doi: 10.17951/a.2019.73.2.21-43

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXIII, NO. 2, 2019 SECTIO A 21–43

FILIPPO BRACCI

Speeds of convergence of orbits of non-elliptic semigroups of holomorphic self-maps of the unit disk

To my friend Yuri Kozitsky on the occasion of his 70th birthday

Abstract. We introduce three quantities related to orbits of non-elliptic continuous semigroups of holomorphic self-maps of the unit disk, the total speed, the orthogonal speed, and the tangential speed and show how they are related and what can be inferred from those.

1. Introduction

Continuous semigroups of holomorphic self-maps of the unit disk $\mathbb{D}$, or for short, semigroups in $\mathbb{D}$, have been studied since the beginning of the previous century and are still a subject of interest, from the dynamical point of view, the analytic point of view, and the geometric point of view, and also, for different applications.

In this paper, we consider *non-elliptic* semigroups in $\mathbb{D}$. For such a non-elliptic semigroup (ϕ_t) it is well known that there exists a unique point

2010 *Mathematics Subject Classification.* Primary 37C10, 30C35; Secondary 30D05, 30C80, 37F99, 37C25.

Key words and phrases. Semigroups of holomorphic functions, hyperbolic geometry, dynamical systems.

Partially supported by PRIN *Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics* n.2017JZ2SW5 and by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

$\tau \in \partial\mathbb{D}$, the Denjoy–Wolff point of (ϕ_t), such that the orbits of (ϕ_t) converge to τ uniformly on compacta.

The main focus of this paper is to attach to any non-elliptic semigroup in $\mathbb{D}$, three quantities, that we call *speeds*, which have interesting properties according to the type and the dynamics of the semigroup.

The first quantity, the *total speed* $v(t)$, is nothing but the hyperbolic distance $\omega(0, \phi_t(0))$ of $\phi_t(0)$ from the origin, for $t \geq 0$. This quantity is pretty much related to the *divergence rate* as defined in [3], and, indeed, the quotient $v(t)/t$ always converges as $t \to \infty$ to the so-called *spectral value* of the semigroup. In particular, for parabolic semigroups, $v(t)/t \to 0$ as $t \to \infty$. We show with an example of a parabolic semigroup of zero hyperbolic step, whose orbits converge non-tangentially to the Denjoy–Wolff point, that for parabolic semigroups there is no better estimate, namely, $v(t)$ converges to ∞ at a speed which is always less than t but can be as close to t as wanted.

The total speed is always bounded from below by $-1/4 \log t$, in the sense that $\liminf[v(t) - \frac{1}{4}\log t] > -\infty$. However, for hyperbolic semigroups, $1/4 \log t$ can be replaced by $(\lambda/2)t$ (where $\lambda > 0$ is the spectral value) and, for parabolic semigroups of positive hyperbolic step, by $\log t$.

The total speed can be decomposed, up to a universal additive constant, as the sum of two other quantities, the *orthogonal speed* $v^o(t)$ and the *tangential speed* $v^T(t)$. This is a general fact of hyperbolic geometry which we prove in Section 3: given a curve $\gamma : [0, +\infty) \to \mathbb{D}$ starting from 0, converging to point $\sigma \in \partial\mathbb{D}$, the *orthogonal projection* of $\gamma(t)$ over $(-1, 1)\sigma$ is the (unique) point $\pi(\gamma(t)) \in (-1, 1)\sigma$ such that

$$\omega(\pi(\gamma(t)), \gamma(t)) = \inf\{\omega(r\sigma, \gamma(t)) : r \in (-1, 1)\}.$$

Then, for all $t \geq 0$,

$$\begin{aligned}\omega(\pi(\gamma(t)), \gamma(t)) + \omega(0, \pi(\gamma(t))) - \frac{1}{2}\log 2 &\leq \omega(0, \gamma(t)) \\ &\leq \omega(\pi(\gamma(t)), \gamma(t)) + \omega(0, \pi(\gamma(t))).\end{aligned}$$

Since $(-1, 1)\sigma$ is a geodesic for the hyperbolic distance, the previous formula can be considered a sort of Pytaghoras' theorem.

