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K. T. HECHT*

New Applications of SO(6)$\supset$(3) Algebras and Their Vector-coherent-state Constructions

Nowe zastosowania algebr SO(6)$\supset$U(3) i konstrukcja ich wektorowych stanów koherentnych

Новое применение алгебр SO(6)$\supset$U(3) и конструкция их векторно-когерентных состояний

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1. Introduction

The recent generalization of standard coherent state theory to a theory of vector cohererent states [1]-[7] has furnished us with a powerful tool for the explicit construction of the irreducible representations of a number of important groups with applications to various branches of physics, [8], [9]. The vector coherent state method is particularly well suited for an analysis of the fermion pair algebra which has important applications in the nuclear shell model and in

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many-fermion systems in general and has long been identified as an SO(2n)⊃U(n) algebra, [10]-[15]. This algebra has recently been discussed by Rowe and Carvalho [16] in terms of the vector coherent state technique. A full implementation of the vector coherent state method, however is limited in practice to those cases where the Wigner-Racah calculus of the core subalgebra, U(n) in the case of the fermion pair algebra SO(2n), is worked out in sufficient detail. One such example is the SO(8)⊃U(4) LST-pairing symmetry for which the pioneering work was carried out by Flowers and Szpikowski [17], [18]. Recently it has been shown that the vector coherent state method can be used to generalize the earlier results to higher seniorities and can thus lead to a more general explicit construction of n-nucleon states in the LST seniority scheme [19]. An almost parallel analysis [20] can be carried out for the Ginocchio SO(8)⊃U(4) symmetry model [21], a fermion pair model with S and D pairs only, which was originally introduced as a "toy" model to study the fermionic foundation of the interacting boson model of Iachello and Arima, but which is gaining new attention in connection with a fermion dynamical symmetry model [22]. Other recent applications of the vector coherent state method involve the neutron-proton quasispin group [23] and the USp(6) group, the latter in connection with attempts [20] to find a more sound fermionic foundation of the rotational or SU(3) limit of the interacting boson model. Both are again symmetries in which pioneering work was carried out by Szpikowski [24]. The importance of the early work of Pomorski and Szpikowski [25] on the USp((n+1)(n+2)) and USp(6) symmetries is being highlighted by recent work on exact boson mappings for nuclear neutron or proton shell model algebras having an SU(3) subalgebra [26]. Many of these symmetries have also been discussed in terms of coherent state theory by Dobaczewski [27] in his functional representation analysis of boson expansion theories.

Very recently it has also been shown that vector coherent state theory can be used to reduce the Wigner calculus for U(n) in the canonical Gel'fand U(n)⊃U(n-1) chain to an exercise in U(n-1) recoupling [28], often with multiplicity-free recoupling coefficients evaluated through permutation group techniques. Indirect applications of vector coherent state theory, through the use of complementary Sp(2d,R) symmetries, have also been carried out in detail in the construction of group theoretically sound orthonormal bases for the nuclear rotational SU(3) ⊃ SO(3) scheme [29] and for the standard Wigner supermultiplet basis [30]. In these applications vector coherent state theory is used to resolve a missing quantum number or inner multiplicity problem.

Introduced originally for the evaluation of matrix elements of the Lie algebras of the discrete series representations of the noncompact Sp(2d,R) groups [1], [31], [32], the vector coherent state method has thus been used to great advantage in a number of other problems. Despite its many uses no specific applications have yet been given for one of the simplest fermion pair algebras, the SO(6) ⊃ U(3) algebra. This is a particularly nice example, since (1) the Wigner-Racah calculus for the U(3) subalgebra is fully worked out [33], (2) only multiplicity-free SU(3) couplings are needed, and (3) the K^2-matrices which are a key feature of the vector coherent state method are all 1-dimensional. It is the purpose of the present note to give two new applications of the SO(6) ⊃ U(3) algebra. The first involves a relativistic quark model of the nucleus, [34]-[37], and will be presented in section 2 together with the details of the vector coherent state construction for this symmetry. The second involves a Ginocchio-type toy model with neutron-proton pairs coupled to $J^{\pi}=1^{+}$ and T=0. It corstitutes part of a search for a fermionic foundation for the 1^{+} neutron-proton scissors mode which has recently been introduced in interacting boson model studies of collective magnetic dipole excitations in deformed nuclei, [38]-[40]. It will be presented in section 3.

