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Forced oscillation of conformable fractional partial delay differential equations with impulses

Abstract. In this paper, we establish some interval oscillation criteria for impulsive conformable fractional partial delay differential equations with a forced term. The main results will be obtained by employing Riccati technique. Our results extend and improve some results reported in the literature for the classical differential equations without impulses. An example is provided to illustrate the relevance of the new theorems.

1. Introduction. In recent years, many researchers found that fractional differential equations are more accurate in describing the mathematical modeling of systems and processes in the field of chemical processes, electrodynamics of computer medium, polymer rheology, mathematical biology, etc. The applications of fractional calculus to biomedical problems are done in the areas of membrane biophysics and polymer viscoelasticity, where the experimentally observed power law dynamics for current-voltage and stress-strain relationships are concisely captured by fractional order differential equations. But the most frequently used definitions involve integration which is nonlocal: Riemann–Liouville derivative & Caputo derivative [5, 9, 13, 22, 27]. Fractional calculus is the study of derivatives and integrals of non-integer order and is the generalized form of classical derivatives and integrals. Those fractional derivatives in the fractional calculus have seemed complicated and lacked some basic properties, like the product rule and the

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chain rule. But in 2014, Khalil et al. [12] introduced a new fractional derivative called the conformable derivative which closely resembles the classical derivative.

In order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulsive differential systems to describe the models. For the basic theory on impulsive differential equations, the reader can refer to the monographs and references [2, 16, 17, 20]. The study of the qualitative behavior of partial differential equations has rapidly expanded in the last few decades, see for example [11, 14, 15, 24, 25, 26, 29, 30, 32] and the references they are cited. In particular, the problem of interval oscillation criteria for integer and fractional order impulsive differential equations have been investigated by few authors, we refer the reader to [3, 4, 19, 28, 31] and the references cited therein.

Recently, the theory of fractional differential equations has been intensively studied by many authors. For example, we mention to the problem of anomalous diffusion [7, 8], the nonlinear oscillation of earthquake which can be modeled with fractional derivative [6], and fluid-dynamic traffic model with fractional derivatives [10] also can be used to eliminate the deficiency arising from the assumption to continuum traffic flow and many other, see also [18, 23] and the references they are cited for recent developments in the description of anomalous transport by fractional dynamics. Following this trend, our aim in this paper is to study oscillation properties of partial differential equation of fractional order of the form

$$(1.1)\qquad \left.\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)g\left(\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)\right)\right]+\sum_{i=1}^{n}q_i(x,t)f_i(u(x,t-\sigma))\\ &\qquad =a(t)\Delta u(x,t)+\sum_{j=1}^{m}a_j(t)\Delta u(x,t-\rho_j)+F(x,t),\\ &\qquad\qquad\qquad\qquad\qquad\qquad t\neq t_k,\ t\geq t_0,\\ &u(x,t_k^+)=\alpha_k(x,t_k,u(x,t_k)),\\ &\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k^+)=\beta_k\left(x,t_k,\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k)\right),\ k=1,2,\dots,\\ &\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times\mathbb{R}_+\equiv G, \end{aligned}\right\}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+=[0,+\infty)$, and $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the conformable partial fractional derivative of order α, $0<\alpha\leq 1$. Equation (1.1) is the enhancement with the boundary condition

$$(1.2)\qquad \frac{\partial u(x,t)}{\partial\gamma}+\mu(x,t)u(x,t)=0 \quad \text{for} \quad (x,t)\in\partial\Omega\times\mathbb{R}_+,$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\mu(x,t) \in C(\partial\Omega \times [0,+\infty), [0,+\infty))$.
In this paper, we assume that the following hypotheses are satisfied:

- (H_1) $r(t) \in C^{\alpha}(\mathbb{R}_+, (0,+\infty))$, $q_i(x,t) \in C(\bar{G}, \mathbb{R}_+)$, $q_i(t) = \min_{x\in\bar{\Omega}} q_i(x,t)$, $i = 1,2,\dots,n$, $f_i \in C(\mathbb{R},\mathbb{R})$ are convex in $\mathbb{R}_+$ with $uf(u) > 0$, $uf_i(u) > 0$ and $\frac{f(u)}{u} \ge \epsilon > 0$, $\frac{f_i(u)}{u} \ge k_i > 0$ for $u \ne 0$, $i = 1,2,\dots,n$, $t-\sigma < t$, $t-\rho_j < t$, for $j = 1,2,\dots,m$.
- (H_2) $F \in C(\bar{G},\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$ with $ug(u) > 0$, $g(u) \le \theta u$ for $u \ne 0$, $g^{-1} \in C(\mathbb{R},\mathbb{R})$ is continuous function with $ug^{-1}(u) > 0$ for $u \ne 0$ and there exists a positive constant η such that $g^{-1}(uv) \le \eta g^{-1}(u)g^{-1}(v)$ for $uv \ne 0$.
- (H_3) $a(t)$, $a_j(t) \in PC(\mathbb{R}_+,\mathbb{R}_+)$, $j = 1,2,\dots,m$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\dots$, and left continuous at $t = t_k$, $k = 1,2,\dots$.
- (H_4) $u(x,t)$ and its derivative $\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\dots$, and left continuous at $t = t_k$, $u(x,t_k) = u(x,t_k^-)$, $\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k^-)$ and $0 < t_1 < \dots < t_k < \dots$, $\lim_{t\to+\infty} t_k = +\infty$.
- (H_5) $\alpha_k, \beta_k \in PC(\bar{\Omega}\times\mathbb{R}_+\times\mathbb{R},\mathbb{R})$, $k = 1,2,\dots$, and there exist positive constants a_k, a_k^*, b_k, b_k^* such that $a_k^* \le a_k \le b_k^* \le b_k$ for $k = 1,2,\dots$ and

$$a_k^* \le \frac{\alpha_k\left(x,t_k,u(x,t_k)\right)}{u(x,t_k)} \le a_k, \quad b_k^* \le \frac{\beta_k\left(x,t_k,\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k)\right)}{\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k)} \le b_k.$$

- (H_6) For any $T \ge 0$ there exist intervals $[c_1,d_1]$ and $[c_2.d_2]$ contained in $[T,\infty)$ such that $c_1 < d_1 \le d_1+\sigma \le c_2 < d_2$, $c_l, d_l \notin \{t_k\}$, $l = 1,2$, $k = 1,2,\dots$, $r(t) > 0$, $q(t) \ge 0$, for $t \in [c_1-\sigma,d_1] \cup [c_2-\sigma,d_2]$ and $F(t)$ has different signs in $[c_1-\sigma,d_1]$ and $[c_2-\sigma,d_2]$. For instance we assume that $F(t) \le 0$ for $t \in [c_1-\sigma,d_1]$ and $F(t) \ge 0$ for $t \in [c_2-\sigma,d_2]$.

