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**An Application of the Generator Coordinate Method
to a Potential Quark Model**

Zastosowanie uogólnionych współrzędnych generujących
do potencjału w modelu kwarków

1. INTRODUCTION

At the present time a number of phenomenological quark models have been developed [1]. These are either bag-like models or nonrelativistic constituent quark models with suitable chosen interactions, sometimes being some kind of approximation of QCD results. In 1984 R. K. Bhaduri *et al* [2, 3] it was proposed that many of the observed features of the excited states spectra for nonstrange baryons could be explained assuming that baryons spherical in the ground state acquire deformation in excited states. This idea has been successfully developed by making use of the Nilsson-type deformed potential in the paper [4]. However, the results obtained in [5] (and references therein) seem to indicate that also in the ground state, one should observe some deformation which cannot be explained within the above deformed mean field model in natural way. In the present paper we propose a little modified generator coordinate method allowing to investigate if baryons are deformed or not, without mean-field approximation. Our preliminary results seem to support the estimations made by R. Davidson and others in [5], i.e. they confirm the possibility of deformation of baryons (rather large) even in their ground state configurations.

2. A GENERAL SCHEME

Let G be a compact group of "motion" (a group of motion introduces the generator coordinates in the generator function [6]) of a nonrelativistic quark model and $T(g)$ its unitary, usually reducible, representation in the state space. The group G can be a non-invariance or symmetry group of the model. The Generator Coordinate Method (GCM) we treat here as a projection method [6] and we do not make use of the integral Hill-Wheeler equation. Our aim is to construct the excited states from a single fixed state $| - \rangle$, e.g. the ground state of the model, via action of the group of motions G . For this purpose we need to define an appropriate generator function. According to a general method a generator function can be written in the form

$$|g\rangle = T(g) | - \rangle. \quad (1)$$

The hermitian, non-negative, Hilbert-Schmidt type overlap operator can be expressed as

$$\mathcal{N} = \int_G dg T(g) | - \rangle \langle - | T(g)^\dagger, \quad (2)$$

where dg denotes a Haar measure normalized to unity, i.e. $\int dg = 1$.

The eigenstates of \mathcal{N} corresponding to non-zero eigenvalues span a subspace K_P of all excited states which can be obtained from $| - \rangle$. Simultaneously, these eigenstates transform according to appropriate irreducible representations of G . It is a consequence of invariance property of \mathcal{N} under group G , namely

$$T(g) \mathcal{N} T(g) = \mathcal{N}. \quad (3)$$

For light systems like mesons or baryons it is important to remove the center of mass motion spurious excitations. A straightforward procedure of constructing $| - \rangle$ and a hamiltonian H directly in the relative coordinate is, if possible, very tedious and time consuming way. Another method which can be applied here is a generalization of a trick proposed in [7].

Let us consider the state $|\Phi_0\rangle$ which can be factorized in the form

$$|\Phi_0\rangle = | - \rangle | \text{CM} \rangle, \quad (4)$$

where $| \text{CM} \rangle$ is the center of mass state and $| - \rangle$ depends only on relative coordinates. The operators $T(g)$ can be relatively easily factorized into the internal T^{int} and the center of mass T^{CM} parts:

$$T(g) = T^{\text{int}}(g)T^{\text{CM}}(g). \quad (5)$$

The simplest way to perform these calculations is to decompose the corresponding Lie algebra onto the internal and center of mass terms. For practical calculations one needs only the objects $|\Phi_0\rangle, |\text{CM}\rangle, T(g), T^{\text{CM}}(g)$.