In case of a non-elliptic semigroup (ϕ_t), we define the *orthogonal speed* $v^o(t) := \omega(0, \pi(\phi_t(0)))$, where π is the orthogonal projection on $(-1, 1)\tau$, where τ is the Denjoy–Wolff point of (ϕ_t). We also define the *tangential speed* $v^T(t) := \omega(\phi_t(0), \pi(\phi_t(0)))$. By the previous formula,

$$v(t) \sim v^o(t) + v^T(t),$$

where, here, $\sim$ means that they have the same asymptotic behavior.

The tangential speed is related to the slope of convergence of orbits. In particular, $v^T(t) \leq C$ for some $C > 0$ and for all $t \geq 0$ if and only if the orbit $[0, \infty) \ni t \mapsto \phi_t(0)$ converges non-tangentially to the Denjoy-Wolff point.

For semigroups, another interesting relation holds, namely, for all $t \geq 0$,

$$v^T(t) \leq v^o(t) + 4\log 2.$$

The previous inequalities imply also that there exist universal constants $C_1, C_2 \in \mathbb{R}$ such that

$$v^o(t) + C_1 \leq v(t) \leq 2v^o(t) + C_2$$

for all $t \geq 0$.

The previous definitions of speeds have Euclidean counterparts and some previous results can be translated in terms of speeds using such a dictionary. It turns out that, for instance, a recent result of D. Betsakos [5] can be rephrased in terms of speeds, namely, for all non-elliptic semigroups, $v^o(t) \geq \frac{1}{4}\log t + C$ for all $t \geq 0$ and a constant $C \in \mathbb{R}$ (while, for parabolic semigroups of positive hyperbolic step, $1/4 \log t$ can be replaced by $1/2 \log t$).

Besides settling the notions of speeds and proving the aforementioned results, in this paper we provide a direct computation of total, orthogonal and tangential speeds in some cases (essentially when the image of the Koenigs function is a vertical angular sector).

The paper ends with a section of open questions which naturally arise from the developed theory.

2. Hyperbolic geometry in simply connected domain

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. We denote by $\kappa_{\mathbb{D}}(z; v)$ the hyperbolic norm of $v \in \mathbb{C}$ at $z \in \mathbb{D}$, namely,

$$\kappa_D(z; v) := \frac{|v|}{1 - |z|^2}.$$

If $\gamma : [0, 1] \to \mathbb{D}$ is a Lipschitz continuous curve, the *hyperbolic length* of γ is

$$\ell_{\mathbb{D}}(\gamma) := \int_0^1 \kappa_{\mathbb{D}}(\gamma(t); \gamma'(t))dt.$$

The integrated distance, *i.e.*, the *hyperbolic distance* in $\mathbb{D}$ is denoted by ω, namely,

$$\omega(z, w) = \inf_{\gamma} \ell_{\mathbb{D}}(\gamma),$$

where γ is any Lipschitz continuous curve joining z and w. It is well known that $\omega(z, w) = \frac{1}{2}\log\frac{1+|T_z(w)|}{1-|T_z(w)|}$, where $T_z(w) = \frac{z-w}{1-\overline{z}w}$ is an automorphism of $\mathbb{D}$.

If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $z \in \Omega$, $v \in \mathbb{C}$, given a Riemann map $f : \mathbb{D} \to \Omega$, we let

$$\kappa_\Omega(z; v) := \kappa_{\mathbb{D}}\left(f^{-1}(z); \frac{v}{f'(f^{-1}(z))}\right).$$

Similarly, we define the hyperbolic length ℓ_Ω of a curve and the hyperbolic distance k_Ω between points of Ω. By Schwarz's Lemma, all these hyperbolic quantities are invariant under biholomorphisms and are decreasing under the action of holomorphic functions.

