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Approximations of self-adjoint C_0-semigroups in the operator-norm topology

Dedicated to Yuri Kozitsky on the occasion of his 70th birthday

Abstract. The paper improves approximation theory based on the Trotter–Kato product formulae. For self-adjoint C_0-semigroups we develop a lifting of the strongly convergent Chernoff approximation (or *product*) formula to convergence in the operator-norm topology. This allows to obtain optimal estimate for the rate of operator-norm convergence of Trotter–Kato product formulae for Kato functions from the class K_2.

1. Introduction

The aim of the paper is to present a new generalised proof of approximation theory developed in [6, 7]. For self-adjoint Trotter–Kato product formulae it allows to obtain optimal estimate for the rate of convergence in operator norm for Kato functions of class K_β, where $\beta = 2$ (see [7]).

Instead of a *double-iteration* procedure of [7] we extend in this paper the *Chernoff approximation formula* [5] and the *Trotter–Neveu–Kato* approximation theorem [8], Theorem IX.2.16, to the operator-norm topology. Essentially we follow here the idea of *lifting* the strongly convergent Chernoff approximation formula to operator-norm convergence [9, 11], whereas majority of results concerning this formula are about the strong operator topology, see, for example, review [2]. In the same vein we quote a recent

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book [1], where different aspects of semigroup convergence in the strong operator topology are presented in great details.

To proceed, we first recall definition of the Kato functions that belong to the class K_β.

Definition 1.1. If a real-valued Borel measurable function $f : [0,\infty) \to [0,1]$ satisfying

$$0 \le f(s) \le 1, \quad f(0) = 1, \quad f'(+0) = -1, \tag{1.1}$$

is such that for any $\varepsilon > 0$ there exists a positive constant $\delta_\varepsilon < 1$ implying

$$f(s) \le 1 - \delta_\varepsilon, \quad s \ge \varepsilon, \tag{1.2}$$

and that for some β, where $1 < \beta \le 2$,

$$[f]_\beta := \sup_{s>0} \frac{|f(s) - 1 + s|}{s^\beta} < \infty, \tag{1.3}$$

then $f \in K_\beta$.

Some elementary examples of functions satisfying Definition 1.1 are

$$f(s) = e^{-s}, \quad f(s) = (1 + k^{-1}s)^{-k}, \quad k > 0. \tag{1.4}$$

Note that the Kato functions of class K_β are not necessarily monotonously decreasing, but it is true in a vicinity of $s = +0$. For more details about different types of Kato functions see Appendix C in [12].

By Definition 1.1 and by the spectral theorem one gets that for any non-negative self-adjoint operator A the *bounded* operator-valued function $t \mapsto f(tA) \in \mathcal{L}(\mathfrak{H})$ is strongly continuous in $\mathbb{R}^+$ and right-continuous on $\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$, that is, $\text{s-lim}_{t\to+0} f(tA) = \mathbb{1}$.

One of the main corollaries of the semigroup approximation results established in the present paper (Theorem 4.5) is the statement about operator norm convergence of the Trotter–Kato product formulae, see Section 5.

Proposition 1.2. *Let $f, g \in K_2$. If A and B are non-negative self-adjoint operators in a separable Hilbert space $\mathfrak{H}$ with domains* dom A *and* dom B *such that the operator sum $C := A + B$ is self-adjoint on* dom C = dom $A \cap$ dom B, *then*

$$\left\| \left[g(tB/n)^{1/2} f(tA/n) g(tB/n)^{1/2} \right]^n - e^{-tC} \right\| = O(n^{-1}), \tag{1.5}$$

$$\left\| \left[f(tA/n) g(tB/n) \right]^n - e^{-tC} \right\| = O(n^{-1}), \tag{1.6}$$

for $n \to \infty$, hold in the operator norm topology. The convergence is locally uniform on $\mathbb{R}^+_0$, but if operator C is strictly positive, it is uniform on $\mathbb{R}^+_0$.

Note that the rates of convergence in (1.5) and in (1.6) are optimal, i.e., they can not be improved in the general setup [7].

2. Chernoff approximation formula: strong operator topology

In this section we give a proof of the Chernoff approximation formula in the strong operator topology, which is alternative to the original one based on the $\sqrt{n}$-Lemma [5]. In conclusion we relax some conditions of the main Theorem 2.3.

Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions $F(t) \leq \mathbb{1}$ such that $F(0) = \mathbb{1}$. We set

$$S(\tau) := \frac{\mathbb{1} - F(\tau)}{\tau}, \quad \tau > 0. \tag{2.1}$$

Then for each $\tau > 0$ bounded operator $S(\tau)$ is self-adjoint and positive.

Let $H \geq 0$ be self-adjoint operator in $\mathfrak{H}$. Then by the *Trotter–Neveu–Kato* convergence theorem we obtain that

$$\text{s-}\lim_{\tau \to +0} (\mathbb{1} + S(\tau))^{-1} = (\mathbb{1} + H)^{-1}, \tag{2.2}$$

holds if and only if

$$\text{s-}\lim_{\tau \to +0} e^{-tS(\tau)} = e^{-tH}, \tag{2.3}$$

uniformly in $t \in \mathcal{I}$ for any closed bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$. In this case we say that this convergence holds *locally uniformly* in $t \in \mathbb{R}_0^+$, whereas if $\mathcal{I} \subset \mathbb{R}^+$, then convergence holds locally uniformly *away* from zero. For example, setting $\tau = t/\eta$ for $\eta \geq 1$, we obtain

$$\text{s-}\lim_{\eta \to +\infty} e^{-tS(t/\eta)} = e^{-tH}, \tag{2.4}$$

locally uniformly away from zero.

To proceed, we need the following elementary estimate:

Lemma 2.1. *For $\lambda \in [0, 1]$ and $r \geq 1$, one has*

$$0 \leq e^{-r(1-\lambda)} - \lambda^r \leq \frac{1}{r}. \tag{2.5}$$

The next assertion serves to *lift* the weak convergence of vectors $\{u_n\}_{n\geq 1}$ in $\mathfrak{H}$ to the strong convergence of this sequence.

Lemma 2.2. *Let $\{u_n\}_{n\geq 1}$ be a weakly convergent sequence of vectors,* w-$\lim_{n\to\infty} u_n = u$*, in a Hilbert space $\mathfrak{H}$. If, in addition,* $\lim_{n\to\infty} \|u_n\| = \|u\|$*, then* s-$\lim_{n\to\infty} u_n = u$.

Proof. Note that

$$\|u_n - u\|^2 = \|u_n\|^2 + \|u\|^2 - 2\,\mathrm{Re}\,(u_n, u).$$

Then by conditions of the lemma this yields s-$\lim_{n\to\infty} u_n = u$. □

Now we are in position to prove the Chernoff approximation formula for self-adjoint semigroups in the *strong operator* topology.