2. The J=0, T=0 Pair Algebra in a Relativistic Quark Model

The relativistic quark model of Bleuler et al. [34]-[37] describes the A-particle nuclear system as a system of 3A quarks in a relativistic bag model. It begins with the observation that the quark level sequence in a relativistic bag exhibits the characteristic features of the Mayer-Jensen shell model, and it contains the basic idea that each color singlet three-quark substructure in the low-lying nuclear states of a nucleus contains one quark pair coupled to J=0 T=0 in its required color anti-triplet state. The J, T-structure of an open-shell nucleus is thus determined by the A quarks not in J=0, T=0-coupled pairs and in particular by N such quarks in the unfilled j-subshell (rather than by 3N quarks). A J=0, T=0 pairing interaction is introduced to separate the "nonnucleonic" excitations, of Δ-type e.g., from the nuclear states so that the problem of too many states for open shell nuclei is avoided. Such a pairing interaction has also been related [37] to the quark-quark interaction derived by t'Hooft [41] from the instanton solution of QCD. Clearly, however, this quark model in its simplest form has many deficiencies. In its most naive form it would predict 2, 3, and 4-nucleon systems dominated by $0s_{1/2}$ configurations, whereas the clustering into three-quark nucleon substructures requires strong excitations into the p and higher shells. For the 3-nucleon system, e.g., a three cluster configuration of three-quark systems contains at most 0.4% of the $(0s_{1/2})^9$ configuration [42]. Recently it has also been shown [43] that the isovector part of the nuclear magnetic moments increases too rapidly compared with the experimental values for high j. An improved quark model of the nucleus of the above type would probably require strong configuration mixing even in heavy nuclei in order to begin to develop the strong spatial correlations into three-quark clusters which seem to be required for real nuclei.

In refs.[34]-[37] the quark-quark pairing interaction is treated in terms of a $U(d) \supset SO(d)$ seniority chain, where $d=2(2j+1)$ for quarks in the last (open) j-shell. Although this can in principle be generalized to $d=2\Sigma(2j+1)$ for mixed configuration calculations, the more general terms of the quark-quark interaction [41] which are clearly needed might be difficult to work out in such a basis. For a more realistic treatment of this model it may therefore be advantageous to use the complementary symmetry, given by an $SO(6) \supset U(3)$

chain; where this symmetry applies universally to all j-shells as well as to mixed configurations.

The SO(6) ⊃ U(3) Lie algebra is now generated by the J=0, T=0, color antitriplet pair-creation and annihilation operators, combined with the color-U(3) subalgebra. To retain the standard notation and normalizations for SO(2n), [16], it will be useful to define the J=0, T=0 pair-creation operators, A_{ik}, in terms of quark creation operators, $a^{\dagger}_{jmm_t,i}$, by

$$A_{ik} = \frac{1}{2}\sum_{jm}\sum_{m_t}(-1)^{j-m+\frac{1}{2}-m_t}\left(a^{\dagger}_{jmm_t,i}\,a^{\dagger}_{j-m-m_t,k} - a^{\dagger}_{jmm_t,k}\,a^{\dagger}_{j-m-m_t,i}\right)$$

$$B_{ik} = (A_{ik})^{\dagger} \qquad (1a)$$

where $A_{ki}=-A_{ik}$, and i,k=1,2,3 are the color indices. A sum over subshells j is included but can be dropped for a pure j^{3N} configuration. Together with the U(3) subalgebra

$$C_{ik} = \sum_{jm}\sum_{m_t} a^{\dagger}_{jmm_t,i}\,a_{jmm_t,k} - \frac{1}{2}\delta_{ik}\,2\sum_{j}(2j+1)\ , \qquad (1b)$$

these operators satisfy the commutation relations

$$[B_{ik},A_{ab}] = \delta_{ka}C_{bi} + \delta_{ib}C_{ak} - \delta_{ia}C_{bk} - \delta_{kb}C_{ai}\ , \qquad (2a)$$

$$[B_{ik},C_{ab}] = \delta_{ka}B_{ib} - \delta_{ia}B_{kb}\ , \qquad (2b)$$

and generate the group SO(6). Although the coherent state realization of the general SO(2n) fermion pair algebra has been given in ref.[16], the present algebra requires an instrinsic state vector $|[\sigma]\alpha\rangle$. It will also be useful to introduce the 3-dimensional vector $\underline{z}$, where z_1, z_2, z_3 are complex variables. In terms of these the vector coherent state is defined by

