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On Construction and Solution of the Higher-order Kortewega—de Vries Equation

O konstrukcji i rozwiązaniu równania wyższego rzędu Kortewega—de Vriesa

О конструкции и решении уравнения высшей степени Кортевега—де Фриса

Dedicated to Professor
Stanisław Szpikowski on occasion
of his 60th birthday

Among the family of nonlinear partial differential equations it is possible to distinguish the class of equations solved via the inverse scattering method. Basing on the Lax's criterion this method is not strictly analytic. So the problem of construction of equations, which may be solved via this method, acquires special significance. One has to find a skew symmetric operator B which is relevant to the appropriate nonlinear partial differential equation and fulfils the criterion of solution of a given equation, the so-called Lax's criterion [1]

$$u_t = \left[L, B \right] \tag{1}$$

where subscript t implies partial differentiation

$$u_t \quad \frac{\partial u}{\partial t}, \tag{2}$$

$[L, B]$ is the commutator

$$[L,B] = LB - BL, \tag{3}$$

and

$$L = -\frac{\partial^2}{\partial x^2} + u(x,t) = -D^2 + u \tag{4}$$

is the well-known Sturm-Liouville operator with u playing the role of the potential depending parametrically on time t. When the commutator $[L, B]$ is the operator of multiplication by a number, equation (1) is equivalent to a certain partial differential equation.

To prove this we take the operator B in the following form

$$B = D^5 + b_1(x)D + b_2(x)D^3 + Db_1(x) + D^3b_2(x) \tag{5}$$

It is easy to note that B is a skew symmetric operator. We try to choose the coefficients b_1 and b_2 in such a way that one commutator $[L, B]$ is equibalent to the operator of multiplication by a number. With this end in mind we equate to zero the coefficients by the operators D, D^2, D^3, D^4. The coefficients closed to D and D are equal to zero in the trivial way. Thus we obtain

$$-4b_{1xx} - 5b_{2xxxx} - 5u_{xxxx} - 6b_2u_{xx} - 6b_{2x}u_x = 0, \tag{6}$$

$$4b_{1x} + 9b_{2xxx} + 10u_{xxx} + 6b_2u_x = 0, \tag{7}$$

$$4b_{2xx} + 5u_{xx} = 0, \tag{8}$$

$$4b_{2x} + 5u_x = 0. \tag{9}$$

In this way equation (1) is reduced to the form

$$-b_{1xxx} - b_{2xxxxx} - 3b_{2x}u_{xx} - u_{xxxxx} - 2b_2u_{xxx} = u_t. \tag{10}$$

Note that equations (8) and (9) are dependent. So from equation (9) we get

$$b_2 = -\frac{5}{4}u + c, \tag{11}$$

where c is any given integration constrant. In view of this equation the set of equations (6), (7) are reduced to the following form

$$16b_{1x} - 5u_{xxx} - 30uu_x + 24\,c\,u_x = 0, \tag{12}$$

$$16b_{1xx} - 5u_{xxxx} - 30u_x^2 - 30uu_{xx} + 24\,cu_{xx} = 0. \tag{13}$$

Since these equations are dependent, from (12) we obtain

$$b_1 = \frac{1}{16}(5u_{xx} + 15u^2 - 24\,c\,u). \tag{14}$$

Substituting equations (11) and (14) into equation (10), after making some manipulations, we see that (10) acquires the form

$$-15u_{xxxxx} - 10uu_{xxx} + 40u_{xx}u_x + 8\,c\,u_{xxx} + 30u^2u_x -$$
$$- 48\,c\,u\,u_x = 16u_t. \tag{15}$$

We have thus obtained in this way the family of nonlinear differential equations parametrized bythe constant c. These equations are called the higher-order Korteweg-de Vries equations.

The most significant use of the nonlinear transformation is the development of the inverse scattering method for exact solution of the above mentioned equation (15). The literature treating the inverse scattering problem is extensive, and the reader is referred to the papers of Nowikow [2], Gel'fand and Levitan [3], Kay and Moses [4], Wadati, Konno, and Ichikawa [5], Fokas and Ablowitz [6].

