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A Recursion Relation for Coefficients of Fractional Parentage in the Seniority Scheme

Rekurencyjne związki między współczynnikami genealogicznymi w schemacie seniority

Рекуррентные соотношения между генеалогическими коэффициентами в модели синьорити

1. Introduction

In the seniority scheme [1], the states of n identical fermions of angular momentum j are classified by the irreducible representations of the groups in the chain

$$SU(2j+1) \supset Sp(2j+1) \supset O(3)$$

and the states defined in this basis may be denoted by $|n(v)\alpha JM\rangle$. Here, v is the seniority, which may be thought of as the number of fermions remaining when as many J = 0 pairs of fermions as possible have been removed from the state. The quantum number α is required only if there is more than one state of angular momentum J and seniority v. For n particles the seniority is restricted by $v = v_{max}, v_{max}-2, \ldots, 1$ or 0, where $v_{max} = j + \frac{1}{2} - |j + \frac{1}{2} - n|$.

An analogous classification for bosons can be made using the chain

$$SU(2j+1) \supset O(2j+1) \supset O(3).$$

For bosons, $v = n, n-2, \ldots, 1$ or 0.

Coefficients of fractional parentage (cfp) are used in both the Nuclear Shell Model [1] and the Interacting Boson Model [2] for the calculation of the matrix elements of one-body operators. (They may also be used in the calculation of two-body operators.) A great advantage of the seniority scheme is that each of these coefficients may be written as the product of two factors, one of which is very simple in form and contains the entire dependence of the coefficient on the particle number n. The residual factor is a cfp between states in which the particle number is equal to the seniority, ie its minimum possible value. These residual factors may be termed "reduced cfp". For any given number of particles the reduced cfp are far fewer in number than the general cfp with $n \geq v$. Moreover, in view of the simple connection between the two sets the greater part of the complexity of a many fermion or boson problem resides in the reduced cfp. This may be especially well appreciated by noting that a method has been developed [3,4] for carrying out full Nuclear Shell Model calculations in a configuration of both neutrons and protons, which employs only reduced cfp for *identical* nucleons, but works in a basis having good angular momentum and isospin.

The particle number dependence is discussed in section 2, using ideas of quasispin [5], which is equivalent to seniority. The recursion relation is introduced in section 3 and applied to some simple illustrative examples in sections 4 and 5.

The reduced matrix elements of single particle creation operators are closely related to cfp. Thus the cfp

$$[j^{n-1}(v_1,\alpha_1 J_1);j|\} j^n(v)\alpha J]$$

is equal to

$$(-)^{n-1}\,(n(v)\alpha J \,\|\, a_j^{+} \,\|\, n-1(v_1)\alpha_1 J_1)/\sqrt{n}$$

for fermions, and to

$$(n(v)\alpha J \,\|\, b_j^{+} \,\|\, n-1(v_1)\alpha_1 J_1)/\sqrt{n}$$

for bosons. These reduced matrix elements rather than the cfp will be used in the remainder of this article.

2. Quasispin Formalism

For fermions of semi-integer angular momentum j we define the quasispin operators by

$$Q_+ = \tfrac{1}{2}\sum_m (-)^{j-m} a_{jm}^{+} a_{j-m}^{+}, \qquad Q_- = Q_+^{+}$$

$$Q_0 = \tfrac{1}{2}\sum_m a_{jm}^{+} a_{jm} - \tfrac{1}{4}(2j+1) \qquad (1)$$

where a_{jm}^{+} is the creation operator for a fermion in the substate m, and m takes the values j, j-1, ..., -j. These operators are scalars, ie they commute with the angular momentum operators of the system. Their commutation relations with each other are exactly those of the analogous angular momentum operators viz.