$$|\underline{z}\rangle = e^{z_1^* A_{23} + z_2^* A_{31} + z_3^* A_{12}}\,|[\sigma]\alpha\rangle\ , \qquad (3)$$

where $[\sigma]\equiv[\sigma_1\sigma_2\sigma_3]$ is the intrinsic U(3)-color symmetry of the state which is entirely free of J=0, T=0-coupled quark pairs, so that

$$B_{ik}\,|[\sigma]\alpha\rangle = 0 \qquad (4)$$

for all subgroup labels α, $\alpha=1, \ldots,$ dimension $[\sigma]$, of the vector $|[\sigma]\alpha\rangle$. In the quark model of the nucleus $[\sigma]=[N00]$ for the "nucleonic" states with 3N quarks in open shells, and $[\sigma]=[N11]$ or $[N\ell\ell]$ for the "nonnucleonic" states involving 1 or ℓ nonnucleonic excitations of Δ or more complicated type.

State vectors $|\psi\rangle$ are then mapped into z-space functional representations, [1]-[3], [16],

$$|\Psi\rangle \rightarrow \Psi_{[\sigma]\alpha}(\underline{z}) = \langle \underline{z}|\Psi\rangle = \langle[\sigma]\alpha|e^{\underline{z}\cdot\underline{B}}|\Psi\rangle, \qquad (5a)$$

with

$$\underline{z}\cdot\underline{B} \equiv \tfrac{1}{2}\epsilon_{ijk} z_i B_{jk}, \qquad (5b)$$

and operators $\mathcal{O}$ are mapped into their z-space realizations, $\Gamma(\mathcal{O})$,

$$\mathcal{O}|\Psi\rangle \rightarrow \Gamma(\mathcal{O})\Psi_{[\sigma]\alpha}(\underline{z}) = \langle[\sigma]\alpha|e^{\underline{z}\cdot\underline{B}}\mathcal{O}|\Psi\rangle$$
$$= \langle[\sigma]\alpha|(e^{\underline{z}\cdot\underline{B}}\mathcal{O}e^{-\underline{z}\cdot\underline{B}})e^{\underline{z}\cdot\underline{B}}|\Psi\rangle = \langle[\sigma]\alpha|\{\mathcal{O}+[\underline{z}\cdot\underline{B},\mathcal{O}]+\ldots\}e^{\underline{z}\cdot\underline{B}}|\Psi\rangle. \qquad (6)$$

This leads to

$$\Gamma(B_{ij}) = \epsilon_{ijk}\partial_k \qquad (7a)$$

$$\Gamma(C_{ij}) = \mathbb{C}_{ij} - z_j\partial_i + \delta_{ij}(z_\rho\partial_\rho) \qquad (7b)$$

$$\Gamma(A_{ij}) = \epsilon_{ijk}\{z_\rho\mathbb{C}_{\rho k} - z_k(\mathrm{tr}\,\mathbb{C}) - z_k(z_\rho\partial_\rho)\}, \qquad (7c)$$

(with $\frac{\partial}{\partial z_k} \equiv \partial_k$), and summation convention for repeated indices.

The group generators have therefore been mapped into a direct sum of a 3-dimensional harmonic oscillator (or Heisenberg-Weyl) algebra, generated by z_k, ∂_k, and an intrinsic U(3) algebra, $\mathbb{C}_{ij}$, which acts only on the components of the intrinsic vector $|[\sigma]\alpha\rangle$. The intrinsic $\mathbb{C}_{ij}$ commute with the z_k and ∂_k. Since the z-space realization, (7), of the operator algebra is a nonunitary or Dyson realization, it will be useful to make a transformation to a unitary or Holstein-Primakoff realization, $\gamma(\mathcal{O})$, via a hermitian U(3)-invariant operator K, ($K^\dagger=K$),

$$\gamma(A_{ij}) = K^{-1}\Gamma(A_{ij})K, \quad \gamma(B_{ij}) = K^{-1}\Gamma(B_{ij})K, \quad \gamma(C_{ij}) = \Gamma(C_{ij}) \tag{8}$$