Let $H = T_{\text{kin}} - T_{\text{CM}} + V_{\text{int}}$ be a translationally invariant hamiltonian expressed in the laboratory frame coordinates. To find the required overlap $\langle - | T^{\text{int}}(g)^+ H T^{\text{int}}(g') | - \rangle$ one can calculate the ratio of two much simpler matrix elements:

$$\langle - | T^{\text{int}}(g)^+ H T^{\text{int}}(g') | - \rangle = \frac{\langle \Phi_0 | T(g')^+ H T(g') | \Phi_0 \rangle}{\langle \text{CM} | T^{\text{CM}}(g^{-1}g') | \text{CM} \rangle} \quad (6)$$

According to the paper [6] the eigenstates $|IM\rangle$ and eigenvalues $\Lambda(I)$ of the overlap operator $\mathcal{N}(2)$ can be written as

$$|IM\rangle = \{\Lambda(I) \dim(I)\}^{-1/2} \sum_K c_{MK}^I P_{MK}^I | - \rangle, \quad (7)$$

where I labels the $\dim(I)$ — dimensional irreducible representation of the group G , M is a set of additional quantum numbers required for states classification, $\Lambda(I)$ and c_{MK}^I are solutions of the equations system:

$$\sum_K c_{MK}^I \left\{ \frac{\langle - | P_{MK}^I | - \rangle}{\dim(I)} - \Lambda(I) \delta_{KK'} \right\} = 0 \quad (8)$$

with the normalization condition $\sum_K |c_{MK}^I|^2 = 1$. The operators P_{MK}^I are standard projection operators:

$$P_{MK}^I = \dim(I) \int_G dg D_{MK}^I(g)^* T(g), \quad (9)$$

where $D^I(g)$ denotes the appropriate unitary matrix irreducible representations of the group G . Using the formula (6) with $H \equiv 1$ one can calculate the matrix elements $\langle - | P_{MK}^I | - \rangle$. To diagonalize the hamiltonian H within the space \mathfrak{R}_P spanned by the vector (7) one needs to calculate the matrix elements $\langle IM | H | I'M' \rangle$ using again the equation (6). In the special case when G is a hamiltonian symmetry group the states (7) are eigenstates of the hamiltonian H projected onto the subspace \mathfrak{R}_P . In the another case when the state $| - \rangle$ belongs to a given irreducible representation of the group G the above formalism can be “married” with the generalized coherent states theory [8] which results can simplify some calculations.

3. NUCLEON ROTATIONAL STATES — A MODEL WITH A SIMPLE GROUP OF MOTION

To calculate a baryon spectrum one needs, in principle, a high order like $SU_C(3) \times SU(6) \times SO_L(3)$ non-invariance group which well separates baryon states. However, to illustrate the method described in the paragraph 2 and find a qualitative behavior of the nucleon energy as a function of a deformation parameter λ we consider only a subgroup $SU_T(2) \times SU_J(2)$, where the indices T and J denote isospin and total angular momentum groups, respectively. Because the generator function should be a singlet in respect to the color group $SU_C(3)$ we do not need take it more as a subgroup of motion into account. The isospin and total angular momentum groups allow to distinguish among states with different (T, J) quantum numbers of a system consisting of u and d quarks. First of all we need to construct a deformed state (4)

$$|\Phi_0(\lambda)\rangle = |\lambda\rangle |CM, \lambda\rangle, \quad (10)$$

where a deformation is introduced through a scaling operator

$$K(\lambda) := \exp(-\lambda \widehat{K}) = K^{\text{int}}(\lambda) K^{\text{cm}}(\lambda) \quad (11)$$

with $\widehat{K} = \sum_{n=1}^3 (z_n \delta_{z_n} + \frac{1}{2})$ and $n = 1, 2$ and 3 denoting z coordinate of the first, second and third quark in a nucleon. Using (11) and undeformed ($\lambda = 0$) state (10) $|\Phi_0\rangle$ one can write the deformed state as

$$|\Phi_0(\lambda)\rangle = K(\lambda) |\Phi_0\rangle. \quad (12)$$

Because the group of motion $G = SU_T(2) \times SU_J(2)$ is a symmetry group of our quark system, to obtain a realistic space of states the vector $|\Phi_0\rangle$ should contain the $SU_F(3)$ octet and it should belong to 56-dimensional irreducible representation of the $SU(6)$ group. Such a state is of the following form

$$|\Phi_0\rangle = |\text{colour singlet}\rangle |\text{spin - isospin}\rangle |\text{space}\rangle, \quad (13)$$