$$\left[Q_+, Q_-\right] = 2Q_0 \qquad \left[Q_0, Q_\pm\right] = \pm Q_\pm .$$

It follows that Q_+, Q_- and Q_0 are the generators of an SU(2) group, the quasispin group. The irreducible representations of this group, charactarised by the quasispin q, span multiplets in which the 2q + 1 states are distinguished by the number of J = 0 pairs which they contain. This is because the "ladder" operators, Q_+ and Q_- of the quasispin create and annihilate respectively pairs of fermions with J = 0. The state $|q, m_q\rangle$ contains $q + m_q$ pairs, where $|m_q| = q, q-1, q-2, \ldots, \frac{1}{2}$ or 0. The total number, n, of fermions in the system is therefore given by

$$n = j + \tfrac{1}{2} + 2m_q. \qquad (2)$$

The seniority, v, which is the number of *unpaired* fermions, is clearly

$$v = n - 2(q + m_q) = j + \tfrac{1}{2} - 2q. \qquad (3)$$

The operators a_{jm}^{+} and $\tilde{a}_{jm} = (-)^{j-m} a_{j,-m}$ form a quasispin doublet, ie

$$\left[Q_0, a_{jm}^{+}\right] = \tfrac{1}{2} a_{jm}^{+} \qquad \left[Q_0, \tilde{a}_{jm}\right] = -\tfrac{1}{2}\tilde{a}_{jm}$$

$$\left[Q_-, \tilde{a}_{jm}\right] = 0 \qquad \left[Q_-, a_{jm}^{+}\right] = \tilde{a}_{jm}$$

$$\left[Q_+, \tilde{a}_{jm}\right] = a_{jm}^{+} \qquad \left[Q_+, a_{jm}^{+}\right] = 0. \qquad (4)$$

The Wigner-Eckart theorem for quasispin implies that the matrix elements of these operators are proportional to SU(2) Clebsch-Gordan coefficients. Thus

$$(n(v)\alpha JM|\, a_{jm}^{+}\, |n-1(v-1)\alpha' J' M')$$

$$= \frac{(q+\tfrac{1}{2}, m_q-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \mid q, m_q)}{(q+\tfrac{1}{2}, -q-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \mid q, -q)} (v(v)\alpha JM|\, a_{jm}^{+}\, |v-1(v-1)\alpha' J' M')$$

$$= \sqrt{\frac{2j + 3 - n - v}{2j + 3 - 2v}}\, (v(v)\alpha JM|\, a_{jm}^{+}\, |v-1(v-1)\alpha' J' M').$$

Application of the Wigner-Eckart theorem [6] for angular momentum then gives immediately

$(n(v)\alpha J \| a_j^{+} \| n-1(v-1)\alpha' J')$

$$= \sqrt{\frac{2j+3-n-v}{2j+3-2v}}\,(v(v)\alpha J \| a_j^{+} \| v-1(v-1)\alpha' J'). \quad (5)$$

Similarly

$(n(v)\alpha JM | a_{jm}^{+} | n-1(v+1)\alpha' J' M')$

$$= \sqrt{\frac{n-v}{2}}\,(v+2(v)\alpha JM | a_{jm}^{+} | v-1(v-1)\alpha' J' M')$$

$$= \sqrt{\frac{n-v}{2j+1-2v}}\,(v(v)\alpha JM | \left[Q_-, a_{jm}^{+}\right] | v-1(v-1)\alpha' J' M').$$

Now, using (4) and applying the Wigner-Eckart theorem, we find

$$(n(v)\alpha J \| a_j^{+} \| n-1(v+1)\alpha' J') \quad (6)$$

$$= (-)^{j+J'-J} \sqrt{\frac{(2J'+1)(n-v)}{(2J+1)(2j+1-2v)}}\,(v+1(v+1)\alpha' J' \| a_j^{+} \| v(v)\alpha J).$$

For bosons, with integer angular momentum j the quasispin operators are defined by equations of the same form as (1) with a_{jm}^{+} and a_{jm} replaced by the boson creation and annihilation operators b_{jm}^{+}, b_{jm}. However, because these obey boson commutation rules

$$\left[b_{jm}, b_{jm}^{+}\right] = \delta_{mm}$$

the commutators of the Q operators are

$$\left[Q_0, Q_\pm\right] = \pm Q_\pm, \qquad \left[Q_+, Q_-\right] = -2Q_0.$$

The group generated by these operators is the non-compact group SU(1,1). The unitary irreducible representations are now of infinite dimension with $|m_q| = q, q+1, q+2, \ldots$. In place of (2) and (3) for the particle number and seniority we have