With this end in mind, let's determine the eigenfunctions of the Sturm-Liouville operator L. As the spectrum remains invariant as u evolves with t, in a complex-valued representation the wave function Ψ has the asymptotic behavior

$$\Psi(x,t) = a(k,t)\exp(-ikx) + b(k,t)\exp(ikx), \quad x \rightarrow -\infty, \tag{16a}$$

$$\Psi(x,t) = \exp(-ikx), \quad x \rightarrow \infty . \tag{16b}$$

The amount reflected b(k) is the reflection coefficient and the amount transmitted a(k) is the transmission coefficient. exp(-ikx) and exp(ikx) represent the left-going and right-going waves, respectively. Nevertheless, for the discrete spectrum the wave function can be written as follows

$$\psi(\alpha_n,x) = b_n(\alpha_n,t)\exp(-\ _nx), \quad x \rightarrow -\infty, \tag{17a}$$

$$\psi(\alpha_n,x)\exp(-\alpha_n x), \quad x \rightarrow \infty . \tag{17b}$$

It can be shown that the function

$$g = \Psi_t + B\Psi \tag{18}$$

is a eigenfunction of the operator L 2 . Let's take in (18) the $x \rightarrow -\infty$ limit. Using (5) and (17), as well as the fact of vanishing of the potential in the infinity,

$$u \xrightarrow[|x| \rightarrow \infty]{} 0,$$

from (18) we obtain

$$g = (2ik^3c - ik - ik^5)\cdot\exp(-ikx). \tag{19}$$

Hence and from equation (18) we can find the time evolution of the function

$$\Psi_t = -B\Psi + (2ick^3 - ik - ik^5)\Psi. \tag{20}$$

From this equation in the limit $x \to -\infty$ we find the evolution of the scattering data

$$a_t = ika, \tag{21}$$

$$b_t = -(2ik^5 - 4ik^3c + ik)b, \tag{22}$$

$$(b_n)_t = 2\alpha_n(1 + {}^4_n) \cdot b_n(\alpha_n, t). \tag{23}$$

Hence we have

$$a(k,t) = a(k,o)\exp(-ikt), \tag{24}$$

$$b(k,t) = b(k,o)\exp\,(4ik^3c - 2ik^5 - ik)t\,, \tag{25}$$

$$b_n(\alpha_n,t) = b_n(\alpha_n,o)\exp\left[2\alpha_n(1+\alpha_n^4)\,t\right], \tag{26}$$

where $a(k,o)$, $b(k,o)$, $b_n(\alpha_n,o)$ are determined from initial data for equation (15).

Let's use now the Gel'fand-Levitan-Marchenko linear integral equation for the case of zero reflection coefficient, $b(k,t) = 0$, and with a kernel determined by the following formula [2]

$$F(x) = \frac{b_n(\alpha_n,t)\exp(-\alpha_n x)}{\dfrac{i\,\partial a(i\alpha_n)}{(i\alpha_n)}} \tag{27}$$

Because the reflection and transmission coefficients are related by conservation of energy:

$$|a|^2 - |b|^2 = 1, \tag{28}$$

we can only define $a(k,t)$

$$a(k,t) = \prod_{n=1}^{N} \frac{k - i_n}{k + i_n} \exp(-ikt). \tag{29}$$

In order to simplify the problem let's suppose that $a(k,0)$ is equal to zero only for $k = i\alpha_n$. Solution of equation (15) is then determined by the formula [2]

$$u(x,t) = -2\left[\ln(\det A)\right]_{xx}, \tag{30}$$

where in this case

$$A = 1 + \frac{b}{i\,a_1(i\alpha_n)}\exp(-2\alpha_n x), \tag{31}$$

$$a_1(i\alpha_n) = \frac{\partial a}{(i\alpha_n)}. \tag{32}$$

Because of (26) and (29) we can write A in the following way

$$A(x,t) = 1 + b_n(k,0)\exp\left\{\alpha_n\left[(2\ {}_n^4 + 1)\,t - 2x\right]\right\}. \tag{33}$$

Then solution of the higher-order Korteweg-de Vries equation is given by

$$u(x,t) = \frac{-2\alpha_n^2}{\cosh\left\{\alpha_n(2\ {}_n^4 + 1)t - 2x - \ln b_n(o)\right\} + 1}. \tag{34}$$

This solution represents soliton moving to the right with the velocity V

$$V = \frac{1 + 2\alpha_n^4}{2} \tag{35}$$

and an amplitude G

$$G = \frac{-4\alpha_n^2\, b_n(o)}{\left[1 + b_n(o)\right]^2}. \tag{36}$$

Basing on the Lax's criterion we have constructed the higher-order Korteweg-de Vries equation which then has been solved via the inverse scattering method. The success of this method for solution of equation (15) can be attributed to two facts. Firstly, the Gel'fand-Levitan-Marchenko equation is linear and the eigenvalues are constants. Secondly, t enters the problem only parametrically.

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STRESZCZENIE

W oparciu o kryterium Laxa skonstruowano piątego rzędu nieliniowe równanie Kortewega-de Vriesa. Odwrotna metoda rozpraszania została zastosowana do znalezienia jednosolitonowego rozwiązania tego równania.

Р Е З Ю М Е

Опираясь на критерий Лакса, сконструировали нелинейное уравнение пятой степени Кортевега-Де Фриса. Обратный метод дисперсии был применен для разыскания односолитонового решения этого уравнения.