where

$$|\text{spin - isospin}\rangle = (3)^{-1/2} (u_\uparrow u_\uparrow d_\downarrow + u_\uparrow d_\downarrow u_\uparrow + d_\downarrow u_\uparrow u_\uparrow)$$

and

$$|\text{space}\rangle = \phi_0(1)\phi_0(2)\phi_0(3).$$

This form is much simpler in applications than the known proton function, see e.g. [9]. In our calculations $\phi_0(k)$, $k = 1, 2, 3$ is assumed to be the 3-dimensional harmonic oscillator ground state function. In this case, the center of mass state is also a gaussian with width $b_{CM}(\hbar/3m\omega)^{1/2}$, where $m = m_u = m_d$ is a quark mass and ω denotes the oscillator frequency and can be written as

$$\eta_0(\vec{R}) = (\pi b_{CM}^2)^{3/4} \exp\left(-\frac{\vec{R}^2}{2b_{CM}^2}\right). \quad (14)$$

where \vec{R} denotes the center of mass position and $R = |\vec{R}|$.

Let $R(\Omega) \in SU_J(2)$ and $\mathcal{G}(\theta) \in SU_T(2)$, where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $\theta = (\theta_1, \theta_2, \theta_3)$ correspond to the appropriate Euler angles. Then the full generator function

$$|\Omega, \theta, \lambda\rangle = R(\Omega)\mathcal{G}(\theta) |\lambda\rangle. \quad (15)$$

After straightforward but rather lengthy calculations, using eq. (6), one can find the eigenstates of the overlap operator \mathcal{N} :

$$|JMTM_T; \lambda\rangle = \{\Lambda(J, T; \lambda)\}^{-1/2} \int_G d\Omega d\theta w_{JMTM_T}(\Omega, \theta) |\Omega, \theta; \lambda\rangle, \quad (16)$$

where $T = 1/2$ or $3/2$ and

$$w_{JMTM_T}(\Omega, \theta) = [(2J + 1)(2T + 1)]^{1/2} D_M^J{}_{1/2}(\Omega)^* D_{M_T}^T{}_{1/2}(\theta)^*,$$

$D_{mk}^j(\alpha, \beta, \gamma) = e^{-im\alpha} d_{mk}^j(\beta) e^{-ik\gamma}$ denote here usual $SU(2)$ Wigner functions. The corresponding eigenvalues can be expressed in the integral form

$$\Lambda(J, T; \lambda) = [3\delta_T(T)]^{-1} \int_0^\pi d\beta \sin \beta \frac{d_{m,m}^J(\beta) d_{m,m}^T(\beta)}{2b^4 g}, \quad (17)$$

$m = 1/2, \beta = \Omega_2$ needed here and in further formulas functions are defined as follows:

$$\begin{aligned} \delta_T(T) &= \delta_T{}_{1/2} + 4\delta_T{}_{3/2}, \\ g_{xx} &= 0.5b^2(1 + \cos^2 \beta + e^{-2\lambda} \sin^2 \beta), \\ g_{zz} &= 0.5b^2(e^{-\lambda} - e^\lambda) \cos \beta \sin \beta, \\ g_{zz} &= 0.5b^2(1 + e^{2\lambda} \sin^2 \beta + \cos^2 \beta) \\ g &= g_{xx}g_{zz} - (g_{xz})^2. \end{aligned} \quad (18)$$

The hamiltonian of a non-relativistic baryon model can be written as

$$H = T_{\text{kin}} - T_{\text{CM}} = V_{\text{int}}, \quad (19)$$

where V_{int} is usually two-body interaction. For our purpose we assume a two-body interaction dependent only on a distances between quarks, denoted r_{ij} :

$$V_{\text{int}} = \sum_{i < j} V(r_{ij}), \quad i, j = 1, 2, 3. \quad (20)$$

The hamiltonian H projected onto the space spanned through the vectors (16) is diagonal in the basis (16). This property allows to find the corresponding eigenvalues calculating directly the diagonal matrix elements of H with a help of eq. (6). The kinetic energy in the center of mass frame for spherical case is $E_{\text{kin0}} = 3\hbar^2/2mb^2$ and for arbitrary deformation λ we obtain