$$n = 2m_q - j - ½$$

$$v = 2q - j - ½. \qquad (7)$$

The number of pairs of bosons with $J = 0$ is given by $m_q - q$. Equations (5) and (6) become in this case

$$(n(v)\alpha J \| b_j^+ \| n-1(v-1)\alpha' J')$$

$$= \sqrt{\frac{n+v+2j-1}{2v+2j-1}}\,(v(v)\alpha J \| b_j^+ \| v-1(v-1)\alpha' J') \qquad (8)$$

and

$$(n(v)\alpha J \| b_j^+ \| n-1(v+1)\alpha' J') \qquad (9)$$

$$= \sqrt{\frac{(n-v)(2J'+1)}{(2j+2v+1)(2J+1)}}\,(-)^{J'+j-J}\,(v+1(v+1)\alpha' J' \| b_j^+ \| v(v)\alpha J).$$

3. Recursion Relation

In order to deduce a recursion relation for the unknown factors in equations (5) and (6), we consider a state of v fermions constructed by adding a fermion to a parent state of $v-1$ fermions and seniority $v-1$, viz.

$$|(v-1)\alpha_1 J_1, j; v\,JM\rangle$$

$$= \sum_{M_1, m} (J_1 M_1, jm | JM)\, a_{jm}^+ \,|v-1(v-1)\alpha_1 J_1 M_1\rangle \qquad (10a)$$

$$= \sum_{v', \alpha} |v(v')\alpha\, JM\rangle (v(v')\alpha\, J \| a_j^+ \| v-1(v-1)\alpha_1 J_1). \qquad (10b)$$

The extra particle has been vector coupled to the parent state to give definite J. Equation (10b) is obtained by intrducing a complete set of states of v fermions, and applying the Wigner-Eckart theorem. It should be noted that the summation over v' in (10b) has only two terms with $v' = v$ and $v-2$. Now using (10a) and (10b) we obtain the scalar product

$$((v-1)\alpha_1' J_1', j; v\, JM \mid (v-1)\alpha_1 J_1, j; v\, JM)$$

in two different forms,

$$\sum_{v',\alpha} (v(v')\alpha J \| a_j^+ \| v-1(v-1)\alpha_1 J_1)(v(v')\alpha J \| a_j^+ \| v-1(v-1)\alpha_1' J_1')$$

$$= \delta(J_1,J_1')\delta(\alpha_1,\alpha_1') + (-)^{J_1+J_1'}\sqrt{(2J_1+1)(2J_1'+1)}\,.\sum_{J_2,\alpha_2} \begin{Bmatrix} j & J_2 & J_1 \\ j & J & J_1' \end{Bmatrix}$$

$$.(v-1(v-1)\alpha_1 J_1 \| a_j^+ \| v-2(v-2)\alpha_2 J_2)$$

$$.(v-1(v-1)\alpha_1' J_1' \| a_j^+ \| v-2(v-2)\alpha_2 J_2)\;. \qquad (11)$$

Finally the term on the left hand side of equation (11) corresponding to $v' = v-2$ is expressed in terms of the grandparent to parent (ie. v-2 to v-1) parentage coefficients using (6) and transferred to the right hand side of the equation. The result is

$$\sum_{\alpha} (v(v)\alpha J \| a_j^+ \| v-1(v-1)\alpha_1 J_1)(v(v)\alpha J \| a_j^+ \| v-1(v-1)\alpha_1' J_1')$$

$$= \delta(J_1,J_1')\delta(\alpha_1,\alpha_1')$$

$$+ (-)^{J_1+J_1'}\sqrt{(2J_1+1)(2J_1'+1)}\;.\sum_{J_2,\alpha_2} \left[\begin{Bmatrix} j & J_1' & J_2 \\ j & J_1 & J \end{Bmatrix} + \frac{2(-)^v\,\delta(J,J_2)}{(2J+1)(2j+5-2n)} \right]$$

$$.(v-1(v-1)\alpha_1 J_1 \| a_j^+ \| v-2(v-2)\alpha_2 J_2)$$

$$.(v-1(v-1)\alpha_1' J_1' \| a_j^+ \| v-2(v-2)\alpha_2 J_2). \qquad (12)$$

and this is the desired recurrence relation.