where the requirement $\gamma(A_{ij}) = (\gamma(B_{ij}))^{\dagger}$ leads, (with $(\partial_k)^{\dagger} = z_k$), to

$$K^{-1}\Gamma(A_{ij})K = K\epsilon_{ijk} z_k K^{-1}, \tag{9a}$$

or
$$\Gamma(A_{ij})K^2 = K^2 \epsilon_{ijk} z_k . \tag{9b}$$

The key to the solution of this equation for K^2 in the Toronto vector coherent state method, [1]-[3], comes through the introduction of an operator, Ω_{op}, with the property

$$[\Omega_{op}, z_i] = \tfrac{1}{2}\epsilon_{ik\ell}\Gamma(A_{k\ell}) = z_\rho \mathbb{C}_{\rho i} - z_i(\mathrm{tr}\,\mathbb{C}) - z_i(z_\rho \partial_\rho). \tag{10}$$

This is satisfied by

$$\Omega_{op} = z_\alpha \partial_\beta \mathbb{C}_{\alpha\beta} - (z_\rho \partial_\rho)(\mathrm{tr}\,\mathbb{C}) - \tfrac{1}{2}(z_\alpha \partial_\alpha)(z_\rho \partial_\rho) + \tfrac{1}{2}(z_\rho \partial_\rho), \tag{11}$$

and eq.(9b) is transformed into

$$(\Omega_{op} z_i - z_i \Omega_{op})K^2 = K^2 z_i . \tag{12}$$

In the z-space realization the orthonormal eigenstates of an SO(6) irreducible representation $[\sigma]$ are given by

$$\left[Z^{[pp0]}(\underline{z}) \times |[\sigma']\rangle\right]^{[h]}_{\alpha_h} = \psi(\underline{z})^{[[\sigma]\times[pp0]][h]}_{\alpha_h}, \tag{13}$$

where the square bracket denotes U(3) coupling, $[\sigma]\times[pp0] \to [h] \equiv [h_1h_2h_3]$, and α_h is a convenient set of subgroup labels for $[h]$. The symmetric polynomial of degree p in the z's, $Z^{[pp0]}(\underline{z})$, must transform according to the U(3) representation $[pp0]$ since the vector $\underline{z}$ transforms according to the antitriplet representation $[110]$. It will also be convenient to use Elliott SU(3) quantum numbers; with $(\lambda_\sigma\mu_\sigma) = (\sigma_1-\sigma_2, \sigma_2-\sigma_3)$, $(\lambda_h\mu_h) = (h_1-h_2, h_2-h_3)$, and with $[pp0] \to (0p)$, $[110] \to (01)$. It is important to note that the product $(\lambda_\sigma\mu_\sigma)\times(0p) \to$

$(\lambda_h\mu_h)$ is multiplicity-free and that p is uniquely determined by the quantum numbers $[\sigma_1\sigma_2\sigma_3]$ and $[h_1h_2h_3]$, so that p serves as a good quantum number, and the K operation is merely multiplication by a normalization factor.

With the U(3) generators built from "intrinsic" components, $\mathbb{C}_{ij}$, and z_j, ∂_i-dependent or "collective" components, $C_{ij}^{coll.}$, with

$$C_{ij}^{coll.} \equiv -z_j\partial_i + \delta_{ij}(z_\rho\partial_\rho), \qquad C_{ij}^{full} = \mathbb{C}_{ij} + C_{ij}^{coll.}, \tag{14}$$

cf. eq.(7b), the Ω operator can be put into the form

$$\Omega_{op} = -C_{\beta\alpha}^{coll.}\mathbb{C}_{\alpha\beta} - \tfrac{1}{2}(z_\alpha\partial_\alpha)(z_\beta\partial_\beta) + \tfrac{1}{2}(z_\beta\partial_\beta)$$

$$= -\tfrac{1}{2}C_{\beta\alpha}^{full}C_{\alpha\beta}^{full} + \tfrac{1}{2}\mathbb{C}_{\beta\alpha}\mathbb{C}_{\alpha\beta} + \tfrac{1}{2}C_{\beta\alpha}^{coll.}C_{\alpha\beta}^{coll.} - \tfrac{1}{2}(z_\alpha\partial_\alpha)(z_\beta\partial_\beta) + \tfrac{1}{2}(z_\beta\partial_\beta). \tag{15}$$