A *geodesics* for the hyperbolic distance is a smooth curve such that the hyperbolic length among any two points of the curve coincide with the hyperbolic distance between the two points. Using the conformal invariance of the hyperbolic distance, it follows studying the case of the unit disk that for every two points there exists a unique (up to parameterization) geodesic joining the two points.

Let $\mathbb{H} := \{w \in \mathbb{C} : \mathsf{Re}\, w > 0\}$ be the right half plane.

Since $\mathbb{H}$ is biholomorphic to $\mathbb{D}$ via a Cayley transform $z \mapsto (1+z)/(1-z)$, one can easily prove that

$$k_{\mathbb{H}}(w_1, w_2) = \frac{1}{2} \log \frac{1 + \left|\frac{w_1 - w_2}{w_1 + \overline{w_2}}\right|}{1 - \left|\frac{w_1 - w_2}{w_1 + \overline{w_2}}\right|}, \quad w_1, w_2 \in \mathbb{H},$$

and

$$\kappa_{\mathbb{H}}(w; v) = \frac{|v|}{2\mathsf{Re}\, w}, \quad w \in \mathbb{H}, \ v \in \mathbb{C}. \tag{2.1}$$

Moreover, one can easily see that both lines parallel to the real axis, and arcs of circles orthogonal to the imaginary axis are geodesics in $\mathbb{H}$.

Finally, using Carathéodory's prime-ends topology (see, e.g., [13]), one can see that for any $z_0 \in \Omega$ and any prime end $\underline{x} \in \partial_C \Omega$ (here $\partial_C \Omega$ denotes the set of prime-ends of Ω endowed with the Carathéodory topology), there exists a unique geodesic $\gamma : [0, +\infty) \to \Omega$, parametrized by hyperbolic arc length, so that $\gamma(0) = z_0$ and $\gamma(t)$ converges to $\underline{x}$ in the Carathéodory topology. Indeed, this is true in $\overline{\mathbb{D}}$ with the Euclidean topology, and since Riemann mappings are isometries for the hyperbolic distance and homeomorphisms for the Carathéodory topology and $\overline{\mathbb{D}}$ is homeomorphic to $\mathbb{D} \cup \partial_C \mathbb{D}$ endowed with the Carathéodory topology, the result follows at once.

The following lemma is a straightforward computation from the very definition:

Lemma 2.1. *Let* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(1) *Let* $0 < \rho_0 < \rho_1$ *and let* $\Gamma := \{\rho e^{i\beta} : \rho_0 \leq \rho \leq \rho_1\}$. *Then,* $\ell_{\mathbb{H}}(\Gamma) = \frac{1}{2\cos\beta} \log \frac{\rho_1}{\rho_0}$. *In particular,* $k_{\mathbb{H}}(\rho_0, \rho_1) = \frac{1}{2} \log \frac{\rho_1}{\rho_0}$.

(2) *Let* $\rho_0, \rho_1 > 0$. *Then,* $k_{\mathbb{H}}(\rho_0, \rho_1 e^{i\beta}) - k_{\mathbb{H}}(\rho_0, \rho_1) \geq \frac{1}{2} \log \frac{1}{\cos\beta}$.

(3) *Let* $\rho_0 > 0$ *and* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then,* $(0, +\infty) \ni \rho \mapsto k_{\mathbb{H}}(\rho e^{i\alpha}, \rho_0 e^{i\beta})$ *has a minimum at* $\rho = \rho_0$, *it is increasing for* $\rho > \rho_0$ *and decreasing for* $\rho < \rho_0$.