For simplicity, we denote

$$I(s) := \max\{j:\ t_0 < t_j < s\},\ r_l := \max\{r(t) : t \in [c_l,d_l]\},\quad l = 1,2,$$
$$L_v(c_l,d_l) = \{v \in C^{\alpha}[c_l,d_l],\quad v(t) \not\equiv 0,\ v(c_l) = v(d_l) = 0,\ l = 1,2\}.$$

For two constants c, $d \ne t_k$ with $c < d$ and a function $\psi \in C([c,d],\mathbb{R})$, we define the operator $\Theta : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Theta_c^d[\psi] = \begin{cases} 0, & I(c) = I(d) \\ \psi(t_{I(c)+1})\theta(c) + \sum_{i=I(c)+2}^{I(d)} \psi(t_i)\varepsilon(t_i), & I(c) < I(d), \end{cases}$$

where

$$\theta(c) = \frac{a_{I(c)+1} - b_{I(c)+1}}{a_{I(c)+1}(t^{\alpha}_{I(c)+1} - c^{\alpha})} \quad \text{and} \quad \varepsilon(t_i) = \frac{a_i - b_i}{a_i(t_i^{\alpha} - t_{i-1}^{\alpha})}.$$

The paper is organized as follows: in Section 2, we present some definitions and results that will be needed later. In Section 3, we establish some interval oscillation criteria for the problem (1.1)–(1.2). An example to illustrate our main results is given. To the best of authors' knowledge there has been no work done on the interval oscillation of conformable fractional impulsive partial delay differential equations.

2. Preliminaries. In this section, we present the basic definitions and the basic lemma that will be used in the proof of the main results.

Definition 2.1. A solution u of the problem (1.1)–(1.2) is a function $u \in C^{\alpha}(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1.1), where

$$t_{-1} := \min\left\{0, \min_{1\le j\le m}\left\{\inf_{t\ge 0} t - \rho_j\right\}\right\}, \quad \hat{t}_{-1} := \min\left\{0, \inf_{t\ge 0} t - \sigma\right\}.$$

Definition 2.2. The solution u of the problem (1.1)–(1.2) is said to be oscillatory in the domain G, if it has arbitrary large zeros. Otherwise it is non-oscillatory.

Definition 2.3 ([12]). Given $f : [0, \infty) \to \mathbb{R}$, the conformable fractional derivative of f of order α is defined by

$$T_{\alpha}(f)(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

If f is α-differentiable in some interval $(0, a)$, $a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Definition 2.4. $I_{\alpha}^{a}(f)(t) = I_1^{a}(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.

Definition 2.5 ([1]). Let f be a function with n variables $x_1, x_2, \ldots, x_n$. The conformable partial derivative of f of order $0 < \alpha \le 1$ in x_i is defined as follows

$$\frac{\partial^{\alpha}}{\partial x_i^{\alpha}} f(x_1, x_2, \ldots, x_n) = \lim_{\epsilon\to 0} \frac{f\big(x_1, x_2, \ldots, x_{i-1}, x_i + \epsilon x_i^{1-\alpha}, \ldots, x_n\big) - f(x_1, x_2, \ldots, x_n)}{\epsilon}.$$

Conformable fractional derivatives have the following properties:

Theorem 2.1. *Let* $\alpha \in (0,1]$ *and* f, g *be* α*-differentiable at some point* $t > 0$. *Then*

(*i*) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$.

(*ii*) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.

(*iii*) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.

(*iv*) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(*v*) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(*vi*) *If* f *is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

For convenience, we introduce the following notations:

$$Y(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx, \quad \text{where} \quad |\Omega| = \int_\Omega dx,$$

and

$$F(t) = \frac{1}{|\Omega|}\int_\Omega F(x,t)dx. \tag{2.1}$$

The following lemma will be the basic tool in proving the main results.

Lemma 2.1. *If the impulsive conformable fractional differential inequality*

$$\left.\begin{aligned} &T_\alpha\left[r(t)g\left(T_\alpha(Y(t))\right)\right] + \sum_{i=1}^{n} k_i q_i(t) Y(t-\sigma) \le F(t), \quad t \ne t_k, \\ &a_k^* \le \frac{Y(t_k^+)}{Y(t_k)} \le a_k, \quad b_k^* \le \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \le b_k, \quad k = 1, 2, \dots \end{aligned}\right\} \tag{2.2}$$

has no eventually positive solution, then every solution of the problem (1.1)–(1.2) *is oscillatory in* G.

Proof. Assume that there exists a nonoscillatory solution $u(x,t)$ of the problem (1.1)–(1.2) and $u(x,t) > 0$. By the assumptions, there exists $t_1 > t_0 > 0$ such that $t - \sigma \ge t_1$ and $t - \rho_j \ge t_1$ for $t \ge t_1 > 0$, and

$$\begin{aligned} &u(x, t-\sigma) > 0 \quad \text{for} \quad (x, t-\sigma) \in \Omega \times [t_1, \infty), \\ &u(x, t-\rho_j) > 0 \quad \text{for} \quad (x, t-\rho_j) \in \Omega \times [t_1, \infty), \ j = 1, 2, \dots, m. \end{aligned}$$

By multiplying both sides of the main equation in (1.1) by $\frac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω, we obtain $t \ge t_1$ and $t \ne t_k$,