Since $(z_\alpha\partial_\alpha)$ has the simple eigenvalue p, the eigenvalue of Ω_{op} in the basis (13) is given by the Casimir invariants of the intrinsic and of the full or final U(3) symmetry.

$$\Omega = -\tfrac{1}{2}C_{\beta\alpha}^{full}C_{\alpha\beta}^{full} + \tfrac{1}{2}\mathbb{C}_{\beta\alpha}\mathbb{C}_{\alpha\beta} + \tfrac{1}{2}p(p+3), \tag{16}$$

with

$$C_{\beta\alpha}^{full}C_{\alpha\beta}^{full} = (h_1-\tfrac{1}{2}\omega)^2 + (h_2-\tfrac{1}{2}\omega)^2 + (h_3-\tfrac{1}{2}\omega)^2 + 2(h_1-h_3), \tag{17}$$

where

$$\omega = 2\sum_j (2j+1), \tag{18}$$

and the h_i are the number of squares in the i^{th} row of the Young tableau describing the final (full) U(3) symmetry, cf. eq.(16). For the most general intrinsic state $[\sigma_1\sigma_2\sigma_3]$ the final tableau $[h_1h_2h_3]$ can in general be obtained by adding a squares to rows 2 and 3, b squares to rows 1 and 3, and c squares to rows 1 and 2 of the intrinsic tableau $[\sigma_1\sigma_2\sigma_3]$; so that

$$h_1 = \sigma_1 + b + c, \quad h_2 = \sigma_2 + a + c, \quad h_3 = \sigma_3 + a + b; \quad a+b+c = p. \tag{19}$$

With this prameterization it is easy to take matrix elements of eq.(12) between states of type (13) and a,b,c on the right, a',b',c' on the left, leading to

$$\Omega_{a+1\,bc} - \Omega_{abc} = \omega - \sigma_2 - \sigma_3 + 2 - a = \frac{K^2_{a+1\,bc}}{K^2_{abc}} \qquad (20a)$$

$$\Omega_{a\,b+1\,c} - \Omega_{abc} = \omega - \sigma_1 - \sigma_3 + 1 - b = \frac{K^2_{a\,b+1\,c}}{K^2_{abc}} \qquad (20b)$$

$$\Omega_{abc+1} - \Omega_{abc} = \omega - \sigma_1 - \sigma_2 - c = \frac{K^2_{abc+1}}{K^2_{abc}} \qquad (20c)$$

With $K^2_{000}=1$, (assuming a normalized intrinsic state), this leads to

$$K^2_{abc} = \frac{(\omega-\sigma_2-\sigma_3+2)!}{(\omega-\sigma_2-\sigma_3+2-a)!}\,\frac{(\omega-\sigma_1-\sigma_3+1)!}{(\omega-\sigma_1-\sigma_3+1-b)!}\,\frac{(\omega-\sigma_1-\sigma_2)!}{(\omega-\sigma_1-\sigma_2-c)!} \qquad (21)$$

Since the unitary form of the z-space operators is given by

$$\gamma(A_{ij}) = K\,\epsilon_{ijk}\,z_k\,K^{-1}, \qquad (22)$$

see eqs.(8) and (9a), the SU(3)-reduced matrix elements of $\gamma(\underline{A})$ between states of type (13) are given by

$$\left(\psi(\underline{z})^{[[\sigma]\times[p+1\,p+1\,0]][\ell']} \,\|\, \gamma(\underline{A}) \,\|\, \psi(\underline{z})^{[[\sigma]\times[pp0]][\ell]}\right)$$

$$= \frac{K_{a'b'c'}}{K_{abc}} \left(\psi(\underline{z})^{[[\sigma]\times[p+1\,p+1\,0]][\ell']} \,\|\, \underline{z} \,\|\, \psi(\underline{z})^{[[\sigma]\times[pp0]][\ell]}\right)$$

$$= \frac{K_{a'b'c'}}{K_{abc}} \begin{bmatrix} [\sigma] & [pp0] & [\ell] \\ [0] & [110] & [110] \\ [\sigma] & [p+1\,p+1\,0] & [\ell'] \end{bmatrix} \left([p+1\,p+1\,0] \,\|\, \underline{z} \,\|\, [pp0]\right), \qquad (23)$$