$$E_{\text{kin}}(J, T; \lambda) = E_{\text{kin0}} \left\{ 1 + \frac{2}{3} e^{-2\lambda} - [18\delta_T \Lambda(J, T; \lambda)]^{-1} \times \right. \\ \left. \times \int_0^\pi d\beta \sin \beta d_{mm}^J(\beta) d_{mm}^T(\beta) \frac{b^2(g_{zz} + e^{-2\lambda} g_{xx})}{(b^4 g)^2} \right\} \quad (21)$$

$m = 1/2$.

The derivation of interaction energy is slightly more involved, however in principle, it is a direct calculation which leads to the following result ($m = 1/2$)

$$E_{\text{int}}(J, T; \lambda) = [2\delta_T \Lambda(J, T; \lambda)]^{-1} \times \\ \times \int_0^\pi d\beta \sin \beta d_{mm}^J(\beta) d_{mm}^T(\beta) V(\beta; \lambda), \quad (22)$$

where

$$V(\beta; \lambda) = e^{-\lambda} [2(\pi^{1/2} b)^3 (2b^4 g)^{1/2}]^{-1} \int_0^\infty d\tau \int_0^\pi d\theta \int_0^{2\pi} d\phi \tau^2 \sin^2 \theta V(\tau) \times \\ \times \exp \left\{ -\frac{1}{2} \tau^2 [g_{xx} \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \varphi / b^2 + \right. \\ \left. + 2e^{-\lambda} g_{xz} \sin \theta \cos \theta + e^{-2\lambda} g_{zz} \cos^2 \theta] \right\}$$

The elementary analysis of vectors (16) shows that the isospin quantum number T must be equal to the baryon spin S which is also a good quantum

number. For rotational nucleon states $T = S = 1/2$ one obtains a mixing of different orbital angular momenta L for a given total angular momentum J , namely $L = J + 1/2$ or $L = J - 1/2$ with $J = 1/2, 3/2, 5/2, \dots$ In addition, because the angular momentum and the scaling operators conserve the parity quantum number, in our model all rotational nucleon states have positive parity. The negative parity states or both parities at the same time one can obtain making use of a little more complicated generator function.

In numerical calculations we have used a simple form of a quark interaction corresponding to one gluon exchange (OGE) plus confinement effect

$$V_{\text{int}} = \sum_{ij} \left(-\frac{\kappa}{r_{ij}} + a_B r_{ij}^n \right) \quad (23)$$

for $n = 1$ and 2 . The parameters of the total hamiltonian are given by the quark mass $m = 340$ MeV and $\kappa = 0,46$ (6) which corresponds to the current quark-gluon coupling constant $\alpha_S = 0,35$. The parameters a_B for $n = 1$ and $n = 2$ are found from fit to the experimental spectrum. In both cases the results are very similar. It is surprising that for $n = 1$ and $n = 2$ in (23) the total energy is related to the deformed nucleon ground state configuration with nearly the same ellipsoide axes ratio approximately equal 3:2. Because of this fact we illustrate our preliminary results only for linear confinement, i.e. $n = 1$. In the figures the deformation parameter is defined as $\beta = e^\lambda$ and represents a ratio of z -axis elongation of a nucleon ellipsoide in respect to the perpendicular one.

In the Fig. 1 the kinetic, OGE and confinement energies are drawn to show their behavior as functions of nucleon deformation. The minimum of the total energy is not too well pronounced what seems to indicate that the vibrational excitations are also possible. To include them into our calculations one needs to consider the deformation parameter λ as an additional collective variable. This problem requires further investigations.

In the Fig. 2 the nucleon rotational spectrum as a function of the confinement strength parameter a_B is shown. To each value of a_B the corresponding equilibrium deformation is indicated. By comparison of the calculated spectrum with experimental one the ground state nucleon configuration has been found for $\beta_{\text{eq}} = 1,55$. Please note, that for $\beta_{\text{eq}} = 1$ corresponding to the spherical case, only $J = 1/2^+$ state exists and the pure rotational excitations are not possible. In the Fig. 2 one can also notice an effect of decreasing of the first excited rotational state energy with increasing of deformation.