(4) *Let* $\theta_0, \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\rho > 0$. *Then* $k_{\mathbb{H}}(\rho e^{i\theta_0}, \rho e^{i\theta_1}) = k_{\mathbb{H}}(e^{i\theta_0}, e^{i\theta_1})$. *Moreover,* $k_{\mathbb{H}}(1, e^{i\theta}) = k_{\mathbb{H}}(1, e^{-i\theta})$ *for all* $\theta \in [0, \pi/2)$ *and* $[0, \pi/2) \ni \theta \mapsto k_{\mathbb{H}}(1, e^{i\theta})$ *is strictly increasing.*

(5) *Let* $\beta_0, \beta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $0 < \rho_0 < \rho_1$. *Then* $k_{\mathbb{H}}(\rho_0 e^{i\beta_0}, \rho_1 e^{i\beta_1}) \geq k_{\mathbb{H}}(\rho_0, \rho_1)$.

(6) *For all* $\rho > 0$ *we have* $k_{\mathbb{H}}(\rho, \rho e^{i\beta}) \leq \frac{1}{2}\log\frac{1}{\cos\beta} + \frac{1}{2}\log 2$.

3. Hyperbolic projections, tangential and orthogonal speeds of curves in the disk

In what follows, for not burdening the notation, we will consider geodesics parameterized by (hyperbolic) arc length, but, as it will be clear, this is not relevant, and any parametrization of geodesics would work as well.

Definition 3.1. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Let $\gamma : \mathbb{R} \to \Omega$ be a geodesic parameterized by arc length. Let $z \in \Omega$. The *hyperbolic projection* $\pi_\gamma(z) \in \gamma(\mathbb{R})$ *of* z *onto* γ is the closest point (in the hyperbolic distance) of γ to z, namely,

$$k_\Omega(\pi_\gamma(z), z) = \min_{t\in\mathbb{R}} k_\Omega(\gamma(t), z).$$

Using conformal invariance, one can easily prove the following:

Proposition 3.2. *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain. Let* $\gamma : \mathbb{R} \to \Omega$ *be a geodesic in* Ω *parameterized by arc length and let* $z \in \Omega$. *Then* $\pi_\gamma(z)$ *is the point of intersection of* γ *with the geodesic* $\tilde{\gamma}$ *containing* z *and intersecting* γ *orthogonally (in the Euclidean sense).*

In particular, by Lemma 2.1(3), if $\rho e^{i\theta} \in \mathbb{H}$, $\rho > 0$ and $\theta \in (-\pi/2, \pi/2)$ and γ denotes the geodesic given by $\gamma(r) = r$, $r > 0$, then

$$\pi_\gamma(\rho e^{i\theta}) = \rho.$$

Although orthogonal projections onto geodesics are not holomorphic maps, they do not increase the hyperbolic distance:

Proposition 3.3. *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain,* $\gamma : \mathbb{R} \to \Omega$ *a geodesic parameterized by arc length. Then for every* $z, w \in \Omega$, *we have*

$$k_\Omega(\pi_\gamma(z), \pi_\gamma(w)) \leq k_\Omega(z, w).$$

Proof. Since the statement is invariant under isometries for the hyperbolic distance, using a univalent map, we can assume $\Omega = \mathbb{H}$ and the image of γ is $(0, +\infty)$. We can write $z = \rho_0 e^{i\beta_0}$ with $\rho_0 > 0$ and $\beta_0 \in (-\pi/2, \pi/2)$ and $w = \rho_1 e^{i\beta_1}$ with $\rho_1 > 0$ and $\beta_1 \in (-\pi/2, \pi/2)$. By Lemma 2.1(3), $\pi_\gamma(z) = \pi_\gamma(\rho_0 e^{i\beta_0}) = \rho_0$ and $\pi_\gamma(w) = \pi_\gamma(\rho_1 e^{i\beta_1}) = \rho_1$. Hence the result follows immediately from Lemma 2.1(5). □

Let $P, Q \in \mathbb{R}^2$ two distinct points, and R any line through P – note that a line is a geodesic for the Euclidean metric. Let $\pi_R(Q)$ denote the (Euclidean) orthogonal projection of Q onto R. By Pythagoras' Theorem, $|P - \pi_R(Q)|^2 + |Q - \pi_R(Q)|^2 = |P - Q|^2$. The next result tells that, in hyperbolic geometry, a Pythagoras' Theorem is true up to a universal constant without squaring the distances:

Proposition 3.4 (Pytaghoras' Theorem in hyperbolic geometry). *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain, $\gamma : \mathbb{R} \to \Omega$ a geodesic parameterized by arc length, $x_0 \in \gamma$ and $z \in \Omega$. Then*

$$k_\Omega(x_0, \pi_\gamma(z)) + k_\Omega(z, \gamma) - \frac{1}{2}\log 2 \leq k_\Omega(x_0, z) \leq k_\Omega(x_0, \pi_\gamma(z)) + k_\Omega(z, \gamma),$$

where $k_\Omega(z, \gamma) := \inf_{t \in \mathbb{R}} k_\Omega(z, \gamma(t)) = k_\Omega(z, \pi_\gamma(z))$.

Proof. Since the statement is invariant under isometries for the hyperbolic distance, using a univalent map, we can transfer our considerations to $\mathbb{H}$, and we can assume that $\gamma(\mathbb{R}) = (0, +\infty)$ and $x_0 = 1$.

Let $z \in \mathbb{H}$, and write $z = \rho e^{i\beta}$ with $\rho > 0$ and $\beta \in (-\pi/2, \pi/2)$. By Lemma 2.1(3), $\pi_\gamma(\rho e^{i\beta}) = \rho$. Hence, by the triangle inequality,

$$k_\mathbb{H}(1, \rho e^{i\beta}) \leq k_\mathbb{H}(1, \rho) + k_\mathbb{H}(\rho, \rho e^{i\beta}) = k_\mathbb{H}(1, \pi_\gamma(\rho e^{i\beta})) + k_\mathbb{H}(\gamma, \rho e^{i\beta}).$$

On the other hand, by Lemma 2.1(2),

$$k_\mathbb{H}(1, \rho e^{i\beta}) \geq k_\mathbb{H}(1, \rho) + \frac{1}{2}\log\frac{1}{\cos\beta}.$$

The previous equation, together with Lemma 2.1(6), gives

$$\begin{aligned} k_\mathbb{H}(1, \rho e^{i\beta}) &\geq k_\mathbb{H}(1, \rho) + \frac{1}{2}\log\frac{1}{\cos\beta} \geq k_\mathbb{H}(1, \rho) + k_\mathbb{H}(\rho, \rho e^{i\beta}) - \frac{1}{2}\log 2 \\ &= k_\mathbb{H}(1, \pi_\gamma(\rho e^{i\beta})) + k_\mathbb{H}(\gamma, \rho e^{i\beta}) - \frac{1}{2}\log 2, \end{aligned}$$

and we are done. □

The previous proposition allows to make sense to the following definition and the subsequent remarks.

Definition 3.5. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $z_0 \in \Omega$. Let $\eta : [0, +\infty) \to \Omega$ be a continuous curve such that $\eta(t)$ converges in the Carathéodory topology of Ω to a prime end $\underline{x} \in \partial_C \Omega$ as $t \to +\infty$. Let $\gamma : (-\infty, +\infty) \to \Omega$ be the geodesic of Ω parameterized by arc length such that $\gamma(0) = z_0$ and $\gamma(t) \to \underline{x}$ in the Carathéodory topology of Ω as $t \to +\infty$. The *orthogonal speed* of η is

$$v^o_{\Omega, z_0}(\eta; t) := k_\Omega(z_0, \pi_\gamma(\eta(t))).$$

The *tangential speed* $v^T_{\Omega, z_0}(\eta; t)$ of η is

$$v^T_{\Omega, z_0}(\eta; t) := k_\Omega(\gamma, \eta(t)).$$

Remark 3.6. Let Ω, z_0, $\underline{x}$, γ and η be as in Definition 3.5.

(1) The orthogonal speed and the tangential speed of a curve do not depend on the parameterization of the geodesic γ. Therefore, the definition of orthogonal speed and tangential speed depend only on Ω, z_0 and $\underline{x}$.