Theorem 2.3. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$ and let $H \geq 0$ be a self-adjoint operator in $\mathfrak{H}$. The convergence*

$$\text{s-}\lim_{\eta \to +\infty} F(t/\eta)^\eta = e^{-tH}, \tag{2.6}$$

holds locally uniformly in t away from zero if and only if the condition (2.2) *is satisfied.*

Proof. First we assume that condition (2.2) is satisfied. Let us show that

$$\text{s-}\lim_{\eta \to +\infty} \left(F(t/\eta)^\eta - e^{-tS(t/\eta)} \right) = 0, \tag{2.7}$$

locally uniformly in $t > 0$. To this end we use the spectral functional calculus for self-adjoint operators to obtain the representation

$$F(\tau)^r - e^{-r(\mathbb{1} - F(\tau))} = \int_{[0,1]} dE_{F(\tau)}(\lambda) \left(\lambda^r - e^{-r(1-\lambda)} \right), \ \tau > 0, \ r \geq 1. \tag{2.8}$$

Then inequality (2.5) yields the estimate

$$\left\| \left(F(\tau)^r - e^{-r(\mathbb{1} - F(\tau))} \right) u \right\| \leq r^{-1} \|u\|, \quad r \geq 1, \quad u \in \mathfrak{H}.$$

Setting $\tau = t/\eta$, $\eta \geq 1$, and $r = t/\tau \geq 1$, we obtain

$$\left\| \left(F(t/\eta)^\eta - e^{-tS(t/\eta)} \right) u \right\| \leq \eta^{-1} \|u\|, \quad \eta \geq 1, \quad u \in \mathfrak{H}.$$

This proves the limit (2.7) locally uniformly in t away from zero. Since condition (2.2) is equivalent to (2.4), the representation

$$F(t/\eta)^\eta - e^{-tH} = F(t/\eta)^\eta - e^{-tS(t/\eta)} + e^{-tS(t/\eta)} - e^{-tH}, \quad \eta \geq 1, \quad t > 0,$$

and (2.7) yield (2.6) .

Conversely, assume that (2.6) is satisfied. Using representation

$$e^{-tS(t/\eta)} - e^{-tH} = F(t/\eta)^\eta - e^{-tH} + e^{-tS(t/\eta)} - F(t/\eta)^\eta,$$

we get from (2.6) and (2.7) that (2.4) holds locally uniformly in t away from zero. For $\tau = t/\eta$, we verify that convergence in (2.3) holds locally uniformly in $t \in \mathbb{R}_0^+$. Then by the Trotter–Neveu–Kato convergence theorem the limit (2.3) implies (2.2). □

The limit (2.6) yields in particular that the semigroup approximation formula

$$\text{s-}\lim_{n \to \infty} F(t/n)^n = e^{-tH}, \tag{2.9}$$

also holds locally uniformly away from zero for the sequence when $\eta = n \in \mathbb{N}$.

We note that formula (2.9) follows from general Banach space Chernoff approximation formula, see Theorem 2.2 in [11], as a particular case for self-adjoint semigroups.

Lemma 2.4. *Let $K(\cdot) : \mathbb{R}^+ \to \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint operators and let H be a non-negative self-adjoint operator. If the weak operator limit:*

$$\text{w-}\lim_{\tau\to+0}(\lambda\mathbb{1} + K(\tau))^{-1} = (\lambda\mathbb{1} + H)^{-1}, \tag{2.10}$$

for each $\lambda > 0$, then it is also true in the strong operator topology:

$$\text{s-}\lim_{\tau\to+0}(\lambda\mathbb{1} + K(\tau))^{-1} = (\lambda\mathbb{1} + H)^{-1}. \tag{2.11}$$

Proof. By virtue of (2.10) we get

$$\lim_{\tau\to+0} \|(\lambda\mathbb{1} + K(\tau))^{-1/2}u\| = \|(\lambda\mathbb{1} + H)^{-1/2}u\|, \quad u \in \mathfrak{H}, \tag{2.12}$$

for $\lambda > 0$. Since

$$(\lambda\,\mathbb{1} + K(\tau))^{-1/2} = \frac{1}{\pi}\int_0^\infty dx\, \frac{1}{\sqrt{x}}\, (x\,\mathbb{1} + \lambda\,\mathbb{1} + K(\tau))^{-1},$$

the limit (2.10) yields w-$\lim_{\tau\to+0}(\lambda\,\mathbb{1} + K(\tau))^{-1/2} = (\lambda\,\mathbb{1} + H)^{-1/2}$. This, together with (2.12) and the lifting Proposition 2.2, imply s-$\lim_{\tau\to+0}(\lambda\,\mathbb{1} + K(\tau))^{-1/2} = (\lambda\,\mathbb{1} + H)^{-1/2}$. Since the product of operators is strongly continuous, this limit yields (2.11). □

Taking into account Lemma 2.4, the conditions of Theorem 2.3 can be relaxed as follows.

Theorem 2.5. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative contractions such that $F(0) = \mathbb{1}$ and let $H \geq 0$ be a self-adjoint operator in $\mathfrak{H}$. The statement* (2.6) *is valid if and only if for each $\lambda > 0$ the condition*

$$\text{w-}\lim_{\tau\to+0}(\lambda\mathbb{1} + S(\tau))^{-1} = (\lambda\mathbb{1} + H)^{-1}, \tag{2.13}$$

is satisfied.

We skip the proof since the line of reasoning is straightforward.

3. Lifting the Chernoff approximation formula to operator-norm topology

A natural question arises: can the limit (2.6) in Theorem 2.3 (or in Theorem 2.5) be lifted to convergence in the operator-norm topology?

First we note that in contrast to *quasi-sectorial* contractions [10], the estimates for *self-adjoint* contraction C in the Chernoff $\sqrt{n}$-Lemma [5] and in its refinement due to the $1/\sqrt[3]{n}$ -Theorem (see Lemma 2.1 and Theorem 3.3 in [11]) can be significantly improved. Namely, the spectral functional calculus of self-adjoint contraction C and Lemma 2.1 yield

$$\left\|C^n - e^{n(C-\mathbb{1})}\right\| = \left\|\int_0^1 dE_C(\lambda)\left(\lambda^n - e^{n(\lambda-1)}\right)\right\| \leq \frac{1}{n}, \quad n \in \mathbb{N}. \tag{3.1}$$

Similarly to the case of the strong operator topology, the next step in the program of lifting the approximation formula to operator-norm topology involves the *lifting* of the Trotter–Neveu–Kato convergence theorem. Therefore, we proceed with the following lemma, which is well suited for *self-adjoint* lifting of this theorem.

Lemma 3.1. *Let K and L be non-negative self-adjoint operators in a Hilbert space $\mathfrak{H}$. Then*

$$\left\|e^{-K} - e^{-L}\right\| \leq c \left\|(\mathbb{1} + K)^{-1} - (\mathbb{1} + L)^{-1}\right\| \tag{3.2}$$

with a constant $c > 0$ independent of operators K and L.