$k = 1, 2, \ldots,$ such that

$$\left.\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)g\left(\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx\right)\right)\right] \\ &\qquad +\frac{1}{|\Omega|}\sum_{i=1}^n\int_\Omega q_i(x,t)f_i(u(x,t-\sigma))dx \\ &= a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx + \frac{1}{|\Omega|}\sum_{j=1}^m a_j(t)\int_\Omega \Delta u(x,t-\rho_j)dx \\ &\qquad +\frac{1}{|\Omega|}\int_\Omega F(x,t)dx. \end{aligned}\right\} \tag{2.3}$$

From Green's formula and boundary condition (1.2) we see that

$$\begin{aligned} \int_\Omega \Delta u(x,t)dx &= \int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\gamma}dS \\ &= -\int_{\partial\Omega}\mu(x,t)u(x,t)dS \le 0 \end{aligned} \tag{2.4}$$

and for $j = 1, 2, \ldots, m$, we have

$$\begin{aligned} \int_\Omega \Delta u(x,t-\rho_j)dx &= \int_{\partial\Omega}\frac{\partial u(x,t-\rho_j)}{\partial\gamma}dS \\ &= -\int_{\partial\Omega}\mu(x,t)u(x,t-\rho_j)dS \le 0, \end{aligned} \tag{2.5}$$

where dS is the surface element on $\partial\Omega$. Moreover by (H_1), it follows that for $i = 1, 2, \ldots, n$,

$$\begin{aligned} \int_\Omega q_i(x,t)f_i(u(x,t-\sigma))dx &\ge q_i(t)\int_\Omega f_i(u(x,t-\sigma))dx \\ &\ge k_iq_i(t)\int_\Omega u(x,t-\sigma)dx. \end{aligned} \tag{2.6}$$

In view of (2.3)–(2.6), we obtain

$$T_\alpha\left[r(t)g\left(T_\alpha(Y(t))\right)\right] + \sum_{i=1}^n k_iq_i(t)Y(t-\sigma) \le F(t), \quad t \ne t_k.$$

For $t \ge t_0$, $t = t_k$, $k = 1, 2, \ldots,$ from the boundary conditions we see that

$$a_k^* \le \frac{u(x,t_k^+)}{u(x,t_k)} \le a_k, \qquad b_k^* \le \frac{\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u(x,t_k)} \le b_k.$$

Multiplying both sides by $\frac{1}{|\Omega|}$, integrating with respect to x over the domain Ω, and using (H_5), we obtain

$$a_k^* \le \frac{Y(t_k^+)}{Y(t_k)} \le a_k, \qquad b_k^* \le \frac{T_\alpha(Y(t_k^+))}{T_\alpha(Y(t_k))} \le b_k.$$

Hence we see that $Y(t)$ is an eventually positive solution of (2.2). This contradicts our assumption and complete the proof. $\square$

3. Main Results. In this section, we establish some new interval oscillation criteria for (1.1) by using the Riccati transformation technique and integral average method. For simplicity, we define

$$\begin{aligned}\Gamma_0(c_l,d_l) := &\int_{c_l}^{t_{I(c_l)+1}} \left[\delta(v'(t))^2 t^{2-2\alpha} r(t) - v^2(t)Q(t)M^l_{I(c_l)}(t)\right] dt\\ &+ \sum_{k=I(c_l)+1}^{I(d_l)-1} \int_{t_k}^{t_{k+1}} \left[\delta(v'(t))^2 t^{2-2\alpha} r(t) - v^2(t)Q(t)M^l_k(t)\right] dt\\ &+ \int_{t_{I(d_l)}}^{d_l} \left[\delta(v'(t))^2 t^{2-2\alpha} r(t) - v^2(t)Q(t)M^l_{I(d_l)}(t)\right] dt\\ &+ \int_{c_l}^{d_l} (1-\alpha)t^{-\alpha} v^2(t)w(t)dt,\end{aligned}$$

where $Q(t) = \sum_{i=1}^{n} k_i q_i(t)$ and

$$M^l_k(t) = \begin{cases} \dfrac{\sigma\alpha}{\sigma\alpha a_k + b_k(t^\alpha - t_k^\alpha)} \dfrac{(t-\sigma)^\alpha - (t_k-\sigma)^\alpha}{t_k^\alpha - (t_k-\sigma)^\alpha}, & t \in (t_k, t_k+\sigma),\\ \dfrac{(t-\sigma)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}, & t \in [t_k+\sigma, t_{k+1}). \end{cases}$$

Theorem 3.1. *Assume that conditions (H_1)–(H_6) hold. Furthermore, assume there exist c_l, d_l satisfying $T \le c_1 < d_1$, $T \le c_2 < d_2$ for any $T \ge 0$.*

(i) $\int_{t_0}^{\infty} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds = \infty$.

(ii) *Assume that $v(t) \in L_v(c_l, d_l)$ is such that*

$$\Gamma_0(c_l,d_l) < r_l \Theta_{c_l}^{d_l}[v^2(t)] \tag{3.1}$$

for $I(c_l) < I(d_l)$, $l = 1, 2$, then every solution of the problem (1.1)–(1.2) *is oscillatory.*

Proof. Assume to the contrary that $u(x,t)$ is a non-oscillatory solution of (1.1)–(1.2). Without loss of generality we may assume that $u(x,t)$ is an eventually positive solution of (1.1)–(1.2). Then there exists $t_1 \ge t_0$ such that $u(x,t) > 0$ for $t \ge t_1$. Therefore, from (2.2) it follows that

$$T_\alpha\left[r(t) g\left(T_\alpha(Y(t))\right)\right] \le F(t) - Q(t)Y(t-\sigma) \quad \text{for } t \in [t_1, \infty). \tag{3.2}$$