where the unitary form of the U(3) 9-[:::] symbol arises in the usual way from the action of the purely collective operator $\underline{z}$ in the basis formed from the coupling of an intrinsic symmetry [σ] with a collective symmetry [pp0]. The SU(3)-reduced matrix element of $\underline{z}$ in its own collective space has the simple oscillator value $[p+1]^{1/2}$. The 9-[:::] symbol with one [0] entry can be expressed in terms of a standard SU(3) Racah coefficient. Finally, since the z-space operator $\gamma(A_{ij})$ is the unitary form of this operator and the $\psi(\underline{z})$ of eq.(13) form an orthonormal set with respect to the z-space scalar product, the result (23) is representation-independent. Transforming to a standard orthonormal basis

$$|[[\sigma]\times[pp0]][\ell],\alpha_\ell\rangle \tag{24}$$

the SU(3)-reduced matrix elements of the J=0, T=0-pair creation operators of eq.(1) are thus given by

$$\langle[[\sigma]\times[p{+}1\; p{+}1\, 0]][\ell'] \,\|\, \underline{A} \,\|\, [[\sigma]\times[pp0]][\ell]\rangle$$

$$= \frac{K_{a'b'c'}}{K_{abc}} U((\lambda_\sigma\mu_\sigma)(0p)(\lambda_{\ell'}\mu_{\ell'})(01);(\lambda_\ell\mu_\ell)(0\,p{+}1))\,[p{+}1]^{1/2}, \tag{25}$$

where the K-ratio is given by eqs.(20), and the multiplicity-free Racah coefficient is known numerically from ref.[33] and analytically from ref.[44] or Appendix IIB of ref.[28].

Finally, the orthonormal basis, eq.(24), can be constructed by the action of a symmetric polynomial of degree p in the standard pair-creation operators $\underline{A}$ of eq.(1a), by using the inverse of eq.(22), (with $K_{000}=1$), to convert

$$[\underline{z}\times\underline{z}\times\ldots\times\underline{z}]^{[pp0]} \longrightarrow$$

$$[K^{-1}\underline{A}K\times K^{-1}\underline{A}K\times\ldots\times K^{-1}\underline{A}K]^{[pp0]} = K^{-1}_{abc}\, Z(\underline{A})^{[pp0]}$$

so that

$$|[[\sigma]\times[pp0]][\ell],\alpha_\ell\rangle = K^{-1}_{abc}\left[Z(\underline{A})^{[pp0]}\times|[\sigma]\rangle\right]^{[\ell]}_{\alpha_\ell} \tag{26}$$

where the replacements $z_1\to A_{23}$, ..., are to be made to convert the symmetric polynomial $Z(\underline{z})$ into the corresponding $Z(\underline{A})$.

The pairing interaction of refs.[34]-[37] can be written as

$$H_P = -g\sum_{\alpha,\beta=1}^{3} A_{\alpha\beta} B_{\alpha\beta} , \tag{27a}$$

and has eigenvalue

$$E_P^{[[\sigma]\times[p+1\,p+1\,0]][\ell']} = -g\left\langle [[\sigma]\times[p+1\,p+1\,0]][\ell'] \| \underline{A} \| [[\sigma]\times[p\,p\,0]][\ell] \right\rangle^2 \tag{27b}$$

The 3N-quark states must be color singlets with $[h']$=[NNN], so that $(\lambda_{h'}, \mu_{h'})=(00)$. The intrinsic symmetry is $[\sigma]$=[N00] for the "nucleonic" states and $[\sigma]=[N\ell\ell]$ for "non-nucleonic" states with ℓ nonnucleonic excitations of Δ or more complicated type. Therefore $p+1=N-\ell$, with $abc=(N-\ell-1)00$. The Racah coefficient with $(\lambda_{h'}, \mu_{h'})=(00)$ has the trivial value, 1. Eqs.(25) and (20a) thus yield

$$E_P = -g(\omega-N+3-\ell)(N-\ell) = -g\left[(\omega-N+3)N - \ell(\omega-\ell+3)\right], \tag{27c}$$

the result obtained through the $U(\omega) \supset SO(\omega)$ symmetry chain in refs.[34]-[37]. However, with the present method it will be easier to evaluate matrix elements of a more general two-body interaction.