Proof. By the Riesz–Dunford functional calculus, one obtains for the difference of exponentials the representation

$$e^{-K} - e^{-L} = \frac{1}{2\pi i}\int_\Gamma dz\, e^{-z}\left((z-K)^{-1} - (z-L)^{-1}\right), \tag{3.3}$$

where the contour Γ is a union of two branches: $\Gamma = \Gamma_0 \cup \Gamma_\infty$, with

$$\begin{aligned} \Gamma_0 &= \{z \in \mathbb{C} : z = e^{i\varphi}, \quad \pi/4 \leq \varphi \leq 2\pi - \pi/4\}, \\ \Gamma_\infty &= \{z \in \mathbb{C} : z = re^{\pm i\pi/4}, \quad r \geq 1\}. \end{aligned} \tag{3.4}$$

From (3.3) we find the representation

$$\begin{aligned} e^{-K} - e^{-L} = \frac{1}{2\pi i}\int_\Gamma dz\, e^{-z}(\mathbb{1}+K)(z-K)^{-1} \times \\ \times \left[(\mathbb{1}+L)^{-1} - (\mathbb{1}+K)^{-1}\right](\mathbb{1}+L)(z-L)^{-1}. \end{aligned} \tag{3.5}$$

Since $(\mathbb{1}+K)(z-K)^{-1} = -\mathbb{1} + (1+z)(z-K)^{-1}$, one gets the estimate

$$\left\|(\mathbb{1}+K)(z-K)^{-1}\right\| \leq 1 + \frac{1+|z|}{\operatorname{dist}(z, \mathbb{R}^+)}.$$

Setting

$$c_\Gamma := \sup_{z\in\Gamma} \frac{1+|z|}{\operatorname{dist}(z, \mathbb{R}^+)} < \infty,$$

we find

$$\sup_{z\in\Gamma}\left\|(\mathbb{1}+K)(z-K)^{-1}\right\| \leq (1 + c_\Gamma), \tag{3.6}$$

where the constant c_Γ depends only on Γ but not on the operator K. Similarly, from (3.6) one also gets

$$\sup_{z\in\Gamma}\left\|(\mathbb{1}+L)(z-L)^{-1}\right\| \leq (1 + c_\Gamma).$$

Using these estimates, we find from (3.5) that

$$\left\|e^{-K} - e^{-L}\right\| \leq c \left\|(\mathbb{1} + K)^{-1} - (\mathbb{1} + L)^{-1}\right\|$$

with

$$c := \frac{1}{2\pi}(1+c_\Gamma)^2 \int_\Gamma |dz|\,|e^{-z}|.$$

Since for $z \in \Gamma_\infty$ the value of $\Re\mathrm{e}\, z > 0$, the integral is convergent and c depends only on the contour Γ. □

The first step towards the proof the *operator-norm* convergence of the Chernoff approximation formula (2.6) would be *lifting* of the strong convergence in (2.2) to the operator-norm convergence. To study the consequence of this lifting we prove the following assertion.

Lemma 3.2. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by (2.1)–(2.2), where H is a non-negative self-adjoint operator in $\mathfrak{H}$. Then the condition*

$$\lim_{\tau\to+0} \|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| = 0, \tag{3.7}$$

is satisfied if and only if

$$\lim_{\eta\to\infty} \sup_{t\in\mathcal{I}} \|(\mathbb{1} + tS(t/\eta))^{-1} - (\mathbb{1} + tH)^{-1}\| = 0, \tag{3.8}$$

for any closed interval $\mathcal{I} \subset \mathbb{R}^+$, i.e., locally uniformly away from zero.

Proof. A straightforward computation shows that

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = t(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}[(\mathbb{1} + S(\tau))^{-1} \\ - (\mathbb{1} + H)^{-1}](\mathbb{1} + H)(\mathbb{1} + tH)^{-1}.$$

Here we used the fact that if $t > 0$ and $\tau > 0$, then for self-adjoint operator $S(\tau)$ the closure

$$\overline{(\mathbb{1} + tS(\tau))^{-1}(\mathbb{1} + S(\tau))} = (\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}.$$

For these values of arguments t and τ we get

$$\|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| \le (1 + 2/t),$$
$$\|(\mathbb{1} + H)(\mathbb{1} + tH)^{-1}\| \le (1 + 2/t).$$

If $\mathcal{I}$ is a closed interval of $\mathbb{R}^+$, for example, $\mathcal{I} := [a, b]$, $0 < a < b < \infty$, then

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \le b(1 + 2/a)^2\|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\|,$$

for $t \in \mathcal{I}$ and $\tau > 0$. Setting $\tau = t/\eta$ we find

$$\|(\mathbb{1} + tS(t/\eta))^{-1} - (\mathbb{1} + tH)^{-1}\| \\ \le b(1 + 2/a)^2\|(\mathbb{1} + S(t/\eta))^{-1} - (\mathbb{1} + H)^{-1}\|. \tag{3.9}$$

Since by (3.7) we obtain for the last factor in the right-hand side of (3.9)

$$\lim_{\eta\to\infty} \sup_{t\in\mathcal{I}} \|(\mathbb{1} + S(t/\eta))^{-1} - (\mathbb{1} + H)^{-1}\| = 0,$$

the estimate (3.9) yields (3.8). The converse is obvious. □

Lemma 3.2 allows to advance in generalisation of self-adjoint Chernoff approximation formula for *operator-norm* convergence.

Theorem 3.3. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by* (2.1)–(2.2)*, where $H \geq 0$ is self-adjoint operator in $\mathfrak{H}$. Then we have*

$$\lim_{\eta\to\infty} \sup_{t\in\mathcal{I}} \big\| F(t/\eta)^\eta - e^{-tH} \big\| = 0, \tag{3.10}$$

for any closed interval $\mathcal{I} \subset \mathbb{R}^+$, if and only if the family $\{S(\tau)\}_{\tau>0}$ satisfies condition (3.7)*.*

Proof. For $t > 0$ and $\eta > 0$, we obviously have the estimate

$$\big\|F(t/\eta)^\eta - e^{-tH}\big\| \leq \big\|F(t/\eta)^\eta - e^{-tS(t/\eta)}\big\| + \big\|e^{-tS(t/\eta)} - e^{-tH}\big\|. \tag{3.11}$$

By the functional calculus of self-adjoint contraction $C := F(\tau)$, Lemma 2.1 yields estimate (3.1), which improves the Chernoff $\sqrt{n}$-Lemma [5], as well as an estimate in [11]. Then (3.1), for continuous variable $n = \eta$, and (3.11), where $S(t/\eta)$ is defined by (2.1), imply

$$\big\|F(t/\eta)^\eta - e^{-tH}\big\| \leq \frac{1}{\eta} + \big\|e^{-tS(t/\eta)} - e^{-tH}\big\|, \quad t > 0, \quad \eta > 0 \ . \tag{3.12}$$

Note that by Lemma 3.1, there is a constant $c > 0$ such that

$$\big\|e^{-tS(t/\eta)} - e^{-tH}\big\| \leq c\big\|(\mathbb{1} + tS(t/\eta))^{-1} - (\mathbb{1} + tH)^{-1}\big\|, \tag{3.13}$$

for $t > 0$, $\eta > 0$. Inserting the estimate (3.13) into (3.12), we obtain

$$\big\|F(t/\eta)^\eta - e^{-tH}\big\| \leq \frac{1}{\eta} + c\big\|(\mathbb{1} + tS(t/\eta))^{-1} - (\mathbb{1} + tH)^{-1}\big\|, \tag{3.14}$$

Then applying Lemma 3.2, we get (3.10).