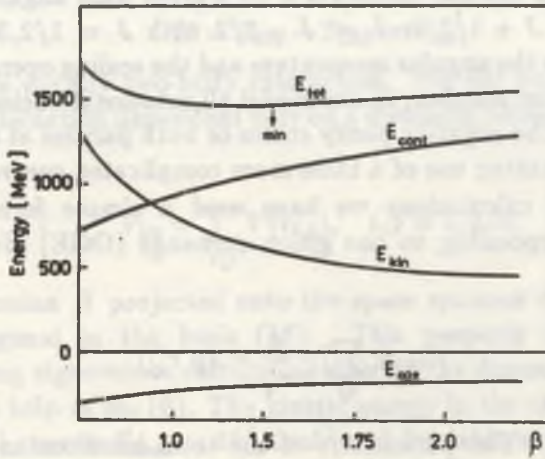


Fig. 1. The kinetic, one-gluon-exchange and confinement energies are plotted as a function of deformation $\beta = e^\lambda$ which represents a ratio of z-axis elongation of a nucleon ellipsoide in respect to the perpendicular one

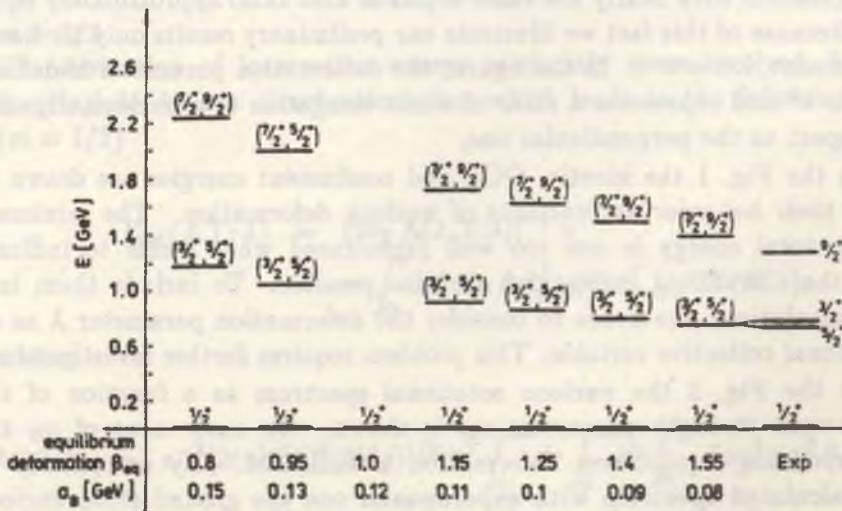


Fig. 2. The nucleon rotational spectrum as a function of the confinement parameter α_0 and the corresponding deformation β . Other parameters as in the text

Because the structure of eigenstates in this model is independent of a particular choice of a Hamilton operator one can draw a conclusion that the deformation is a general feature of all baryon states. It is supported by the observation that in our calculations one can also consider the Δ rotational states which are also deformed though, because of too poor hamiltonian in this schematic calculations the $N - \Delta$ splitting is much too small (now we are not interested in improving of this situation with our simple group of motion and the schematic interaction). For example, for Δ deformed state $T = S = 3/2$ and $J = 3/2$ the angular momentum quantum number L can be equal 0 or 2. This structure can explain a possible E2/M1 transitions in the reaction $\Delta \rightarrow N + \gamma$ suggested in the paper [5]. This reaction requires a detail considerations and will be the topic of a subsequent paper.

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STRESZCZENIE

Przebadano i zastosowano metodę współrzędnej generującej w przypadku modelu kwarków o dziwności zerowej.

Wstępne wyniki wskazują na istnienie barionów zdeformowanych już w ich stanie podstawowym. Zgadza się to z niezerowym stosunkiem E2/M1 w reakcji $\Delta \rightarrow N + \gamma$.