3. A Ginocchio-type Model Built from 1^+ Fermion-pairs

In the Ginocchio S, D fermion-pair algebras the single nucleon creation operators, $a^\dagger_{jm}$ for a mixed configuration of j-values are given in terms of pseudo angular momenta $\underline{k}$ and $\underline{i}$, with $\underline{k}+\underline{i}=\underline{j}$. With k=1, i=3/2 the single particle j-quantum numbers take on the values j=1/2, 3/2, 5/2. Similarly, with k=2, i=3/2 : j=1/2, 3/2, 5/2, 7/2

with k=1, i=1/2 and 7/2 : j=1/2, 3/2, 5/2, 7/2, 9/2
with k=1, i=3/2 and 9/2 : j=1/2, 3/2, 5/2, 7/2, 9/2, 11/2
with k=1 and 5, i=3/2 : j=1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2

The i=3/2 algebras with k-spins coupled to two-fermion value K=0 lead to identical fermion S and D pairs with I=0 and 2 which generate an SO(8)$\supset$U(4) fermion pair algebra. The k=1 algebras with i-spins coupled to two-fermion value I=0, on the other hand, generate a USp(6)$\supset$U(3) S,D fermion pair algebra. In ref.[20] an attempt was made to increase the maximum allowed values of the SU(3) quantum numbers ($\lambda\mu$) of this algebra by combining the collective S,D-pair ($\lambda\mu$)'s with an intrinsic ($\lambda_\sigma\mu_\sigma$). However, instead of an increase in the maximum possible values of $\lambda+\mu$, the introduction of intrinsic ($\lambda_\sigma\mu_\sigma$)'s led to a decrease instead.

In view of recent IBM studies of collective magnetic dipole excitations in deformed nuclei [38]-[40] it may be of some interest to construct a Ginocchio-type fermion pair algebra generated by 1^+ proton-neutron pairs. Such an algebra is generated by the $J^\pi=1^+$ T=0 pair creation and annihilation operators

$$A_{mm'}=\frac{1}{2}\sum_{im_i}\sum_{m_t}(-1)^{i-m_i+\frac{1}{2}-m_t}\left(a^\dagger_{1mim_im_t}\,a^\dagger_{1m'i-m_i-m_t}-a^\dagger_{1m'im_im_t}\,a^\dagger_{1mi-m_i-m_t}\right)$$

with (28)

$$A_{m'm}=-A_{mm'},\qquad B_{mm'}=(A_{mm'})^\dagger,$$

that is, by a k=1 algebra with two-particle I spins coupled coherently to I=0 and two-particle isospin T=0 which restricts the two-particle K spin to the single value K=1, (with $M_K=1$, 0, -1 for mm'=10, 1-1, 0-1). Together with the U(3) subalgebra

$$C_{mm'}=\sum_{im_i}\sum_{m_t}a^\dagger_{1mim_im_t}\,a_{1m'im_im_t}-\frac{1}{2}\delta_{mm'}\,\omega\,, \tag{29a}$$

with
$$\omega=2\sum_i(2i+1)\,, \tag{29b}$$

these operators generate an SO(6)$\supset$U(3) algebra with commutation relations given by eqs.(2).

Since the SO(6)$\supset$U(3) state construction given in section

2 was completely general it applies to any SO(6)⊃U(3) algebra. The matrix elements of the pair-operators can be read from eq.(25), and the state vector construction follows at once from eqs.(20) and (26). The possible (λμ)-values are shown in table 1 for the k=1 i=3/2, j=1/2 3/2 5/2-shell for the case [σ]=[000], that is with no intrinsic U(3) excitations. In this case a=b=0; the maximum possible c-value, c=ω, follows at once from eq.(20c). Table 1 shows the (λμ)-values for the full O(6) symmetry with SO(6) particle and hole branches. The table also compares the 1^+ T=0-pair group with the S,D-pair group with USp(6) symmetry. It is interesting to note that both reach the same limiting λ+μ-values of 8. (Since the S,D-pair group applies to identical nucleons the neutron and proton (40) representations can be coupled to resultant (80)). For arbitrary excitations in the j=1/2 3/2 5/2 shell, on the other hand, the Pauli principle permits excitations as high as λ+μ=12 in the Elliott SU(3) model. Eqs.(20) also show that the introduction of an intrinsic U(3) symmetry lowers (rather than raises) the maximum possible λ+μ-value, a phenomenon already observed for the USp(6) S,D-pair symmetry, [20]. Both the 1^+ T=0-pair and the S,D-pair groups thus differ radically from the Sp(6,R) symmetry, [1] [31], where the combination of intrinsic and collective excitations serves to increase the λ+μ-values. Table 2 shows the possible (λμ)-values for the j=1/2 3/2 5/2 shell with one pair coupled to an intrinsic SU(3) symmetry $(\lambda_\sigma\mu_\sigma)$=(01). (This intrinsic symmetry would be quite natural in higher shells with more than one single-particle i-spin where there would be more than one antisymmetrically coupled pair with I=0). Again, eqs.(20) can be used to see that in this case, with $\sigma_1=\sigma_2=1$, the maximum c-value is ω-2, leading to highest SU(3) representations of (0,ω-1) with abc=00ω-2 and (1,ω-2) with abc=01ω-2. Apart from its inability to reach highly rotational (λμ)-values the 1^+, T=0 fermion pair symmetry also suffers from another deficiency. It contains no operators which lead naturally to M1 transition probabilities proportional to the isovector $(g_p-g_n)^2$ factor. The search for a sound fermionic foundation of the IBM 1^+ scissors-mode excitation may therefore have to continue.