Conversely, let us assume (3.10). Note that

$$\big\|e^{-tS(t/\eta)} - e^{-tH}\big\| \leq \big\|F(t/\eta)^\eta - e^{-tH}\big\| + \big\|F(t/\eta)^\eta - e^{-tS(t/\eta)}\big\|,$$

for $t > 0$ and $\eta > 0$. Applying to the last term the spectral representation for $F(t)$ and Lemma 2.1 for the corresponding integrand, we find for $t > 0$ and $\eta > 0$

$$\big\|e^{-tS(t/\eta)} - e^{-tH}\big\| \leq \big\|F(t/\eta)^\eta - e^{-tH}\big\| + \frac{1}{\eta} \, . \tag{3.15}$$

Then by assumption (3.10) the estimate (3.15) yields

$$\lim_{\eta\to\infty} \sup_{t\in\mathcal{I}} \big\|e^{-tS(t/\eta)} - e^{-tH}\big\| = 0, \tag{3.16}$$

locally uniformly away from zero. Hence, the limit $\lim_{\eta\to\infty}\left\|e^{-tS(t/\eta)} - e^{-tH}\right\| = 0$, or equivalently $\lim_{\tau\to+0}\left\|e^{-tS(\tau)} - e^{-tH}\right\| = 0$, holds for any $t > 0$.

Now, using representation:

$$(3.17) \qquad (\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1} = \int_0^\infty ds\; e^{-s}\left(e^{-sS(\tau)} - e^{-sH}\right),$$

we obtain the estimate

$$(3.18) \qquad \|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| \le \int_0^\infty ds\; e^{-s}\left\|e^{-sS(\tau)} - e^{-sH}\right\|.$$

Let $\Phi_\tau(s) := e^{-s}\left\|e^{-sS(\tau)} - e^{-sH}\right\|$. Since $S(\tau) \ge 0$ and $H \ge 0$, one gets $\Phi_\tau(s) \le 2\,e^{-s} \in L^1(\mathbb{R}_0^+)$ and $\lim_{\tau\to+0}\Phi_\tau(s) = 0$. Then limit $\lim_{\tau\to+0}$ in the right-hand side of (3.18) is zero by the Lebesgue *dominated convergence* theorem, that yields (3.7). □

Lemma 3.4. *Let $\{X_n\}_{n>0}$ be a sequence of bounded non-negative self-adjoint operators such that $\lim_{n\to\infty}\|X_n - X\| = 0$ for a linear operator X. Then*
(i) $X \in \mathcal{L}(\mathfrak{H})$ and $X = X^* \ge 0$;
(ii) for any continuous function $g(\cdot) : [0, \|X\|] \longrightarrow \mathbb{R}$ one gets

$$\lim_{n\to\infty}\|g(X_n) - g(X)\| = 0.$$

Proof. (i) This part is a straightforward corollary of the properties of the sequence $\{X_n\}_{n\ge1}$.
(ii) Note that $\lim_{n\to\infty}\|X_n - X\| = 0$ implies $\lim_{n\to\infty}\|X_n^n - X^n\| = 0$ for $n \in \mathbb{N}$, and estimate $\|X_n\| \le \|X\| + \delta$ for any $\delta > 0$, where $n > N(\delta)$. Then

$$\lim_{n\to\infty}\|p(X_n) - p(X)\| = 0,$$

for any polynomial $p : [0, \|X\|] \longrightarrow \mathbb{R}$.

By the Weierstrass theorem, polynomials are dense in the set of continuous functions $C_\infty([0, \|X\|])$ in topology $\|\cdot\|_\infty$ of *uniform* convergence. Thus, for any given $\varepsilon > 0$, we can find polynomial $p(\cdot)$ such that $\|g - p\|_\infty = \sup_{x\in[0,\|X\|]}|g(x) - p(x)| < \varepsilon/3$. Then by spectral representation for operators X and X_n, we obtain for $n > N(\varepsilon)$:

$$\|g(X) - p(X)\| < \varepsilon/3\,,$$
$$\|g(X_n) - p(X_n)\| < \varepsilon/3\,.$$

Now taking $n > N(\delta) \wedge N(\varepsilon)$, one gets $\|p(X_n) - p(X)\| < \varepsilon/3$, which consequently yields the estimate $\|g(X_n) - g(X)\| < \varepsilon$ and therefore proves the lemma. □

Corollary 3.5. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family*

$\{S(\tau)\}_{\tau>0}$ be defined by (2.1)–(2.2)*, where H is non-negative self-adjoint operator in $\mathfrak{H}$. If*

$$\lim_{\eta\to\infty}\big\|F(t_0/\eta)^\eta - e^{-t_0 H}\big\| = 0, \tag{3.19}$$

holds for some $t_0 > 0$, then (3.10) *holds for any closed interval $\mathcal{I} \subset \mathbb{R}^+$.*

Proof. We use the representation

$$F(t/\eta)^\eta = (F(t_0/\nu)^\nu)^{t/t_0}, \quad \nu := \frac{t_0}{t}\,\eta, \quad t > 0. \tag{3.20}$$

Let $X := e^{-t_0 H}$ and $X_\nu := F(t_0/\nu)^\nu$. Then by assumption (3.19), $\lim_{\nu\to\infty}\|X_\nu - X\| = 0$. Now, let function $x \mapsto g(x) := x^{t/t_0}$ be defined for $x \geq 0$. Then by Lemma 3.4, we obtain

$$\lim_{\nu\to\infty}\big\|X_\nu^{t/t_0} - X^{t/t_0}\big\| = 0\,,$$

and by virtue of representation (3.20) it follows that $\lim_{\eta\to\infty}\big\|F(t/\eta)^\eta - e^{-tH}\big\| = 0$ holds for any $t \in \mathbb{R}^+$.

Now, proceeding as above in the proof of Theorem 3.3, one deduces (3.18), which yields (3.7). Finally, applying Theorem 3.3, we obtain (3.10) for any closed interval $\mathcal{I} \subset \mathbb{R}^+$. □

Since by definition of $\{e^{-sS(\tau)}\}_{s\geq 0}$ and by C_0-semigroup property of $\{e^{-sH}\}_{t\geq 0}$ the corresponding strong limits: s-$\lim_{s\to+0}$, are well-defined, the $\lim_{\tau\to+0}\big\|e^{-sS(\tau)} - e^{-sH}\big\| = 0$ in (3.18) is valid also for $s = 0$. A question arises: what happens if the condition (3.16) is satisfied uniformly for *any* bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$?

Theorem 3.6. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by* (2.1)–(2.2)*, where H is non-negative self-adjoint operator in $\mathfrak{H}$. Then the convergence*

$$\lim_{\tau\to+0}\sup_{t\in\mathcal{I}}\big\|e^{-tS(\tau)} - e^{-tH}\big\| = 0, \tag{3.21}$$

holds for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$ if and only if the condition

$$\lim_{\tau\to 0}\sup_{t\in\mathcal{I}}\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| = 0, \tag{3.22}$$

is valid for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$.

