x)) e^{c_\phi \psi(x)} \leq 1,$$

see (3.6), which was used in the final step. The continuity of the map $\gamma \mapsto LF^\phi(\gamma)$ follows by the very definition of the topology of Γ^ψ.

4.2.5 Proof of Lemma 3.2.6

We fix t and u and then define

$$H_s(\gamma) = \sum_{x \in \gamma} g_s(x) \psi(x), \quad H^+(\gamma) = \max\{H_{t+u}(\gamma); H_t(\gamma)\},$$

$$H^-(\gamma) = \min\{H_{t+u}(\gamma); H_t(\gamma)\}.$$

Then

$$
\begin{aligned}
|F^{\phi_{t+u}}(\gamma) - F^{\phi_t}(\gamma)| &= e^{-H^+(\gamma)}\left[e^{H^+(\gamma)-H^-(\gamma)} - 1\right] \qquad (4.123)\\
&\leq \max\{F^{\phi_{t+u}}(\gamma); F^{\phi_t}(\gamma)\}\sum_{x\in\gamma}|g_{t+u}(x) - g_t(x)|\psi(x)\\
&\leq \frac{2u}{\delta_*}\Psi(\gamma)\prod_{x\in\gamma}(1 - c_\phi(t+u)\psi(x))\\
&\leq \frac{2u}{e\delta_* c_\phi(t+u)}\prod_{x\in\gamma}(1 - c_\phi(t+u)\psi(x))e^{c_\phi(t+u)\psi(x)}\\
&\leq \frac{2ue^{t+u}}{e\delta_* c_\phi},
\end{aligned}
$$

which completes the proof, see (3.14), (3.13) and (4.122).

4.2.6 Proof of Lemma 3.2.7

As in (4.122), for fixed t and u we have

$$
\begin{aligned}
|(LF^{\phi_{t+u}})(\gamma) - (LF^{\phi_t})(\gamma)| &\leq K_1(\gamma) + K_2(\gamma) + K_3(\gamma), \qquad (4.124)\\
K_1(\gamma) &:= \sum_{x\in\gamma}\left|F^{\phi_{t+u}}(\gamma\setminus x) - F^{\phi_t}(\gamma\setminus x)\right|\left|(\Phi\phi_{t+u})(x) - \phi_{t+u}(x)\right|,\\
K_2(\gamma) &:= \sum_{x\in\gamma}F^{\phi_t}(\gamma\setminus x)\left|(\Phi\phi_{t+u})(x) - (\Phi\phi_t)(x)\right|,\\
K_3(\gamma) &:= \sum_{x\in\gamma}F^{\phi_t}(\gamma\setminus x)\left|\phi_{t+u}(x) - \phi_t(x)\right|.
\end{aligned}
$$

By (3.8) and (4.118) we have

$$
\frac{1}{1 - c_\phi(t+u)\psi(x)} \leq \frac{1}{1 - c_\phi\psi(x)} \leq \frac{1}{1 - \psi(x)} \leq \frac{1}{\delta_*}.
$$

Then proceeding as in obtaining the second inequality in (4.123), we arrive at

$$\left|F^{\phi_{t+u}}(\gamma\setminus x)-F^{\phi_t}(\gamma\setminus x)\right| \leq \frac{2u}{\delta_*}\Psi(\gamma\setminus x)\prod_{y\in\gamma\setminus x}(1-c_\phi(t+u)\psi(x))$$

$$\leq \frac{2u}{\delta_*^2}\Psi(\gamma)\prod_{y\in\gamma}(1-c_\phi(t+u)\psi(x)) \qquad (4.125)$$

Next, by (3.7) and (3.8) we have

$$|(\Phi\phi_{t+u})(x)-\phi_{t+u}(x)| \leq |1-(\Phi\phi_{t+u})(x)|+|1-\phi_{t+u}(x)| \leq 2\psi(x).$$