The use of this relation is straightforward. The right hand side of equation (12) contains only the grandparent to parent coefficients, which are assumed known from the previous step of the calculation. The left hand side is clearly a matrix whose rows and columns are labelled by the parents $\alpha_1 J_1$. If we assume that, as is often the case, α is redundant this matrix has a single eigenvalue equal to n, the remainder all being zero. The parentage coefficients for the *offspring* state of n fermions are the components of the corresponding eigenvector normalised to norm n. In fact, as the matrix is factorable they can be obtained together with their relative phases from the matrix elements in a single row. If however ther are r offspring states distinguished by $\alpha = 1,2,\ldots,r$ the matrix will have r eigenvalues equal to n and the parentage coefficients are given by r orthogonal eigenvectors each with norm n. This procedure defines the states $|\, v(v)\alpha JM\rangle$ to within a unitary transformation, this arbitrariness being reflected in the degeneracy of the eigenvalues.

For bosons an analogous derivation leads to

$$\sum_{\alpha} (v(v)\alpha J \,\|\, b_j^{\dagger} \,\|\, v-1(v-1)\alpha_1 J_1)(v(v)\alpha J \,\|\, b_j^{\dagger} \,\|\, v-1(v-1)\alpha_1' J_1')$$

$$= \delta(J_1,J_1')\delta(\alpha_1,\alpha_1') + (-)^{J_1+J_1'}\sqrt{(2J_1+1)(2J_1'+1)}$$

$$\sum_{\alpha_2, J_2} \left[\begin{Bmatrix} j & J_1 & J_2 \\ j & J_1' & J \end{Bmatrix} - \frac{2\delta(J,J_2)}{(2J+1)(2n+2j-3)} \right] \qquad (13)$$

$$.(v-1(v-1)\alpha_1 J_1 \,\|\, b_j^{\dagger} \,\|\, v-2(v-2)\alpha_2 J_2)$$

$$.(v-1(v-1)\alpha_1' J_1' \,\|\, b_j^{\dagger} \,\|\, v-2(v-2)\alpha_2 J_2).$$

4. Application to Fermions

For v = 1 and 2 equation (12) gives immediately

$(1(1)j \,\|\, a_j^{\dagger} \,\|\, 0(0)0) = 1$ and

$$(2(2)J \| a_j^{+} \| 1(1)j) = \sqrt{2} \text{ for } J=2,4,6,\ldots,2j-1$$
$$= 0 \text{ otherwise.} \qquad (14)$$

For states of seniority 3 ($j>\frac{1}{2}$) equations (12) and (14) now give

$$\sum_{\alpha} (3(3)\alpha J \| a_j^{+} \| 2(2)J_1)(3(3)\alpha J \| a_j^{+} \| 2(2)J_1')$$

$$= \delta(J_1,J_1') + 2\sqrt{(2J_1+1)(2J_1'+1)}\left[\begin{Bmatrix} j & j & J_1 \\ j & J & J_1' \end{Bmatrix} - \frac{2\delta(J,j)}{(4j^2-1)}\right] \qquad (15)$$

where $J_1, J_1' = 2,4,6,\ldots,2j-1$, subject to $|j - J| \le J_1, J_1' \le j + J$. The trace of this matrix must be $3r$, where r is the number of states of spin J and seniority 3 in the configuration j^3. Thus

$$r = \frac{1}{3} \sum_{J_1=|J-j|}^{J+j} \left[1 + 2(2J_1+1)\left[\begin{Bmatrix} j & j & J_1 \\ j & J & J_1 \end{Bmatrix} - \frac{2\delta(j,J)}{(4j^2-1)}\right]\right] \qquad (16)$$

where J_1 runs over even values.