Proof. By conditions of theorem and by Lemma 3.1, we obtain from (3.13) the estimate

$$\sup_{t\in\mathcal{I}}\big\|e^{-tS(\tau)} - e^{-tH}\big\| \leq c\sup_{t\in\mathcal{I}}\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\|\,,$$

for $\tau > 0$ and for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$. This estimate and condition (3.22) imply the convergence in (3.21).

Conversely, assume (3.21). Note that by representation (3.17), one gets

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \int_0^\infty ds\ e^{-s}\big(e^{-s\,tS(\tau)} - e^{-s\,tH}\big)\,, \quad t \geq 0\,,$$

that yields the estimate

$$\big\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\big\| \leq \int_0^\infty ds\ e^{-s}\big\|e^{-s\,tS(\tau)} - e^{-s\,tH}\big\|,$$

for $\tau > 0$ and $t \geq 0$.

Now, let $0 < \varepsilon < 1$ and let $N_\varepsilon := -\ln(\varepsilon/2)$. Then

$$\int_{N_\varepsilon}^\infty ds\ e^{-s}\big\|e^{-s\,tS(\tau)} - e^{-s\,tH}\big\| \leq \varepsilon,$$

for $\tau > 0$ and $t \geq 0$. Hence,

$$\big\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\big\| \leq \int_0^{N_\varepsilon} ds\ e^{-s}\big\|e^{-stS(\tau)} - e^{-stH}\big\| + \varepsilon,$$

that yields

$$\sup_{t\in\mathcal{I}}\big\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\big\| \leq \sup_{\substack{t\in\mathcal{I}\\ s\in[0,N_\varepsilon]}}\big\|e^{-s\,tS(\tau)} - e^{-s\,tH}\big\| + \varepsilon,$$

for $\tau > 0$ and for any bounded interval $\mathcal{I}$ of $\mathbb{R}_0^+$. Applying (3.21), we obtain

$$\lim_{\tau\to+0}\sup_{t\in\mathcal{I}}\big\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\big\| \leq \varepsilon,$$

for any $\varepsilon > 0$. This completes the proof of (3.22). □

Now we are in position to prove another version of Theorem 3.3 for the operator-norm Chernoff approximation formula. We relax the restriction $\mathcal{I} \subset \mathbb{R}^+$ to condition $\mathcal{I} \subset \mathbb{R}_0^+$, but for (3.8) instead of (3.7).

Theorem 3.7. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by (2.1)–(2.2), where H is non-negative self-adjoint operator in $\mathfrak{H}$. Then*

$$\lim_{\eta\to\infty}\sup_{t\in\mathcal{I}}\big\|F(t/\eta)^\eta - e^{-tH}\big\| = 0, \tag{3.23}$$

for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$ if and only if

$$\lim_{\eta\to\infty}\sup_{t\in\mathcal{I}}\big\|(\mathbb{1} + tS(t/\eta))^{-1} - (\mathbb{1} + tH)^{-1}\big\| = 0, \tag{3.24}$$

is satisfied for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$.

Proof. By (3.14) and by assumption (3.24), we obtain the limit (3.23). Conversely, using (3.15) and assumption (3.23), one gets (3.21) for $\tau = t/\eta$ and for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$. Then application of Theorem 3.6 yields (3.24). □

4. Operator-norm approximation and estimates of the rate of convergence

Theorem 3.7 admits further modifications. In particular, it allows establishing estimates for the rate of operator-norm convergence.

Theorem 4.1. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by* (2.1)–(2.2)*, where H is non-negative self-adjoint operator in $\mathfrak{H}$.*
(i) *If $\rho \in (0,1]$ and there is a constant $M_\rho > 0$ such that the estimate*

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq M_\rho \left(\frac{\tau}{t}\right)^\rho , \tag{4.1}$$

holds for $\tau, t \in (0,1]$ and $0 < \tau \leq t$, then there is a constant $c_\rho > 0$ such that the estimate

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq c_\rho \left(\frac{\tau}{t}\right)^\rho , \tag{4.2}$$

is valid for $0 < \tau \leq t \leq 1$.
(ii) *If $\rho \in (0,1)$ and there is a constant c_ρ such that* (4.2) *holds, then there is a constant $M_\rho > 0$ such that the estimate* (4.1) *is valid for $0 < \tau \leq t \leq 1$.*

Proof. (i) By Lemma 3.1, there is a constant $c > 0$ such that

$$\left\|e^{-tS(\tau)} - e^{-tH}\right\| \leq c\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\|, \tag{4.3}$$

for $\tau, t > 0$. Using (4.1), we obtain

$$\left\|e^{-tS(\tau)} - e^{-tH}\right\| \leq c\, M_\rho \left(\frac{\tau}{t}\right)^\rho .$$

If $0 < \tau \leq t$, the inequality (2.5) and the spectral representation for $F(\tau)$ (2.8) yield

$$\left\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\right\| \leq \frac{\tau}{t}. \tag{4.4}$$

Then estimate

$$\left\|F(\tau)^{t/\tau} - e^{-tH}\right\| \leq \left\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\right\| + \left\|e^{-tS(\tau)} - e^{-tH}\right\| \tag{4.5}$$

gives

$$\left\|F(\tau)^{t/\tau} - e^{-tH}\right\| \leq \frac{\tau}{t} + c\, M_\rho \left(\frac{\tau}{t}\right)^\rho ,$$

for $\tau, t \in (0,1]$ with $0 < \tau \leq t$. Since for $\rho \in (0,1]$ one has $\tau/t \leq (\tau/t)^\rho$, this implies

$$\left\|F(\tau)^{t/\tau} - e^{-tH}\right\| \leq (1 + c\, M_\rho) \left(\frac{\tau}{t}\right)^\rho . \tag{4.6}$$

Setting $c_\rho := 1 + c\, M_\rho$, we prove (4.2) for $\rho \in (0,1]$.

(ii) To prove (4.1), we use the representation

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \int_0^\infty dx\, e^{-x}\big(e^{-xtS(\tau)} - e^{-xtH}\big),$$

for $\tau, t > 0$. Then we get

$$(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} = \sum_{n=0}^\infty \int_n^{n+1} dx\, e^{-x}\big(e^{-xtS(\tau)} - e^{-xtH}\big).$$

Substitution $x = y + n$ yields

$$\begin{aligned}&(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\\ &\qquad= \sum_{n=0}^\infty e^{-n} \int_0^1 dy\, e^{-y}\big(e^{-(y+n)tS(\tau)} - e^{-(y+n)tH}\big).\end{aligned}$$