We use the latter estimate and (4.125) to obtain

$$\begin{aligned} K_1(\gamma) &\leq \frac{4u}{\delta_*^2}\Psi^2(\gamma)\prod_{y\in\gamma}(1-c_\phi(t+u)\psi(x)) \qquad (4.126)\\ &\leq \frac{16u}{(e\delta_*c_\phi(t+u))^2}\prod_{y\in\gamma}(1-c_\phi(t+u)\psi(x))\,e^{c_\phi(t+u)\psi(x)}\\ &\leq \frac{16u}{(e\delta_*c_\phi)^2}e^{2(t+u)}. \end{aligned}$$

By (3.14) we have

$$\begin{aligned} K_2(\gamma) &\leq \frac{1}{\delta_*}F^{\phi_t}(\gamma)\sum_{x\in\gamma}|(\Phi\phi_{t+u})(x)-(\Phi\phi_t)(x)| \qquad (4.127)\\ &\leq \frac{2un_*m}{\delta_*}\Psi(\gamma)F^{\phi_t}(\gamma) \leq \frac{2un_*m}{e\delta_*c_\phi}e^t. \end{aligned}$$

Similarly,

$$K_3(\gamma) \leq \frac{1}{\delta_*}F^{\phi_t}(\gamma)\sum_{x\in\gamma}|\phi_{t+u}(x)-\phi_t(x)| \leq \frac{2u}{e\delta_*c_\phi}e^t. \qquad (4.128)$$

Now we use (4.126), (4.127), (4.128) in (4.124), and thus obtain (3.17) with

$$C_\phi = \frac{2u(n_*m+1)}{e\delta_*c_\phi}+\frac{16u}{(e\delta_*c_\phi)^2},$$

which completes the proof.

4.2.7 Proof of Lemma 3.2.10

The proof of the stated inclusion will be done by showing that each F^ϕ, $\phi \in C_\psi(X)$, can be obtained as an $\| \cdot \|_L$-limit of the elements of $\mathcal{D}^0(L)$. Namely, we are going to show that

$$\|\lambda F^\phi_\lambda - F^\phi\|_L \to 0, \qquad \text{as} \quad \lambda \to +\infty. \tag{4.129}$$

To this end, with the help of the first equality in (3.20) we write

$$\left|\lambda(LF^\phi_\lambda)(\gamma) - (LF^\phi)(\gamma)\right| = \left|\int_0^{+\infty} \left[(LF^{\phi_t})(\gamma) - (LF^\phi)(\gamma)\right] e^{-\lambda t} dt\right| \tag{4.130}$$

$$\leq \int_0^{+\infty} \left|(LF^{\phi_{\epsilon s}})(\gamma) - (LF^\phi)(\gamma)\right| e^{-s} ds, \quad \epsilon = 1/\lambda.$$

Now we use here (3.17) with $t = 0$, $u = \epsilon s$ and obtain for $\epsilon < 1/2$ the following estimate

$$\text{LHS}(4.130) \leq \epsilon C_\phi \int_0^{+\infty} s e^{-s(1-2\epsilon)} ds = \frac{\epsilon}{(1-2\epsilon)^2} C_\phi \to 0, \quad \text{as } \epsilon \to 0. \tag{4.131}$$

Next, by (3.20) – and then by (3.16) – we get

$$\|\lambda F^\phi_\lambda - F^\phi\| = \|LF^\phi_\lambda\| \leq \int_0^{+\infty} \|LF^{\phi_t}\| e^{-\lambda t} dt \tag{4.132}$$

$$\leq \frac{2}{e\delta_* c_\phi} \int_0^{+\infty} e^{-(\lambda-1)t} dt = \frac{1}{\lambda - 1}\left(\frac{2}{e\delta_* c_\phi}\right),$$

where we have used the fact that $c_{\phi_t} = c_\phi(t) = c_\phi e^{-t}$, see (4.118). Now (4.129) readily follows by (4.131) and (4.132). The second part of the statement follows by Remark 3.2.8.