Table 1

Possible $(\lambda\mu)$-Values. Comparison of neutron-proton 1^+-pair and identical-nucleon S,D-pair groups for the k=1, i=3/2, $(j=1/2\ 3/2\ 5/2)^n$ configurations. $(\lambda_\sigma\mu_\sigma)=(00)$.

1^+ T=0-Pair Group
O(6)-symmetry (444)

n	Possible $(\lambda\mu)$	
24	—	(00)
22	—	(10)
20	—	(20)
18	—	(30)
16	(08)	(40)
14	(07)	(50)
12	(06)	(60)
10	(05)	(70)
8	(04)	(80)
6	(03)	—
4	(02)	—
2	(01)	—
0	(00)	—

S,D Pair Group
USp(6)-symmetry (222)
(Identical nucleons)

n	Possible $(\lambda\mu)$		
12	(00)		
10	(02)		
8	(04)	(20)	
6		(22)	(00)
4	(40)	(02)	
2	(20)		
0	(00)		

Table 2

The 1^+ T=0-pair group with intrinsic SU(3)-symmetry $(\lambda_\sigma\mu_\sigma)=(01)$
O(6)-symmetry (433)

n	Possible $(\lambda\mu)$			
22			(10)	
20			(20)	(01)
18			(30)	(11)
16		(16)	(40)	(21)
14	(07)	(15)	(50)	(31)
12	(06)	(14)	(60)	(41)
10	(05)	(13)	(70)	(51)
8	(04)	(12)		(61)
6	(03)	(11)		
4	(02)	(10)		
2	(01)			

4. Concluding Remarks

The SO(6) ⊃U(3) fermion pair algebra is the simplest non-trivial fermion pair algebra. It has the following attractive features: The Wigner-Racah calculus for its U(3) subgroup is fully available through the computer code of Draayer and Akiyama [33]. Only multiplicity-free SU(3)-couplings are needed for the SO(6) ⊃U(3) state construction. The K^2-matrices are all 1-dimensional, so that the state construction by vector coherent state mthods can be carried out in complete analytical form. Of the two new examples given for the SO(6)⊃U(3) symmetry, the relativistic quark model of the nucleus proposed by the Bonn group may be an application where the use of SO(6)⊃U(3) vector coherent state methods may simplify detailed calculations. It is hoped that further useful examples of the simple SO(6)⊃U(3) symmetry will be discovered.

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STRESZCZENIE

Dyskutowano w pracy dwa nowe zastosowania algebry Lie par fermionowych w łańcuchu SO(6) ⊃ U(3): (1) relatywistyczny model kwarkowy jądra zaproponowany przez grupę w Bonn i (2) model typu Ginocchia zbudowany z par fermionowych $J=1^{+}$ i $T=0$. Zostały skonstruowane w pełni analityczne stany w obrębie koherentnej teorii.

P E З Ю M E

В работе рассматриваются два новые применения алгебры Ли фермионных пар в цепочке SO(6) ⊃ U(3): 1° релятивистская кварковая модель ядра предложена группой из Бонн и 2° модель типа Гинoккио построена из фермионовских пар $J = 1^{+}$ и $T = 0$ Сконстуированы полностью аналитические состояния в рамках когерентной теории.