(2) If $\Omega, \Omega' \subsetneq \mathbb{C}$ are simply connected domains, $z_0 \in \Omega$, $z_0' \in \Omega'$ and $f : \Omega \to \Omega'$ is a biholomorphism such that $f(z_0) = z_0'$, then $v^o_{\Omega,z_0}(\eta; t) = v^o_{\Omega',z_0'}(f \circ \eta; t)$ and $v^T_{\Omega,z_0}(\eta; t) = v^T_{\Omega',z_0'}(f \circ \eta; t)$ for all $t \geq 0$. This follows immediately since f is an isometry for the hyperbolic distances of Ω and Ω'.

The actual orthogonal speed and tangential speed of a curve depend on the base point chosen, but, asymptotically they do not:

Lemma 3.7. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $z_0, z_1 \in \Omega$. Then for every $\underline{x} \in \partial_C\Omega$ and for every continuous curve $\eta : [0, +\infty) \to \Omega$ converging to $\underline{x}$ in the Carathéodory topology of Ω, we have*

(1) $\lim_{t\to+\infty} v^o_{\Omega,z_0}(\eta; t) = +\infty$,
(2) $\lim_{t\to+\infty} |v^T_{\Omega,z_0}(\eta; t) - v^T_{\Omega,z_1}(\eta; t)| = 0$,
(3) $\limsup_{t\to+\infty} |v^o_{\Omega,z_0}(\eta; t) - v^o_{\Omega,z_1}(\eta; t)| \leq k_\Omega(z_0, z_1)$.

Proof. By Remark 3.6(2), up to composing with a biholomorphism from $\mathbb{H}$ to Ω, we can assume $\Omega = \mathbb{H}$, $z_0 = 1$ and $\underline{x}$ is the prime end of $\mathbb{H}$ which corresponds to "∞", namely, the prime end defined by the null chain $\{(n+1)e^{i\theta} : |\theta| < \pi/2\}_{n\in\mathbb{N}}$. Hence, $\lim_{t\to+\infty} |\eta(t)| = +\infty$. Moreover, the geodesic in $\mathbb{H}$ which joins 1 to $\underline{x}$ is $\gamma_0(r) := r$, $r \in (0, +\infty)$. While, the geodesic in $\mathbb{H}$ which joins $z_1 := x + iy$ to $\underline{x}$ is $\gamma_1(r) := r + iy$, $r \in (0, +\infty)$.

From Lemma 2.1(3), we have $\pi_{\gamma_0}(\eta(t)) = |\eta(t)|$. This shows in particular that

$$v^o_{\mathbb{H},1}(\eta; t) = k_\mathbb{H}(1, \pi_{\gamma_0}(\eta(t))) = k_\mathbb{H}(1, |\eta(t)|) \to +\infty,$$

as $t \to +\infty$, and (1) follows.

On the other hand, using the automorphism $z \mapsto z - iy$ which maps γ_0 onto γ_1 and taking into account that it is an isometry for $k_\mathbb{H}$, we see that $\pi_{\gamma_1}(\eta(t)) = |\eta(t) - iy| + iy$.

Therefore,

$$\begin{aligned}|v^T_{\mathbb{H},1}(\eta; t) - v^T_{\mathbb{H},x+iy}(\eta; t)| &= |k_\mathbb{H}(\eta(t), \pi_{\gamma_0}(\eta(t))) - k_\mathbb{H}(\eta(t), \pi_{\gamma_1}(\eta(t)))| \\ &\leq k_\mathbb{H}(\pi_{\gamma_0}(\eta(t)), \pi_{\gamma_1}(\eta(t))) = k_\mathbb{H}(|\eta(t)|, |\eta(t) - iy| + iy).\end{aligned}$$

Taking into account that $\lim_{t\to+\infty} |\eta(t)| = +\infty$, a direct computation shows that

$$\lim_{t\to+\infty} k_\mathbb{H}(|\eta(t)|, |\eta(t) - iy| + iy) = 0, \tag{3.1}$$

and hence (2) follows.