Thus $T_\alpha(Y(t)) \ge 0$ or $T_\alpha(Y(t)) < 0$, $t \ge t_1$, for some $t_1 \ge t_0$. We now claim that

$$T_\alpha(Y(t)) \ge 0 \quad \text{for } t \ge t_1. \tag{3.3}$$

Suppose not, then $T_\alpha(Y(t)) < 0$ and there exists $t_2 \in [t_1, \infty)$ such that $T_\alpha(Y(t_2)) < 0$. Since $r(t)g(T_\alpha(Y(t)))$ is strictly decreasing in $[t_1, \infty)$, it is clear that

$$r(t)g\left(T_\alpha(Y(t))\right) < r(t_2)g\left(T_\alpha(Y(t_2))\right) := -c,$$

where $c > 0$ is a constant. For $t \in [t_2, \infty)$, after integrating the above inequality from t_2 to t, we have

$$Y(t) \le Y(t_2) - c_1 \int_{t_2}^{t} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to \infty$, we get $\lim_{t\to+\infty} Y(t) = -\infty$. This contradiction shows that (3.3) holds. Define the Riccati transformation

$$w(t) := \frac{r(t)g(T_\alpha(Y(t)))}{Y(t)}. \tag{3.4}$$

It follows from (2.2) that $w(t)$ satisfies

$$T_\alpha(w(t)) \le \frac{F(t)}{Y(t)} - Q(t)\frac{Y(t-\sigma)}{Y(t)} - \frac{w^2(t)}{\delta r(t)}.$$

By assumption (H_6), we can choose c_1, $d_1 \ge t_0$ such that $r(t) > 0$, $q(t) \ge 0$ for $t \in [c_1 - \sigma, d_1]$ and $F(t) \le 0$ for $t \in [c_1 - \sigma, d_1]$. From (2.2), we can easily see that

$$T_\alpha(w(t)) \le -Q(t)\frac{Y(t-\sigma)}{Y(t)} - \frac{w^2(t)}{\delta r(t)}. \tag{3.5}$$

For $t = t_k,\ k = 1, 2, \dots,$ we have

$$w(t_k^+) := \frac{r(t_k^+)g(T_\alpha(Y(t_k^+)))}{Y(t_k^+)} \le \frac{b_k}{a_k} w(t_k). \tag{3.6}$$

First, we consider the case when $I(c_1) < I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{I(c_1)+1}, t_{I(c_1)+2}, \dots, t_{I(d_1)}$. Choose $v(t) \in L_v(c_1, d_1)$. Multiplying both sides of (3.5) by $v^2(t)$ and integrating the resulting inequality from c_1 to d_1, we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{I(c_1)+1}} v^2(t)t^{1-\alpha}w'(t)dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} v^2(t)t^{1-\alpha}w'(t)dt + \dots \\
&\quad + \int_{t_{I(d_1)}}^{d_1} v^2(t)t^{1-\alpha}w'(t)dt \\
&\le -\int_{c_1}^{t_{I(c_1)+1}} v^2(t)\frac{w^2(t)}{\delta r(t)}dt - \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} v^2(t)\frac{w^2(t)}{\delta r(t)}dt - \dots \\
&\quad - \int_{t_{I(d_1)}}^{d_1} v^2(t)\frac{w^2(t)}{\delta r(t)}dt - \int_{c_1}^{t_{I(c_1)+1}} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_{I(c_1)+1}}^{t_{I(c_1)+1}+\sigma} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \\
&-\int_{t_{I(c_1)+1}+\sigma}^{t_{I(c_1)+2}} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt - \dots \\
&-\int_{t_{I(d_1)-1}+\sigma}^{t_{I(d_1)}} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt - \int_{t_{I(d_1)}}^{d_1} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt.
\end{aligned}
$$

Using integration by parts on the left-hand side, and noting that $v(c_1) = v(d_1) = 0$, we get

$$
\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} v^2(t_k)t_k^{1-\alpha}\left[w(t_k) - w(t_k^+)\right] \\
&\le -\int_{c_1}^{d_1}\left[\frac{v(t)w(t)}{\sqrt{\delta r(t)}} - v'(t)t^{1-\alpha}\sqrt{\delta r(t)}\right]^2 dt \\
&\quad -\int_{c_1}^{t_{I(c_1)+1}} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \\
&\quad -\sum_{k=I(c_1)+1}^{I(d_1)-1}\Bigg[\int_{t_k}^{t_k+\sigma} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \\
&\qquad\qquad + \int_{t_k+\sigma}^{t_{k+1}} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt\Bigg] \\
&\quad -\int_{t_{I(d_1)}}^{d_1} v^2(t)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt + \int_{c_1}^{d_1} t^{2-2\alpha}\delta r(t)(v'(t))^2 dt \\
&\quad +\int_{c_1}^{d_1}(1-\alpha)t^{-\alpha}p^2(t)w(t)dt.
\end{aligned} \tag{3.7}
$$

We consider several cases to estimate $\frac{Y(t-\sigma)}{Y(t)}$.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$, since $t_{k+1} - t_k > \sigma$, we consider two subcases:

Case 1.1: If $t \in [t_k + \sigma, t_{k+1}]$, then $t - \sigma \in [t_k, t_{k+1} - \sigma]$ and there are no impulsive moments in $(t - \sigma, t)$. Then, for any $t \in [t_k + \sigma, t_{k+1}]$ we have

$$
Y(t) - Y(t_k^+) = T_\alpha(Y(\xi))\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right), \quad \xi \in (t_k, t).
$$

From this,

$$
Y(t) \ge T_\alpha(Y(\xi))\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right).
$$

We obtain

$$\frac{T_\alpha(Y(t))}{Y(t)} < \frac{\alpha}{t^\alpha - t_k^\alpha}.$$

Integrating it from $t-\sigma$ to t, we have

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{(t-\sigma)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}.$$

Case 1.2: If $t \in (t_k, t_k+\sigma)$, then $t-\sigma \in (t_k-\sigma, t_k)$ and there is an impulsive moment t_k in $(t-\sigma, t)$. Similarly to Case 1.1, we obtain

$$Y(t) - Y(t_k-\sigma) = T_\alpha(Y(\xi_1))\left(\frac{t^\alpha - (t_k-\sigma)^\alpha}{\alpha}\right), \quad \xi_1 \in (t_k-\sigma, t_k]$$

or

$$\frac{T_\alpha(Y(t))}{Y(t)} < \frac{\alpha}{\delta}\frac{1}{t^\alpha - (t_k-\sigma)^\alpha}.$$

Integrating it from $t-\sigma$ to t, we get

$$\frac{Y(t-\sigma)}{Y(t_k)} > \frac{(t-\sigma)^\alpha - (t_k-\sigma)^\alpha}{t_k^\alpha - (t_k-\sigma)^\alpha} > 0, \quad t \in (t_k, t_k+\sigma). \tag{3.8}$$