As an example consider the case $J = j$, so that $J_1 = 2,4,6,\ldots,2j-1$. Equation (16) gives $r = 0$ for $j \le \frac{7}{2}$ and $r = 1$ for $j = \frac{9}{2}, \frac{11}{2}$ and $\frac{13}{2}$. Thus for any of these last three j-values equation (12) gives

$$(3(3)j \| a_j^{+} \| 2(2)2j-1) = \sqrt{1 - \frac{4(4j-1)}{(4j^2-1)} + 2(4j-1)\begin{Bmatrix} j & j & 2j-1 \\ j & j & 2j-1 \end{Bmatrix}}$$

$$= \sqrt{1 - \frac{4(4j-1)}{(4j^2-1)} + \frac{(2j)!(2j-1)!}{(4j-2)(4j-4)!}}. \qquad (17)$$

(NB. This expression vanishes for all $j < \frac{9}{2}$.) The remaining coefficients are given by

$$(3(3)j \parallel a_j^+ \parallel 2(2)J_1)$$

$$= 2\sqrt{(2J_1+1)(4j-1)}\left[\begin{Bmatrix} j & j & J_1 \\ j & j & 2j-1 \end{Bmatrix} - \frac{2}{(4j^2-1)}\right] / (3(3)j \parallel a_j^+ \parallel 2(2)2j-1)$$

for $J_1 = 2, 4, 6, \ldots, 2j - 3$. (18)

4. Application to Bosons

A system of n identical p-bosons ($j = 1$) has total angular momentum given by $J = n, n - 2, n - 4, \ldots, 0$ or 1. Thus since a state of $v = 0$ has $J = 0$, the only possible J for a state of $v = 2$ is 2. Similarly three bosons have $J = 1$ or 3 and since $J = 1$ must correspond to $v = 1$, it follows that $J = 3$ corresponds to $v = 3$. It may be inferred that, in general $J = v$ for p-bosons. Equation (13) then leads immediately to

$$(J(J)J \parallel b_1^+ \parallel J-1(J-1)J-1) = \sqrt{J}.$$

Now using (8) and (9) we find the familiar results,

$$(n(J)J \parallel b_1^+ \parallel n-1(J-1)J-1) = \sqrt{\frac{J(n+J+1)}{2J+1}}$$

$$(n(J)J \parallel b_1^+ \parallel n-1(J+1)J+1) = \sqrt{\frac{(n-J)(J+1)}{2J+1}}. \qquad (19)$$

The coefficients in (19) are closely related to the reduced matrix elements of $\underline{r}$ between eigenstates of a spherical harmonic oscillator. They are also relevant to IBM3 [7], which is a version of the IBM involving isovector bosons. In this case, J would represent isospin rather than angular momentum.

The case $j = 2$ is of particular interest in relation to the IBM. In this model it is assumed that some low-lying collective states of nuclei can be represented as states of a system of s-bosons ($j = 0$) and d-bosons ($j = 2$). Calculations in the model are usually performed using the seniority basis for the d-bosons. The states are therefore defined by $|N,n,v,JM\rangle$, where N is the total number of bosons, n the number of d-bosons and v the d-boson seniority. Specific formulas for the parentage coefficients which are important in the "vibrational limit" of the model, ie close to the scheme defined above, have been given in reference [2] (eg. figure 5). These results are readily given by equation (13). Moreover application to transitions involving the X' band is straightforward. In this case we find, using d^+ for convenience instead of b_2^+

$$(v(v)\,2v-4 \,\|\, d^+ \,\|\, v-1(v-1)\,2v-2) =$$

$$(v-1)\sqrt{1+(4v-7)\left[\begin{Bmatrix}2 & 2v-4 & 2v-2\\ 2 & 2v-4 & 2v-2\end{Bmatrix}-\frac{2}{(2v+1)(4v-7)}\right]} \qquad (20)$$

and

$$(v(v)\,2v-4 \,\|\, d^+ \,\|\, v-1(v-1)J_1) =$$

$$(-)^{J_1}\sqrt{(2J_1+1)(4v-7)}\left[\begin{Bmatrix}2 & 2v-4 & 2v-2\\ 2 & 2v-4 & J_1\end{Bmatrix}-\frac{2}{(2v+1)(4v-7)}\right]$$

$$\cdot\frac{(v-1,(v-1)J_1 \,\|\, d^+ \,\|\, v-2,(v-2)\,2v-4)^2}{(v(v)\,2v-4 \,\|\, d^+ \,\|\, v-1,(v-1)2v-2)}$$

for $J_1 = 2v-2, 2v-4, 2v-5, 2v-6$. (21)

The results are shown in the figure which provides an extension to figure 5 of reference [2] by taking $n_d = v-1$. The reduced matrix elements of reference [2] differ by a factor of $\sqrt{4v-7}$ from those of (20) and (21). This factor has been included in the figure.