4.2.8 Proof of Corollary 3.2.11

The stated closedness follows by (3.23), whereas the density of $\mathcal{D}(L)$ is a consequence of Lemma 3.2.10 and (3.18). For $\phi \in C_\psi(X)$ and $\lambda > 0$, define

$$R_\lambda F^\phi = F^\phi_\lambda = \int_0^{+\infty} F^{\phi_t} e^{-\lambda t} dt,$$

see (3.19). Then

$$\|R_\lambda F^\phi\| \leq 1/\lambda, \tag{4.133}$$

which allows one to continue R_λ to all $F \in E(\Gamma^\psi)$ since $E^0(\Gamma^\psi)$ is dense therein. At the same time, by (3.20) it follows that

$$(\lambda - L)R_\lambda F^\phi = F^\phi,$$

which can be continued to all $F \in E(\Gamma^\psi)$. Thus, R_λ is the resolvent of L, whose norm can be estimated by means of (4.133). The property $R_\lambda : E(\Gamma^\psi) \to \mathcal{D}(L)$ can be proved by taking a Cauchy sequence $\{F^{\phi_n}\}_{n\in\mathbb{N}} \subset E^0(\Gamma^\psi)$, and then showing that the sequence $\{(\lambda - L)F_\lambda^{\phi_n}\}_{n\in\mathbb{N}} \subset \mathcal{D}^0(L)$ is a Cauchy sequence in $\|\cdot\|_L$. This can be done similarly as in the proof of Lemma 3.2.10.

4.2.9 Proof of Theorem 3.3.1

Corollary 3.2.11 and (4.133) allows one to apply here the celebrated Hille-Yosida theorem, see [3, page 134], by which it follows that $(L, \mathcal{D}(L))$ is the generator of a C_0-semigroup, say $S = \{S(t)\}_{t\geq 0}$, of bounded linear operators on $E(\Gamma^\psi)$ such that the operator norm of each $S(t)$ satisfies $\|S(t)\| \leq 1$. Then the existence of the solution in question in the form $F_t = S(t)F_0$ is a standard fact, see [3, Theorem 3.1.12, page 115]. If F_0 belongs to the core of $\mathcal{D}(L)$, i.e., $F_0 \in \mathcal{D}^0(L)$, see (3.21), the solution can be obtained in an explicit form. In this case, in view of the linearity of $S(t)$, we take $F_0 = F_\lambda^\phi$ for some $\lambda > 1$ and $\phi \in C_\psi(X)$. Then the solution is

$$F_t = S(t)F_0 = S(t)F_\lambda^\phi = F_\lambda^{\phi_t} = F_\lambda^{Q_t(\phi)} = \int_0^{+\infty} F^{\phi_{t+s}} e^{-\lambda s} ds, \tag{4.134}$$

see Remark 3.2.3. That is, for F_0 in the core of $\mathcal{D}(L)$, the action of S on F_0 is obtained by applying the semigroup of nonlinear operators acting in the space of continuous functions defined on the basic space X. Then, in the subcritical case, the stated convergence follows by the concluding statement of Lemma 3.2.2.

4.2.10 Proofs of Proposition 3.3.5 and Lemma 3.3.4

Proof of Proposition 3.3.5. First, we rewrite (1.34) in the form

$$\mu_{t_2}(F) = \mu_{t_1}(F) + \int_{t_1}^{t_2} \mu_s(LF)ds, \qquad 0 \leq t_1 < t_2. \tag{4.135}$$

For $F = F^\phi_\lambda$, $\phi \in C_\psi(X)$, $\lambda > 1$, see (3.21), by (3.15), and then by (3.19) and (3.20), we have $\|LF^\phi_\lambda\| \leq 2$. Then by (4.135) we get

$$|\mu_{t_2}(F^\phi_\lambda) - \mu_{t_1}(F^\phi_\lambda)| \leq 2|t_2 - t_1|.$$

For $F = \sum_n \alpha_n F^{\phi_n}_{\lambda_n} \in \mathcal{D}^0(L)$, this yields

$$|\mu_{t_2}(F) - \mu_{t_1}(F)| \leq 2\left(\sum_n |\alpha_n|\right)|t_2 - t_1|.$$

Now for $F = F^\phi$, $\phi \in C_\psi(X)$, by (3.16) we have

$$|\mu_{t_2}(F^\phi) - \mu_{t_1}(F^\phi)| \leq \frac{2}{e\delta_* c_\phi}|t_2 - t_1|.$$