For any $t \in (t_k, t_k+\sigma)$ we have

$$Y(t) - Y(t_k^+) \le T_\alpha(Y(t_k^+))\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right).$$

Using the impulsive conditions in equation (1.1), we get

$$Y(t) - a_kY(t_k) < b_kT_\alpha(Y(t_k))\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right)$$

$$\frac{Y(t)}{Y(t_k)} < b_k\frac{T_\alpha(Y(t_k))}{Y(t_k)}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right) + a_k.$$

Using $\frac{T_\alpha(Y(t_k))}{Y(t_k)} < \frac{1}{\sigma}$, we obtain

$$\frac{Y(t)}{Y(t_k)} < a_k + \frac{b_k}{\sigma}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right).$$

That is,

$$\frac{Y(t_k)}{Y(t)} > \frac{\sigma\alpha}{\sigma\alpha a_k + b_k(t^\alpha - t_k^\alpha)}. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{\sigma\alpha}{\sigma\alpha a_k + b_k(t^\alpha - t_k^\alpha)}\frac{(t-\sigma)^\alpha - (t_k-\sigma)^\alpha}{t_k^\alpha - (t_k-\sigma)^\alpha} \ge 0.$$

Case 2: If $t \in [c_1, t_{I(c_1)+1}]$, we consider three subcases:
Case 2.1: If $t_{I(c_1)} > c_1 - \sigma$ and $t \in [t_{I(c_1)} + \sigma, t_{I(c_1)+1}]$, then $t - \sigma \in$

$[t_{I(c_1)}, t_{I(c_1)+1} - \sigma]$ and there are no impulsive moments in $(t-\sigma, t)$. Proceeding as in Case 1.1 and using the Mean-value Theorem on $(t_{I(c_1)}, t_{I(c_1)+1}]$, we get

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{(t-\sigma)^\alpha - t^\alpha_{I(c_1)}}{t^\alpha - t^\alpha_{I(c_1)}} > 0, \quad t \in [t_{I(c_1)} + \sigma, t_{I(c_1)+1}].$$

Case 2.2: If $t_{I(c_1)} > c_1 - \sigma$ and $t \in [c_1, t_{I(c_1)} + \sigma)$, then $t - \sigma \in [c_1 - \sigma, t_{I(c_1)})$ and there is an impulsive moment $t_{I(c_1)}$ in $(t-\sigma, t)$. Making a similar analysis as in Case 1.2, we have

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{\sigma\alpha}{\sigma\alpha a_{I(c_1)} + b_{I(c_1)}(t^\alpha - t^\alpha_{I(c_1)})} \frac{(t-\sigma)^\alpha - (t_{I(c_1)} - \sigma)^\alpha}{t^\alpha_{I(c_1)} - (t_{I(c_1)} - \sigma)^\alpha} \geq 0,$$

$t \in (c_1, t_{I(c_1)} + \sigma)$.
Case 2.3: If $t_{I(c_1)} < c_1 - \sigma$, then for any $t \in [c_1, t_{I(c_1)+1}]$, $t - \sigma \in [c_1 - \sigma, t_{I(c_1)+1} - \sigma]$ and there are no impulsive moments in $(t-\sigma, t)$. Working as in Case 1.1, we get

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{(t-\sigma)^\alpha - t^\alpha_{I(c_1)}}{t^\alpha - t^\alpha_{I(c_1)}} > 0, \quad t \in [c_1, t_{I(c_1)+1}].$$

Case 3: For $t \in (t_{I(d_1)}, d_1]$ we consider three subcases:
Case 3.1: If $t_{I(d_1)} + \sigma < d_1$ and $t \in [t_{I(d_1)} + \sigma, d_1]$, then $t - \sigma \in [t_{I(d_1)}, d_1 - \sigma]$ and there are no impulsive moments in $(t-\sigma, t)$. Using a similar analysis as in Case 2.1, we have

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{(t-\sigma)^\alpha - t^\alpha_{I(d_1)}}{t^\alpha - t^\alpha_{I(d_1)}} > 0, \quad t \in [t_{I(d_1)} + \sigma, d_1].$$

Case 3.2: If $t_{I(d_1)} + \sigma < d_1$ and $t \in [t_{I(d_1)}, t_{I(d_1)} + \sigma)$, then $t - \sigma \in [t_{I(d_1)} - \sigma, t_{I(d_1)})$ and there is an impulsive moment $t_{I(d_1)}$ in $(t-\sigma, t)$. Using a similar analysis as in Case 2.2, we obtain

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{\sigma\alpha}{\sigma\alpha a_{I(d_1)} + b_{I(d_1)}(t^\alpha - t^\alpha_{I(d_1)})} \frac{(t-\sigma)^\alpha - (t_{I(d_1)} - \sigma)^\alpha}{t^\alpha_{I(d_1)} - (t_{I(d_1)} - \sigma)^\alpha} \geq 0.$$

Case 3.3: If $t_{I(d_1)} + \sigma \geq d_1$, then for any $t \in (t_{I(d_1)}, d_1]$ we get $t - \sigma \in (t_{I(d_1)} - \sigma, d_1 - \sigma]$ and there is an impulsive moment $t_{I(d_1)}$ in $(t-\sigma, t)$. Proceeding as in Case 3.2, we get

$$\frac{Y(t-\sigma)}{Y(t)} > \frac{\sigma\alpha}{\sigma\alpha a_{I(d_1)} + b_{I(d_1)}(t^\alpha - t^\alpha_{I(d_1)})} \frac{(t-\sigma)^\alpha - (t_{I(d_1)} - \sigma)^\alpha}{t^\alpha_{I(d_1)} - (t_{I(d_1)} - \sigma)^\alpha} \geq 0.$$

Combining all these cases, we have

$$\frac{Y(t-\sigma)}{Y(t)} > \begin{cases} M^1_{I(c_1)}(t) & \text{for } t \in [c_1, t_{I(c_1)+1}], \\ M^1_k(t) & \text{for } t \in (t_k, t_{k+1}],\ k = I(c_1)+1, \dots, I(d_1)-1, \\ M^1_{I(d_1)}(t) & \text{for } t \in (t_{I(d_1)+1}, d_1]. \end{cases}$$