Now, using the triangle inequality,

$$\begin{aligned}|v^o_{\mathbb{H},1}(\eta;t)-v^o_{\mathbb{H},x+iy}(\eta;t)| &= |k_{\mathbb{H}}(1,\pi_{\gamma_0}(\eta(t)))-k_{\mathbb{H}}(x+iy,\pi_{\gamma_1}(\eta(t)))|\\ &= |k_{\mathbb{H}}(1,\pi_{\gamma_0}(\eta(t)))-k_{\mathbb{H}}(x+iy,\pi_{\gamma_0}(\eta(t)))\\ &\quad + k_{\mathbb{H}}(x+iy,\pi_{\gamma_0}(\eta(t)))-k_{\mathbb{H}}(x+iy,\pi_{\gamma_1}(\eta(t)))|\\ &\leq k_{\mathbb{H}}(1,x+iy)+k_{\mathbb{H}}(\pi_{\gamma_0}(\eta(t)),\pi_{\gamma_1}(\eta(t)))\\ &= k_{\mathbb{H}}(1,x+iy)+k_{\mathbb{H}}(|\eta(t)|,|\eta(t)-iy|+iy),\end{aligned}$$

and thus (3) follows from (3.1). □

The reason for the name "tangential speed" follows from the following property:

Proposition 3.8. *Let $\eta:[0,+\infty)\to\mathbb{D}$ be a continuous curve converging to a point $\sigma\in\partial\mathbb{D}$. Let*

$$t_0 := \inf\{s\geq 0 : \mathsf{Re}\,(\overline{\sigma}\eta(t))\geq 0\ \forall t\in[s,+\infty)\}.$$

Then for all $t\geq t_0$,

$$\left|\omega(0,\eta(t))-\frac{1}{2}\log\frac{1}{1-|\eta(t)|}\right|\leq\frac{1}{2}\log 2,$$

$$\left|v^o_{\mathbb{D},0}(\eta;t)-\frac{1}{2}\log\frac{1}{|\sigma-\eta(t)|}\right|\leq\frac{1}{2}\log 2,$$

$$\left|v^T_{\mathbb{D},0}(\eta;t)-\frac{1}{2}\log\frac{|\sigma-\eta(t)|}{1-|\eta(t)|}\right|\leq\frac{3}{2}\log 2.$$

Proof. Since $\eta(t)\to\sigma$ as $t\to+\infty$, it follows that $t_0<+\infty$.

The first equation follows immediately from the very definition of ω. Indeed, for every $t\geq 0$,

$$\left|\omega(0,\eta(t))-\frac{1}{2}\log\frac{1}{1-|\eta(t)|}\right|=\frac{1}{2}\log(1+|\eta(t)|)<\frac{1}{2}\log 2.$$

happened

In order to prove the other two equations, up to change η with $\overline{\sigma}\eta$, we can assume without loss of generality that $\sigma=1$. Let $C:\mathbb{D}\to\mathbb{H}$ be the Cayley transform given by $C(z)=\frac{1+z}{1-z}$. For every $t\geq 0$, let us write $\rho_t e^{i\theta_t}:=C(\eta(t))$, with $\rho_t>0$ and $\theta_t\in(-\pi/2,\pi/2)$. This implies in particular, that $\rho_t\geq 1$ for all $t\geq t_0$. Then, for $t\geq t_0$, we have

$$\begin{aligned}v^o_{\mathbb{D},0}(\eta;t) &= v^o_{\mathbb{H},1}(\rho_t e^{i\theta_t};t)=k_{\mathbb{H}}(1,\rho_t)=\frac{1}{2}\log\rho_t\\ &= \frac{1}{2}\log|C(\eta(t))^o|=\frac{1}{2}\log\frac{|1+\eta(t)|}{|1-\eta(t)|},\end{aligned}\tag{3.2}$$

where, the first equality follows from Remark 3.6(2), the second equality follows from the definition of orthogonal speed and since the orthogonal projection of $\rho_t e^{i\theta_t}$ onto the geodesic $(0,+\infty)$ is ρ_t by Lemma 2.1(3), and the third equality follows from Lemma 2.1(1).