The extension of the latter to the linear combinations of F^{ϕ_n} can be done similarly as above. □

Proof of Lemma 3.3.4. By Remark 3.2.8 we know that $E^0(\Gamma^\psi)$ is *bp*-dense in $B_{\mathrm{b}}(\Gamma^\psi)$. Then the measurability of $t \mapsto \mu_t(F)$, $F \in B_{\mathrm{b}}(\Gamma^\psi)$ follows by the continuity (hence, measurability) just proved. □

4.2.11 Proof of Theorem 3.3.3

In view of Lemma 3.3.4, it remains to establish the existence and uniqueness of solutions of (1.34) with $F \in \mathcal{D}(L)$. First we prove existence. For $F \in \mathcal{D}(L)$ and $t > 0$, we have $F_t = S(t)F$, see (4.134). Then we set

$$\mu_t(F) = \mu(F_t) = \mu(S(t)F), \qquad \mu \in \mathcal{P}(\Gamma^\psi).$$

This, in particular, means $\mu_s(F_t) = \mu_{s+t}(F)$, and also

$$\mu_t(F^\phi_\lambda) = \mu(F^{\phi_t}_\lambda), \qquad \mu_t(F^\phi) = \mu(F^{\phi_t}), \tag{4.136}$$

holding for all $\lambda > 1$ and $\phi \in C_\psi(X)$, see also (3.24). To prove that $t \mapsto \mu_t$ solves (4.135), we take $F = F^\phi_\lambda \in \mathcal{D}^0(L)$, and then get by (3.20) the following

$$\begin{aligned} \int_{t_1}^{t_2} \mu_s(LF^\phi_\lambda)ds &= -\int_{t_1}^{t_2} \mu_s(F^\phi)ds + \int_{t_1}^{t_2} \mu_s(\lambda F^\phi_\lambda)ds \qquad (4.137) \\ &= -\int_{t_1}^{t_2} \mu_s(F^\phi)ds + \int_{t_1}^{t_2}\int_0^{+\infty} \lambda e^{-\lambda t}\mu_s(F^{\phi_t})dsdt, \end{aligned}$$

where we used also Fubini's theorem. Then by (4.136) and the flow property we get $\mu_s(F^{\phi_t}) = \mu_{s+t}(F^\phi)$ and then use this in the second summand (name it Υ) of the last line of (4.137), then integrate by parts and obtain

$$\begin{aligned}
\Upsilon &= \int_{t_1}^{t_2} \mu_s(F^\phi)ds + \int_{t_1}^{t_2} \frac{d}{ds}\left(\int_0^{+\infty} e^{-\lambda t}\mu_{s+t}(F^\phi)dt\right)ds \\
&= \int_{t_1}^{t_2} \mu_s(F^\phi)ds + \int_{t_1}^{t_2} \frac{d}{ds}\left(\int_0^{+\infty} e^{-\lambda t}\mu_s(F_t^\phi)dt\right)ds \\
&= \int_{t_1}^{t_2} \mu_s(F^\phi)ds + \int_{t_1}^{t_2} \frac{d}{ds}\mu_s(F_\lambda^\phi)ds \\
&= \int_{t_1}^{t_2} \mu_s(F^\phi)ds + \mu_{t_2}(F_\lambda^\phi) - \mu_{t_1}(F_\lambda^\phi).
\end{aligned}$$

Now we plug this in (4.137) and get that property of $t \mapsto \mu_t(F)$, $F \in \mathcal{D}^0(L)$, solves (4.135). For $F \in \mathcal{D}(L)$, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^0(L)$ be such that $\|F - F_n\|_L \to 0$ as $n \to +\infty$. Then

$$\begin{aligned}
\left|\mu_{t_2}(F) - \mu_{t_1}(F) - \int_{t_1}^{t_2} \mu_s(LF)ds\right| &\le |\mu_{t_2}(F - F_n)| + |\mu_{t_1}(F - F_n)| \\
&+ \int_{t_1}^{t_2} |\mu_s(LF - LF_n)|\, ds \le (t_2 - t_1 + 2)\|F - F_n\|_L,
\end{aligned}$$

which yields that $t \mapsto \mu_t(F)$, $F \in \mathcal{D}(L)$ also solves (4.135).