Since

$$\begin{aligned}&e^{-(y+n)\,t\,S(\tau)} - e^{-(y+n)\,t\,H}\\ &\qquad= \big(e^{-n\,t\,S(\tau)} - e^{-n\,t\,H}\big)e^{-y\,t\,S(\tau)} + e^{-n\,t\,H}\big(e^{-y\,t\,S(\tau)} - e^{-y\,t\,H}\big),\end{aligned}$$

and

$$e^{-n\,tS(\tau)} - e^{-n\,tH} = \sum_{k=0}^{n-1} e^{-ktS(\tau)}\big(e^{-tS(\tau)} - e^{-tH}\big)e^{-(n-k-1)tH},$$

we get

$$\begin{aligned}&(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\\ &= \sum_{n=0}^\infty e^{-n}\Bigg\{\Bigg(\sum_{k=0}^{n-1} e^{-ktS(\tau)}\big(e^{-tS(\tau)} - e^{-tH}\big)e^{-(n-k-1)tH}\Bigg)\int_0^1 dy\, e^{-y}\, e^{-ytS(\tau)}\\ &\quad + e^{-ntH}\int_0^1 dy\, e^{-y}\,\big(e^{-ytS(\tau)} - e^{-ytH}\big)\Bigg\}.\end{aligned}$$

Hence, we obtain for $\tau, t > 0$ the estimate

$$\begin{aligned}&\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\|\\ &\quad\le \sum_{n=0}^\infty e^{-n}\Big\{n\,\big\|e^{-tS(\tau)} - e^{-tH}\big\| + \int_0^1 dy\, e^{-y}\,\big\|e^{-ytS(\tau)} - e^{-ytH}\big\|\Big\}.\end{aligned} \tag{4.7}$$

Note that assumption (4.2) and estimate (4.4) yield

$$\big\|e^{-tS(\tau)} - e^{-tH}\big\| \le (1 + c_\rho)\left(\frac{\tau}{t}\right)^\rho, \tag{4.8}$$

for $0 < \tau \le t \le 1$. To treat the last term in (4.7) we use decomposition

$$
\begin{aligned}
&\int_0^1 dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\| \\
&= \int_{\tau/t}^1 dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\| + \int_0^{\tau/t} dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\|.
\end{aligned} \tag{4.9}
$$

Hence, by (4.8) we obtain

$$
\big\|e^{-ytS(\tau)} - e^{-ytH}\big\| \le (1 + c_\rho)\left(\frac{\tau}{ty}\right)^\rho,
$$

for $\tau, t, y \in (0, 1]$ and $\tau/t \le y$. This yields the estimate

$$
\int_{\tau/t}^1 dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\| \le (1 + c_\rho) \int_0^1 dy\, e^{-y} y^{-\rho} \left(\frac{\tau}{t}\right)^\rho, \tag{4.10}
$$

for $0 < \tau \le t \le 1$ and $\rho \in (0, 1)$. Since $\rho < 1$, one gets

$$
\int_0^{\tau/t} dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\| \le\ 2\left(\frac{\tau}{t}\right)^\rho. \tag{4.11}
$$

Taking into account (4.10) and (4.11), we obtain from (4.9) the estimate

$$
\int_0^1 dy\, e^{-y}\, \big\|e^{-ytS(\tau)} - e^{-ytH}\big\| \le \left((1+c_\rho) \int_0^1 dy\, e^{-y} y^{-\rho} + 2\right) \left(\frac{\tau}{t}\right)^\rho. \tag{4.12}
$$

Finally, using (4.8) and (4.12), one gets for (4.7) the estimate

$$
\begin{aligned}
&\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \le \\
&\qquad \sum_{n=0}^{\infty} e^{-n}\Big\{n\,(1 + c_\rho) + (1 + c_\rho) \int_0^1 dy\, e^{-y} y^{-\rho} + 2\Big\} \left(\frac{\tau}{t}\right)^\rho.
\end{aligned}
$$

Now setting

$$
M_\rho := \sum_{n=0}^{\infty} e^{-n}\Big\{n\,(1 + c_\rho) + (1 + c_\rho) \int_0^1 dy\, e^{-y} y^{-\rho} + 2\Big\}, \tag{4.13}
$$

we obtain the estimate (4.1) for $0 < \tau \le t \le 1$. □

In Theorem 4.1(i) it is shown that for $\rho = 1$ the condition (4.1) implies (4.2). Since integral in (4.13) diverges for $\rho = 1$, it is *unclear* whether the converse is also true. Hence, Theorem 4.1(ii) does not cover this case.

Note that the setting $\tau = t/\eta$ transforms inequality (4.2) into

$$
\sup_{t\in[0,1]} \big\|F(t/\eta)^\eta - e^{-tH}\big\| \le c_\rho\, \frac{1}{\eta^\rho}, \quad \eta \ge 1. \tag{4.14}
$$

This inequality gives the convergence rate estimate for restricted interval: $t \in (0, 1]$, and *local* conditions: $0 < \tau \le t \le 1$. The same conditions yield generalisation of (4.14) to *any* bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$.

Theorem 4.2. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by (2.1)–(2.2), where H is a non-negative self-adjoint operator in $\mathfrak{H}$.*

If for some $\rho \in (0, 1]$ there is a constant $M_\rho > 0$ such that the estimate (4.1) holds for $\tau, t \in (0, 1]$ and $0 < \tau \le t$, then for any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$ there is a constant $c_\rho^{\mathcal{I}} > 0$ such that the estimate

$$\sup_{t\in\mathcal{I}}\left\|F(t/\eta)^\eta - e^{-tH}\right\| \le C_\rho^{\mathcal{I}}\,\frac{1}{\eta^\rho}, \tag{4.15}$$

holds for $\eta \ge 1$.

Proof. Let $N \in \mathbb{N}$ such that $\mathcal{I} \subseteq [0, N]$. Then representation

$$\begin{aligned}F(t/N\eta)^{N\eta} &- e^{-t\,H}\\ &= \sum_{k=0}^{N-1} e^{-k\ tH/N}(F(t/N\eta)^\eta - e^{-t\,H/N})F(t/N\eta)^{(N-1-k)\eta},\end{aligned}$$

yields the estimate

$$\left\|F(t/N\eta)^{N\eta} - e^{-t\,H}\right\| \le N\left\|F(t/N\eta)^\eta - e^{-t\,H/N}\right\|.$$

Let $t' := t/N$ and $\tau' := t'/\eta$, $\eta \ge 1$. Then $0 < \tau' \le t' \le 1$. Applying Theorem 4.1, we find that

$$\left\|F(t/N\eta)^\eta - e^{-tH/N}\right\| = \left\|F(\tau')^{t'/\tau'} - e^{-t'H}\right\| \le c_\rho\left(\frac{\tau'}{t'}\right)^\rho.$$

This implies for $0 < \tau' \le t' \le 1$, i.e., for $t \le N$ the estimate

$$\left\|F(t/N\eta)^{N\eta} - e^{-tH}\right\| \le c_\rho N\left(\frac{\tau'}{t'}\right)^\rho.$$

Since $\tau' = t'/\eta$, then for $\eta' := N\eta \ge 1$ we get, cf. (4.14),

$$\left\|F(t/\eta')^{\eta'} - e^{-tH}\right\| \le c_\rho N^{1+\rho}\left(\frac{1}{\eta'}\right)^\rho, \quad t \in [0, N].$$

Setting $c_\rho^{[0,N]} := c_\rho N^{1+\rho}$, we get the proof of the theorem for $\mathcal{I} = [0, N]$. Since for any bounded interval $\mathcal{I}$ one can always find a $N \in \mathbb{N}$ such that $\mathcal{I} \subseteq [0, N]$, this completes the proof. □

To extend this result to $\mathcal{I} = \mathbb{R}_0^+$ one needs *global* conditions for $0 < \tau \le t < \infty$.