Hence, and since $r(t)g(T_\alpha(Y(t)))$ is non-increasing in $(c_1, t_{I(c_1)+1}]$, by (3.7) we have

$$\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} v^2(t_k)t_k^{1-\alpha}\left[w(t_k)-w(t_k^+)\right] \\
&\le \int_{c_1}^{t_{I(c_1)+1}} \left[(v'(t))^2 t^{2-2\alpha}\delta r(t) - v^2(t)Q(t)M^1_{I(c_1)}(t)\right] dt \\
&\quad + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(v'(t))^2 t^{2-2\alpha}\delta r(t) - v^2(t)Q(t)M^1_k(t)\right] dt \\
&\quad + \int_{t_{I(d_1)}}^{d_1} \left[(v'(t))^2 t^{2-2\alpha}\delta r(t) - v^2(t)Q(t)M^1_{I(d_1)}(t)\right] dt \\
&\quad + \int_{c_1}^{d_1} (1-\alpha)t^{-\alpha}v^2(t)w(t)dt.
\end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned}
Y(t) > Y(t) - Y(c_1) &= T_\alpha(Y(\xi_2))\left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right) \\
&\ge \frac{r(t)(T_\alpha(Y(t)))}{r(\xi_2)}\left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right), \quad \xi_2 \in (c_1, t).
\end{aligned}$$

Letting $t \to t^-_{I(c_1)+1}$, it follows that

$$w(t_{I(c_1)+1}) < \frac{r_1}{t^\alpha_{I(c_1)+1} - c_1^\alpha}. \tag{3.11}$$

Similarly, we can prove that on $(t_{k-1}, t_k]$, $k = I(c_1)+2, \ldots, I(d_1)$,

$$w(t_k) < \frac{r_1}{t_k^\alpha - t_{k-1}^\alpha}. \tag{3.12}$$

Hence, from (3.11) and (3.12), we have

$$\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} v^2(t_k)t_k^{1-\alpha}w(t_k)\left[\frac{a_k - b_k}{a_k}\right] \\
&\ge r_1\left[v^2(t_{I(c_1)+1})t^{1-\alpha}_{I(c_1)+1}\frac{a_{I(c_1)+1} - b_{I(c_1)+1}}{a_{I(c_1)+1}}\frac{1}{t^\alpha_{I(c_1)+1} - c_1^\alpha}\right. \\
&\quad \left. + \sum_{k=I(c_1)+1}^{I(d_1)} v^2(t_k)t_k^{1-\alpha}\frac{a_k - b_k}{a_k}\frac{1}{t_k^\alpha - t_{k-1}^\alpha}\right] \ge r_1\Theta_{c_1}^{d_1}[v^2(t)].
\end{aligned} \tag{3.13}$$

Thus we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} v^2(t_k)t_k^{1-\alpha}w(t_k)\left[\frac{a_k-b_k}{a_k}\right] \geq r_1\Theta_{c_1}^{d_1}[v^2(t)].$$

Therefore, using (3.10), we get

$$\begin{aligned}
&\int_{c_1}^{t_{I(c_1)+1}} \left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^1_{I(c_1)}(t)\right]dt\\
&+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^1_k(t)\right]dt\\
&+\int_{t_{I(d_1)}}^{d_1}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^1_{I(d_1)}(t)\right]dt\\
&+\int_{c_1}^{d_1}(1-\alpha)t^{-\alpha}v^2(t)w(t)dt > r_1\Theta_{c_1}^{d_1}[v^2(t)],
\end{aligned}$$

which contradicts (3.1). If $I(c_1)=I(d_1)$ then $r_1\Theta_{c_1}^{d_1}[v^2(t)]=0$ and there are no impulsive moments in $[c_1,d_1]$. Similarly to the proof of (3.10), we obtain

$$\int_{c_1}^{d_1}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^1_{I(c_1)}(t)+v^2(t)(1-\alpha)t^{-\alpha}w(t)\right]dt>0.$$

This again contradicts our assumption. Finally, if $u(x,t)$ is eventually negative, we can consider $[c_2,d_2]$ instead of $[c_1,d_1]$ and get a contradiction. The proof is complete. □

Next, we establish new oscillation criteria for (1.1)–(1.2) by using the integral average method used in [21] for ordinary differential equations. Let $D=\{(t,s): t_0\leq s\leq t\}$, then the functions $H_1,H_2\in C(D,\mathbb{R})$ are said to belong to the class $\mathcal{H}$ if

- (H_7) $H_1(t,t)=H_2(t,t)=0$, $H_1(t,s)>0, H_2(t,s)>0$ for $t>s$ and
- (H_8) H_1 and H_2 have partial derivatives $\frac{\partial H_1}{\partial t}$ and $\frac{\partial H_2}{\partial s}$ on D such that

$$\frac{\partial H_1}{\partial t}=h_1(t,s)H_1(t,s),\qquad \frac{\partial H_2}{\partial s}=-h_2(t,s)H_2(t,s)$$

where $h_1,h_2\in L_{loc}(D,\mathbb{R})$.

We put

$$\begin{aligned}
\Gamma_{1,l}=&\int_{c_l}^{t_{I(c_l)+1}}H_1(t,c_l)Q(t)M^l_{I(c_l)}(t)dt\\
&+\sum_{k=I(c_l)+1}^{I(\lambda_l)-1}\int_{t_k}^{t_{k+1}}H_1(t,c_l)Q(t)M^l_k(t)dt+\int_{t_{I(\lambda_l)}}^{\lambda_l}H_1(t,c_l)Q(t)M^l_{I(d_l)}(t)dt
\end{aligned}$$

$$+\int_{c_l}^{\lambda_l} H_1(t,c_l)\left[\frac{w(t)}{\delta r(t)} - t^{1-\alpha}h_1(t,c_l) - (1-\alpha)t^{-\alpha}\right] w(t)dt$$

and

$$\begin{aligned}\Gamma_{2,l} = &\int_{\lambda_l}^{t_{I(\lambda_l)+1}} H_2(d_l,t)Q(t)M_{I(\lambda_l)}^l(t)dt \\ &+ \sum_{k=I(\lambda_l)+1}^{I(d_l)-1} \int_{t_k}^{t_{k+1}} H_2(d_l,t)Q(t)M_k^l(t)dt \\ &+ \int_{t_{I(d_l)}}^{d_l} H_2(d_l,t)Q(t)M_{I(d_l)}^l(t)dt \\ &+ \int_{\lambda_l}^{d_l} H_2(d_l,t)\left[\frac{w(t)}{\delta r(t)} + t^{1-\alpha}h_2(d_l,t) - (1-\alpha)t^{-\alpha}\right] w(t)dt.\end{aligned}$$