Assume now that $t \mapsto \tilde{\mu}_t$ is another solution of (1.34), and hence of (4.135), satisfying $\tilde{\mu}_t|_{t=0} = \mu$. By Proposition 3.3.5 the map $t \mapsto \tilde{\mu}(F)$, $F \in \mathcal{D}^0(L)$ is Lipschitz-continuous. Then, for each $\lambda > 1$ and $\phi \in C_\psi(X)$, we have

$$d\tilde{\mu}_s(F_\lambda^\phi) = \tilde{\mu}_s(LF_\lambda^\phi)ds,$$

holding for Lebesgue-almost all $s \ge 0$. Then

$$\begin{aligned}
-\lambda \int_0^t e^{-\lambda s}\tilde{\mu}_s(F_\lambda^\phi)ds &= \int_0^t \tilde{\mu}_s(F_\lambda^\phi)de^{-\lambda s} \\
&= \tilde{\mu}_t(F_\lambda^\phi)e^{-\lambda t} - \tilde{\mu}_0(F_\lambda^\phi) - \int_0^t e^{-\lambda s}\tilde{\mu}_s(LF_\lambda^\phi)ds
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\mu}_t(F_\lambda^\phi)e^{-\lambda t} - \tilde{\mu}_0(F_\lambda^\phi) - \lambda \int_0^t e^{-\lambda s}\tilde{\mu}_s(F_\lambda^\phi)ds \\
&\quad + \int_0^t e^{-\lambda s}\tilde{\mu}_s(F^\phi)ds.
\end{aligned}
$$

This yields

$$
\mu(F_\lambda^\phi) = \tilde{\mu}_0(F_\lambda^\phi) = \tilde{\mu}_t(F_\lambda^\phi)e^{-\lambda t} + \int_0^t e^{-\lambda s}\tilde{\mu}_s(F^\phi)ds, \qquad \lambda > 1,
$$

which after passing to the limit $t \to +\infty$ leads to

$$
\mu(F_\lambda^\phi) = \int_0^{+\infty} e^{-\lambda s}\tilde{\mu}_s(F^\phi)ds, \tag{4.138}
$$

that holds for all $\lambda > 1$. By the very definition in (4.136) the map $t \mapsto \mu_t(F^\phi)$ is continuous; the continuity of $t \mapsto \tilde{\mu}_t(F^\phi)$ was established in Proposition 3.3.5. Both maps are bounded. By (3.19) and (4.136), and then by (4.138), the Laplace transforms of both these maps coincide. Therefore, by Lerch's theorem $\mu_t(F^\phi) = \tilde{\mu}_t(F^\phi)$ for all $t > 0$ and $\phi \in C_\psi(X)$. As mentioned above, see Proposition 1.1.7, the class of functions $\{F^\phi : \phi \in C_\psi(X)\}$ is separating, that means $\mu_t = \tilde{\mu}_t$, $t > 0$ and hence the stated uniqueness. The proof the weak convergence $\mu_t \Rightarrow \mu_s$ follows by (4.136) and the fact that $\{F^\phi : \phi \in C_\psi(X)\}$ is also convergence determining, see again Proposition 1.1.7. It remains to prove that $\mu_t \Rightarrow \mu_\infty$ as $t \to +\infty$. Since the set $\{F^\phi : \phi \in C_\psi(X)\}$ is convergence determining, to this end it is enough to show that $\mu_t(F^\phi) \to \mu_\infty(F^\phi) = 1$, holding for all $\phi \in C_\psi(X)$. By (4.136) and the concluding statement of Theorem 2.3.5 we have

$$
\lim_{t\to+\infty} \mu_t(F^\phi) = \lim_{t\to+\infty} \mu(F^{\phi_t}) = \mu(F_\infty) = 1,
$$

which completes the whole proof.

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