Theorem 4.3. *Let $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ be a measurable family of non-negative self-adjoint contractions such that $F(0) = \mathbb{1}$. Let self-adjoint family $\{S(\tau)\}_{\tau>0}$ be defined by (2.1)–(2.2), where H is a non-negative self-adjoint operator in $\mathfrak{H}$.*

If for some $\rho \in (0,1]$ *there is a constant* $M_\rho > 0$ *such that the estimate* (4.1) *holds for* $0 < \tau \leq t < \infty$, *then there is a constant* $c_\rho^{\mathbb{R}^+} > 0$ *such that the estimate*

$$\sup_{t \in \mathbb{R}_0^+} \left\| F(t/\eta)^\eta - e^{-tH} \right\| \leq c_\rho^{\mathbb{R}^+} \frac{1}{\eta^\rho}\,, \tag{4.16}$$

holds for $\eta \geq 1$.

Proof. The line of reasoning that leads from (4.3) to the estimate (4.6) is obviously still valid if we assume $0 < \tau \leq t < \infty$. Then setting $\tau := t/\eta$, we deduce from (4.6)

$$\left\| F(t/\eta)^\eta - e^{-tH} \right\| \leq c_\rho^{\mathbb{R}^+} \frac{1}{\eta^\rho}\,, \quad \eta \geq 1,$$

where $c_\rho^{\mathbb{R}^+} := 1 + c\, M_\rho$ and $t \in \mathbb{R}_0^+$. □

For the case $\rho = 1$ the assumption (4.1) can be simplified and reduced to t-independent *canonical* form (3.7). To use Theorem 4.1 and Theorem 4.2, we return to *local* conditions: $0 < \tau \leq t \leq 1$.

Theorem 4.4. *Let* $F(\cdot) : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(\mathfrak{H})$ *be a measurable family of non-negative self-adjoint contractions such that* $F(0) = \mathbb{1}$. *Let self-adjoint family* $\{S(\tau)\}_{\tau > 0}$ *be defined by* (2.1)–(2.2), *where* H *is a non-negative self-adjoint operator in* $\mathfrak{H}$.

If there is a constant $M_1 > 0$ *such that the estimate*

$$\|(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}\| \leq M_1 \tau\,, \tag{4.17}$$

holds for $\tau \in (0,1]$, *then for any bounded interval* $\mathcal{I} \subset \mathbb{R}_0^+$ *there is a constant* $c_1^{\mathcal{I}} > 0$ *such that the estimate*

$$\sup_{t \in \mathcal{I}} \left\| F(t/\eta)^\eta - e^{-tH} \right\| \leq c_1^{\mathcal{I}} \frac{1}{\eta}\,, \tag{4.18}$$

holds for $\eta \geq 1$.

Proof. For $t > 0$ the identity

$$\begin{aligned}&(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1} \\ &= t(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}[(\mathbb{1} + S(\tau))^{-1} - (\mathbb{1} + H)^{-1}](\mathbb{1} + H)(\mathbb{1} + tH)^{-1},\end{aligned}$$

yields estimate

$$\begin{aligned}&\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \\ &\quad \leq M_1 \tau\, t\, \|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| \|(\mathbb{1} + H)(\mathbb{1} + tH)^{-1}\|\,,\end{aligned} \tag{4.19}$$

where we used condition (4.17).

Let $0 < t \leq 1$. Then $\|(\mathbb{1} + S(\tau))(\mathbb{1} + tS(\tau))^{-1}\| \leq 1/t$ and $\|(\mathbb{1} + H)((\mathbb{1} + tH)^{-1}\| \leq 1/t$. Therefore (4.19) implies estimate (4.1) for $0 < \tau \leq t \leq 1$ and $\rho = 1$. By virtue of Theorem 4.1 we obtain (4.14). Finally, applying

Theorem 4.2 for $\rho = 1$ we extend the proof of (4.18) to any bounded interval $\mathcal{I} \subset \mathbb{R}_0^+$. □

To extend Theorem 4.4 (case $\rho = 1$) to infinite interval $\mathcal{I} = \mathbb{R}_0^+$ we add more conditions, including a *global* one.

Theorem 4.5. *Let in addition to conditions of Theorem 4.4 the operator* $H \geq \mu\mathbb{1}$, $\mu > 0$. *Moreover, we assume that for any* $\varepsilon \in (0, 1]$ *there exists* $\delta_\varepsilon \in (0, 1)$ *such that*

$$0 \leq F(\tau) \leq (1 - \delta_\varepsilon)\mathbb{1}, \tag{4.20}$$

is valid for $\tau \geq \varepsilon$, *cf. Definition 1.1. If there is constant* $M_1 > 0$ *such that the* (4.17) *holds for* $\tau \in (0, \varepsilon)$, *then there exists constant* $c_1^{\mathbb{R}^+} > 0$ *such that estimate* (4.18) *is valid for infinite interval* $\mathcal{I} = \mathbb{R}_0^+$.

Proof. Since (4.17) implies the resolvent-norm convergence of $\{S(\tau)\}_{\tau>0}$, when $\tau \to +0$, and since $H \geq \mu\mathbb{1}$, there exists $0 < \mu' \leq \mu$ such that $S(\tau) \geq \mu'\mathbb{1}$ for $\tau \in (0, \varepsilon)$, where $\varepsilon \leq 1$.

On the other hand, (4.19) yields that for $t > 0$

$$\begin{aligned} &\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| = \\ &\leq M_1 \frac{\tau}{t} \|(\mathbb{1} + S(\tau))(\mathbb{1}/t + S(\tau))^{-1}\| \|(\mathbb{1} + H)(\mathbb{1}/t + H)^{-1}\| . \end{aligned} \tag{4.21}$$

Since $S(\tau) \geq \mu'\mathbb{1}$, $\tau \in (0, \varepsilon)$, and $H \geq \mu I$, for $t > 0$ we obtain estimates

$$\begin{aligned} &\|(\mathbb{1} + S(\tau))(\mathbb{1}/t + S(\tau))^{-1}\| \leq \frac{1 + \mu'}{\mu'} , \\ &\|(\mathbb{1} + H)(\mathbb{1}/t + H)^{-1}\| \leq \frac{1 + \mu}{\mu} . \end{aligned} \tag{4.22}$$

By (4.21) these estimates give

$$\|(\mathbb{1} + tS(\tau))^{-1} - (\mathbb{1} + tH)^{-1}\| \leq M_1^{\mathbb{R}^+} \frac{\tau}{t} , \tag{4.23}$$

for $\tau \in (0, \varepsilon)$ and $0 < t < \infty$. Here $M_1^{\mathbb{R}^+} := M_1 (1 + \mu')(1 + \mu)/\mu'\mu$.