Theorem 3.2. *Assume that conditions* (H_1)–(H_6) *hold. Furthermore, assume that there exist* c_l, d_l *satisfying with* $c_1 < \lambda_1 < d_1 \le c_2 < \lambda_2 < d_2$. *If there exists* H_1, $H_2 \in \mathcal{H}$ *such that* (H_7), (H_8) *hold and*

$$\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1} > \Lambda(H_1,H_2;c_l,d_l), \tag{3.14}$$

where

$$\begin{aligned}&\Lambda(H_1,H_2;c_l,d_l) \\ &= -\left\{\frac{r_l}{H_1(\lambda_l,c_l)}\Theta_{c_l}^{\lambda_l}[H_1(\cdot,c_l)] + \frac{r_l}{H_2(d_l,\lambda_l)}\Theta_{\lambda_l}^{d_l}[H_2(d_l,\cdot)]\right\},\end{aligned} \tag{3.15}$$

then every solution of (1.1)–(1.2) *is oscillatory.*

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the problem (1.1)–(1.2). Notice that whether or not there are impulsive moments in $[c_1,\lambda_1]$ and $[\lambda_1,d_1]$, we should consider the following cases $I(c_1) < I(\lambda_1) < I(d_1)$, $I(c_1) = I(\lambda_1) < I(d_1)$, $I(c_1) < I(\lambda_1) = I(d_1)$ and $I(c_1) = I(\lambda_1) = I(d_1)$.

Moreover, the impulsive moments of $Y(t-\sigma)$ involve the following two cases: $t_{I(\lambda_l)} + \sigma > \lambda_l$ and $t_{I(\lambda_l)} + \sigma \le \lambda_l$. Consider the case $I(c_1) < I(\lambda_1) < I(d_1)$, with $t_{I(\lambda_l)} + \sigma > \lambda_l$. For this case, the impulsive moments are $t_{I(\lambda_1)+1}$, $t_{I(\lambda_1)+2}, \ldots,$ $t_{I(d_1)}$ in $[\lambda_1,d_1]$. Multiplying both sides of (3.5) by $H_1(t,c_1)$ and integrating from c_1 to λ_1, we obtain

$$\begin{aligned}\int_{c_1}^{\lambda_1} H_1(t,c_1)T_\alpha(w(t))dt \le &-\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \\ &-\int_{c_1}^{\lambda_1} H_1(t,c_1)\frac{w^2(t)}{\delta r(t)}dt.\end{aligned}$$

Applying integration by parts on the R.H.S of first integral we get,

$$\begin{aligned}
&\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \\
&+\int_{c_1}^{\lambda_1}\left(\frac{w(t)}{\delta r(t)}-t^{1-\alpha}h_1(t,c_1)-(1-\alpha)t^{-\alpha}\right)w(t)H_1(t,c_1)dt \\
\le &-\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[w(t_k)-w(t_k^+)\right]-H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1).
\end{aligned} \tag{3.16}$$

As in the proof of Theorem 3.1, we divide the interval $[c_1,\lambda_1]$ into several parts and calculate the estimation of $Y(t-\sigma)/Y(t)$, to obtain

$$\begin{aligned}
\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{Y(t-\sigma)}{Y(t)}dt \ge &\int_{c_1}^{t_{I(c_1)+1}} H_1(t,c_1)Q(t)M^1_{I(c_1)}(t)dt \\
&+\sum_{k=I(c_1)+1}^{I(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)M^1_k(t)dt \\
&+\int_{t_{I(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{I(\lambda_1)}(t)dt.
\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{I(c_1)+1}} H_1(t,c_1)Q(t)M^1_{I(c_1)}(t)dt \\
&+\sum_{k=I(c_1)+1}^{I(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t_k,c_1)Q(t)M^1_k(t)dt \\
&+\int_{t_{I(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{I(\lambda_1)}(t)dt \\
&+\int_{c_1}^{\lambda_1}\left[\frac{w(t)}{\delta r(t)}-t^{1-\alpha}h_1(t,c_1)-(1-\alpha)t^{-\alpha}\right]w(t)H_1(t,c_1)dt \\
\le &-\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k)-H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1).
\end{aligned} \tag{3.18}$$

On the other hand, multiplying both sides of (3.5) by $H_2(d_1,t)$, integrating from λ_1 to d_1 and following a similar procedure as above, we get

$$
\begin{aligned}
&\int_{\lambda_1}^{t_{I(\lambda_1)+1}} H_2(d_1,t)Q(t)M^1_{I(\lambda_1)}(t)dt \\
&+\sum_{k=I(\lambda_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} H_2(d_1,t_k)Q(t)M^1_k(t)dt \\
&+\int_{t_{I(d_1)}}^{d_1} H_2(d_1,t)Q(t)M^1_{I(d_1)}(t)dt \\
&+\int_{\lambda_1}^{d_1}\left[\frac{w(t)}{\delta r(t)}+t^{1-\alpha}h_2(d_1,t)-(1-\alpha)t^{-\alpha}\right]w(t)H_2(d_1,t)dt \\
\le& -\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\frac{a_k-b_k}{a_k}\right]w(t_k)+H_2(d_1,\lambda_1)\lambda_1^{1-\alpha}w(\lambda_1).
\end{aligned}
\tag{3.19}
$$

Dividing (3.18) and (3.19) by $H_1(\lambda_1,c_1)$ and $H_2(d_1,\lambda_1)$, respectively and summing the resulting inequalities, we get

$$
\begin{aligned}
&\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1} \\
&\le-\left[\frac{1}{H_1(\lambda_1,c_1)}\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right. \\
&\qquad\left.+\frac{1}{H_2(d_1,\lambda_1)}\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right].
\end{aligned}
\tag{3.20}
$$