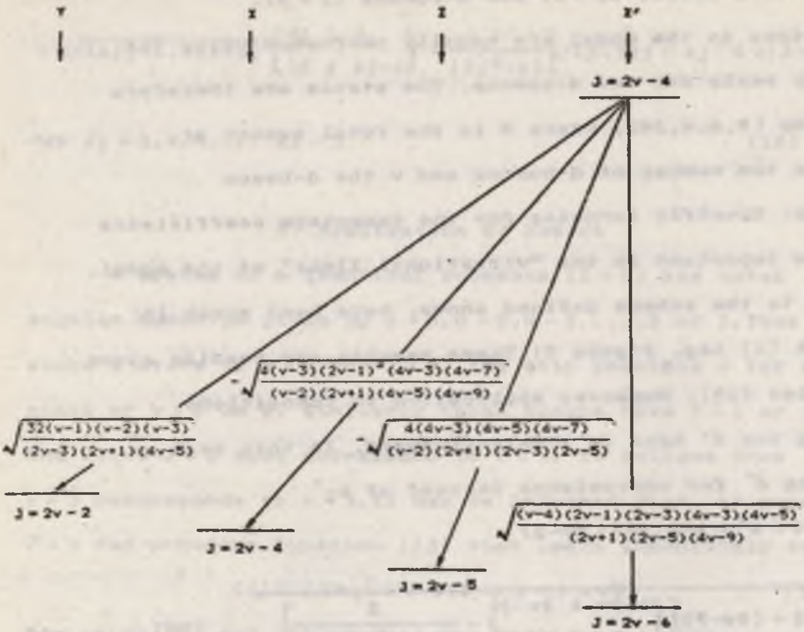

Figure. Reduced matrix elements of $d^{\dagger}$.

Another of the three dynamical symmetries [8] of the IBM, namely the O(6) limit, corresponds to a seniority scheme. The relevant group chain is

$$U(6) \supset O(6) \supset O(5) \supset O(3)$$

and the states are labelled by $|N(v)\sigma JM\rangle$, where v is the O(6) seniority, and the O(5) or d-boson seniority is now denoted by σ. The parentage coefficients

$$(v(v)\sigma J \| d^{\dagger} \| v-1, (v-1)\sigma_1 J_1)$$

and

$$(v(v)\sigma J \| s^{\dagger} \| v-1, (v-1)\sigma J)$$

can be calculated from a modified form of equation (13). For this case the factor $(2v + 2j - 3)$ in the denominator in (13) becomes $2v + 2$. The states defined in this scheme do not have definite numbers of d-bosons as the pairs are of the form $(d^{+}.d^{+}) - (s^{+}s^{+})$. However as this expression is invariant under O(5) as well as O(6), σ remains a good quantum number.

A similar six-boson problem occurs in IBM4 [9], where the bosons have either T = 1, S = 0 or T = 0, S = 1. Here T is the isospin and S is an intrinsic spin possessed by the bosons in addition to their orbital angular momentum of 0 or 2. As long as only symmetric orbital states are considered the spin-isospin variables can be treated in isolation. In this case it is more natural to exploit the well known isomorphism between O(6) and SU(4) and write the group chain as

$$U(6) \supset SU(4) \supset SU(2)_T \times SU(2)_S .$$

Application of equation (13) to this case leads to the SU(4) Wigner coefficients given in table A2.1 of reference [10].

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STRESZCZENIE

W artykule dyskutuje się ścieżki rekurencyjne między współczynnikami genealogicznymi w schemacie seniority. Wyznaczona zależność zredukowanych współczynników od liczby cząstek pozwala prosto określać elementy macierzowe operatorów niszczenia i rodzenia w układach fermionów bądź bozonów.

РЕЗЮМЕ

В статье обсуждаются рекуррентные соотношения между генеалогическими коэффициентами в модели синьорити. Полученная зависимость приведенных коэффициентов от числа частиц позваляет легко находить матричные элементы операторов рождения и аннигиляции в фермионных системах.