Note that if $\tau/t \leq 1$, then (4.23), for $0 < \tau \leq t < \infty$ and $\tau \in (0, \varepsilon)$, satisfies conditions of Theorem 4.3. Indeed, for $\tau/t \leq 1$ and spectral representation for $F(\tau) \geq 0$ we obtain the estimate

$$\left\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\right\| \leq \frac{\tau}{t} , \quad 0 < \tau \leq t < \infty,$$

which together with (4.3) for (4.23) and (4.5) allow to extend the result (4.18) of Theorem 4.4 to the case $0 < \tau \leq t < \infty$, for $\tau = t/\eta$, $\eta \geq 1$. Since, $\tau = t/\eta < \varepsilon$, this yields

$$\left\|F(t/\eta)^\eta - e^{-tH}\right\| \leq \widehat{c}_1^{\,\mathbb{R}^+} \frac{1}{\eta} , \tag{4.24}$$

where $\widehat{c}_1^{\,\mathbb{R}^+} := 1 + c\, M_1^{\mathbb{R}^+}$ for interval $t \in [0, \varepsilon\eta)$.

Now let $t \geq \varepsilon\eta$. Then by assumption (4.20) we have

$$\|F(t/\eta)^\eta\| \leq (1-\delta_\varepsilon)^\eta = e^{\eta \ln(1-\delta_\varepsilon)}, \quad t \geq \eta\varepsilon. \tag{4.25}$$

Note that $H \geq \mu\mathbb{1}$ implies $\|e^{-tH}\| \leq e^{-\eta\varepsilon\mu}$ for $t \geq \eta\varepsilon$. This together with (4.24) and (4.25) yield the estimate

$$\left\|F(t/\eta)^\eta - e^{-tH}\right\| \leq \widehat{c}_1^{\mathbb{R}^+}\frac{1}{\eta} + e^{\eta\ln(1-\delta_\varepsilon)} + e^{-\eta\varepsilon\mu},$$

for $\varepsilon > 0$, cf. (4.23) and for *any* $t \geq 0$.

Since $\widetilde{c}_1 := \sup_{\eta\geq 1}\eta(e^{\eta\ln(1-\delta_\varepsilon)} + e^{-\eta\varepsilon\mu}) < \infty$, there exists constant $c_1^{\mathbb{R}^+} := \widehat{c}_1^{\mathbb{R}^+} + \widetilde{c}_1$ such that (4.18) is valid for $\eta \geq 1$ and infinite interval $\mathcal{I} = \mathbb{R}_0^+$ $\square$

5. Concluding remarks

1. Let the Kato functions $f, g \in K_2$ (Definition 1.1) and a measurable family of non-negative self-adjoint contractions with $F(0) = \mathbb{1}$ be defined by

$$F(t) := g(tB)^{1/2} f(tA) g(tB)^{1/2}, \quad t \geq 0.$$

Here A and B are positive self-adjoint operators in a Hilbert space $\mathfrak{H}$ with domains $\operatorname{dom} A$ and $\operatorname{dom} B$ such that the operator sum $C := A + B$ is self-adjoint on $\operatorname{dom} C = \operatorname{dom} A \cap \operatorname{dom} B$.

Then by (1.2) the family $\{F(t)\}_{t\geq 0}$ satisfies condition (4.20), i.e., Theorem 4.5 yields (1.5) in Proposition 1.2 for $H = C$ and for discrete choice of continuous parameter: $\eta = n$, where $n \in \mathbb{N}$.

2. To prove convergence of the *sequences* of *non-self-adjoint* approximants (1.6), we note that for $n \in \mathbb{N}$ and $t \geq 0$:

$$(f(tA/n)g(tB/n))^n = f(tA/n)g(tB/n)^{1/2}F(t/n)^{n-1}g(tB/n)^{1/2}.$$

Using the representation

$$\begin{aligned}
&(f(tA/n)g(tB/n))^n - e^{-tC} \\
&\quad = f(tA/n)g(tB/n)^{1/2}(F(t/n)^{n-1} - e^{-tC})g(tB/n)^{1/2} \\
&\qquad + f(tA/n)g(tB/n)^{1/2}e^{-tC}(g(tB/n)^{1/2} - \mathbb{1}) \\
&\qquad + f(tA/n)(g(tB/n)^{1/2} - \mathbb{1})e^{-tC} + (f(tA/n) - \mathbb{1})e^{-tC},
\end{aligned}$$

we obtain the following estimate:

$$\begin{aligned}
\left\|(f(tA/n)g(tB/n))^n - e^{-tC}\right\| &\leq \left\|F(t/n)^{n-1} - e^{-tC}\right\| \\
&\quad + 2\left\|(\mathbb{1} - g(tB/n)^{1/2})e^{-tC}\right\| + \left\|(\mathbb{1} - f(tA/n))e^{-tC}\right\|.
\end{aligned}$$

Since $\left\|(\mathbb{1} - g(tB/n)^{1/2})e^{-tC}\right\| \leq \left\|(\mathbb{1} - g(tB/n))e^{-tC}\right\|$, we obtain

$$\begin{aligned}
\left\|(f(tA/n)g(tB/n))^n - e^{-tC}\right\| &\leq \left\|F(t/n)^{n-1} - e^{-tC}\right\| \\
&\quad + 2\left\|(\mathbb{1} - g(tB/n))e^{-tC}\right\| + \left\|(\mathbb{1} - f(tA/n))e^{-tC}\right\|.
\end{aligned}$$

Note that by Theorem 4.5, (4.24), one gets for $c_1^{\mathbb{R}^+} > 1$ and $\eta = n - 1 \geq 1$,

$$\left\|F(t/n)^{n-1} - e^{-tC}\right\| \leq c_1^{\mathbb{R}^+} \, \frac{2}{n}\,, \quad n \geq 2\,,\ t \geq 0. \tag{5.1}$$

On the other hand, since $f, g \in K_2$ and $C = A + B$, one obtains estimates:

$$\begin{aligned} \left\|(\mathbb{1} - f(tA/n))e^{-tC}\right\| &\leq C_C^A\ \gamma[f]\ \frac{1}{n}\,, \\ \left\|(\mathbb{1} - g(tB/n))e^{-tC}\right\| &\leq C_C^B\ \gamma[g]\ \frac{1}{n}\,, \end{aligned} \tag{5.2}$$

where $\gamma[f] := \sup_{x>0}(1-f(x))/x$ and similarly for g. Collecting inequalities (5.1) and (5.2), we get for some $\Gamma > 0$ the estimate

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tC}\right\| \leq \Gamma\ \frac{1}{n}\,,$$

that proves in (1.6) the asymptotic for $n \to \infty$.

3. The proof of optimality of the asymptotic (1.5) and (1.6) is a subtle matter, see [7]. To this aim, one has to establish for convergence an estimate from below and also an example, where the operator-norm convergence is broken if operator $A+B$ is not self-adjoint, but only essentially self-adjoint.

In the present paper we developed the *lifting* topology of convergence for self-adjoint Chernoff approximation. It yields optimal estimate for the rate of convergence for Trotter–Kato product formulae. For non-self-adjoint case one uses other schemes essentially based on analyticity of semigroups, see [3, 4]. The results for quasi-sectorial contractions [10] improved by the $1/\sqrt[3]{n}$ -Theorem [11], are still not sufficiently refined to yield optimality for estimates of the rate of convergence.

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