Using a similar method as in (3.12), we obtain

$$
\left.
\begin{aligned}
&-\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\le -r_1\Theta_{c_1}^{\lambda_1}[H_1(\cdot,c_1)] \\
&-\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\le -r_1\Theta_{\lambda_1}^{d_1}[H_2(d_1,\cdot)].
\end{aligned}
\right\}
\tag{3.21}
$$

From (3.20) and (3.21), we obtain

$$
\begin{aligned}
\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1} &\le -\left\{r_1\Theta_{c_1}^{\lambda_1}[H_1(\cdot,c_1)]+r_1\Theta_{\lambda_1}^{d_1}[H_2(d_1,\cdot)]\right\} \\
&\le \Lambda(H_1,H_2;c_l,d_l),
\end{aligned}
$$

which is a contradiction to condition (3.14). In the case when $u(x,t)<0$, we take the interval $[c_2,d_2]$ instead of the interval $[c_1,d_1]$ and proceeding as in the proof of this case, we get a contradiction. The proof is complete. □

In the following, we present an example to illustrate the results.

Example 1. Consider the following impulsive partial differential equation

$$\left.\begin{aligned}
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left(\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t)\right)+\frac{m}{2}u\left(x,t-\frac{\pi}{8}\right)+\frac{3m}{2}u\left(x,t-\frac{\pi}{8}\right)\\
&\qquad=\frac{2t}{3}\Delta u(x,t)+\frac{5t}{3}\Delta u(x,t-\pi)+F(x,t),\\
&\qquad\qquad\qquad t\neq 2k\pi\pm\frac{\pi}{4},\ t\geq t_0,\\
&u(x,t_k^+)=\frac{1}{2}u(x,t_k),\\
&\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t_k^+)=\frac{3}{2}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t_k),\quad k=1,2,\dots,
\end{aligned}\right\}\tag{3.22}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$u_x(0,t)+u(0,t)=u_x(\pi,t)+u(\pi,t)=0,\quad t\neq 2k\pi\pm\frac{\pi}{4},\quad k=1,2,\dots.$$

Here $\alpha=\frac{1}{2}$, $r(t)=\frac{1}{2}$, $q(t)=m/4$, $q_1(t)=3m/4$, $m>0$ is a positive constant, $f(u)=f_1(u)=g(u)=2u$,

$$F(x,t)=2me^{-x}\left(\sin\left(t-\frac{\pi}{8}\right)+\frac{1}{4}\cos t\right),$$

$a_k=1/2$, $b_k=3/2$. Let $\sigma=\frac{\pi}{8}$, $t_{k+1}-t_k=\frac{\pi}{2}>\frac{\pi}{8}$. Also, for any $T>0$ we choose k large enough such that $T<c_1=4k\pi-\frac{\pi}{2}<d_1=4k\pi$ and $c_2=4k\pi+\frac{\pi}{8}<d_2=4k\pi+\frac{\pi}{2}$, $k=1,2,\dots$. Then there is an impulsive movement $t_k=4k\pi-\frac{\pi}{4}$ in $[c_1,d_1]$ and an impulsive moment $t_{k+1}=4k\pi+\frac{\pi}{4}$ in $[c_2,d_2]$. For $\epsilon=\epsilon_1=1/2$ we have $Q(t)=m/2$, and we take $v(t)=\sin 4t$, $t_{I(c_1)}=4k\pi-\frac{7\pi}{4}$, $t_{I(d_1)}=4k\pi-\frac{\pi}{4}$. Then by simple calculation, the left-hand side of equation (3.1) becomes the following:

$$\begin{aligned}
&\int_{c_l}^{t_{I(c_l)+1}}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^l_{I(c_l)}(t)\right]dt\\
&+\sum_{k=I(c_l)+1}^{I(d_l)-1}\int_{t_k}^{t_{k+1}}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^l_k(t)\right]dt\\
&+\int_{t_{I(d_l)}}^{d_l}\left[\delta(v'(t))^2t^{2-2\alpha}r(t)-v^2(t)Q(t)M^l_{I(d_l)}(t)\right]dt\\
&+\int_{c_l}^{d_l}(1-\alpha)t^{-\alpha}v^2(t)w(t)dt
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}} \left[16t\cos^2 4t - \frac{1}{2} m \sin^2 4t \left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}} \right)\right] dt \\
&\quad + \int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}} \left[16t\cos^2 4t - \frac{1}{2} m \sin^2 4t \left(\frac{\frac{\pi}{16}\left[t^{\frac{1}{2}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}}\right]^{-1}}{\frac{\pi}{32} + \frac{3}{2}} \right)\right. \\
&\qquad \left. \times \left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{3\pi}{8}\right)^{\frac{1}{2}}}{\left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{3\pi}{8}\right)^{\frac{1}{2}}} \right) \right] dt \\
&\quad + \int_{4k\pi-\frac{\pi}{8}}^{4k\pi} \left[16t\cos^2 4t - \frac{1}{2} m \sin^2 4t \left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}} \right)\right] dt \\
&\quad + \frac{1}{2}\int_{4k\pi-\frac{\pi}{2}}^{4k\pi} t^{-1}\sin^2 4t \cot t\, dt \\
&\simeq\ 148.099 - m\ 0.26613.
\end{aligned}
$$

Since $I(c_1) = k-1$, $I(d_1) = k$, $r_1 = 2$, we have

$$
r_1 \Theta_{c_1}^{d_1}[v^2(t)] = 2\left[\frac{a_{I(c)+1} - b_{I(c)+1}}{a_{I(c)+1}(t^{\alpha}_{I(c_1)+1} - c^{\alpha})} \sin^2(4t_{I(c_1)+1}) \right].
$$

Note that condition (3.1) is satisfied in $[c_1, d_1]$ if

$$
148.0775 < m\ 0.26613
$$

and we can choose the constant m very large enough so that (3.22) holds. Therefore, condition (3.1) is satisfied in $[c_1, d_1]$. We can work similarly for $t \in [c_2, d_2]$. Hence, by Theorem 3.1, every solution of (3.22) is oscillatory. In fact, $u(x,t) = e^{-x}\sin t$ is one of such solutions.

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