Therefore, by (3.2), and taking into account that for $t \geq t_0$, we have $|1+\eta(t)| \geq 1 + \mathsf{Re}\,\eta(t) \geq 1$,

$$\left| v^o_{\mathbb{D},0}(\eta;t) - \frac{1}{2}\log\frac{1}{|1-\eta(t)|}\right| = \frac{1}{2}\log|1+\eta(t)| \leq \frac{1}{2}\log 2.$$

As for the last inequality, from Proposition 3.4, we have

$$\omega(0,\eta(t)) - v^o_{\mathbb{D},0}(\eta;t) \leq v^T_{\mathbb{D},0}(\eta;t) \leq \omega(0,\eta(t)) - v^o_{\mathbb{D},0}(\eta;t) + \frac{1}{2}\log 2,$$

and using the previous two inequalities for the estimates of $\omega(0,\eta(t))$ and $v^o_{\mathbb{D},0}(\eta;t)$, we get the result. □

Remark 3.9. As a consequence of the previous proposition, we have that if $\eta : [0,+\infty) \to \mathbb{D}$ is a continuous curve such that $\lim_{t\to+\infty}\eta(t) = \sigma \in \partial\mathbb{D}$, then η converges to σ non-tangentially if and only if $\limsup_{t\to+\infty} v^T_{\mathbb{D},0}(\eta;t) < +\infty$.

4. Continuous non-elliptic semigroups of holomorphic self-maps of the unit disk

In this paper we consider only non-elliptic (continuous) semigroups of holomorphic self-maps of the unit disk. We refer the reader to, e.g., [1, 2, 7, 16, 22, 24, 19, 20, 4, 8, 9, 10, 18, 21, 23, 25, 26, 27, 28] for all unproved statements and more on the subject.

A *continuous non-elliptic semigroups of holomorphic self-maps of the unit disk*, or just a *non-elliptic semigroup* for short, is a family (ϕ_t) such that for every $t \geq 0$, $\phi_t : \mathbb{D} \to \mathbb{D}$ is holomorphic, with no fixed point in $\mathbb{D}$ for $t > 0$, $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \geq 0$, $\phi_0(z) = z$ for all $z \in \mathbb{D}$ and $[0,+\infty) \ni t \mapsto \phi_t$ is continuous with respect to the topology of uniform convergence on compacta of $\mathbb{D}$.

If (ϕ_t) is a non-elliptic semigroup in $\mathbb{D}$, there exists a point $\tau \in \partial\mathbb{D}$, the *Denjoy–Wolff point* of (ϕ_t) such that $\lim_{t\to\infty}\phi_t(z) = \tau$ for all $z \in \mathbb{D}$, and the convergence is uniform on compacta.

Moreover, the angular derivative $\phi_t'(\tau)$ of ϕ_t at τ exists for all $t \geq 0$ and there exists $\lambda \geq 0$, *the spectral value of* (ϕ_t) such that

$$\phi_t'(\tau) = e^{-\lambda t}$$

for all $t \geq 0$.

If (ϕ_t) is a semigroup in $\mathbb{D}$, there exists an (essentially unique) *holomorphic model* $(\Omega, h, z+it)$, where $h : \mathbb{D} \to \mathbb{C}$ is univalent, $h(\mathbb{D})$ is starlike at infinity (namely, $h(\mathbb{D}) + it \subseteq h(\mathbb{D})$ for all $t \geq 0$) and $h(\phi_t(z)) = h(z) + it$ for all $z \in \mathbb{D}$ and $t \geq 0$. Moreover, $\Omega = \bigcup_{t\geq 0} h(\mathbb{D}) - it$ and we have the following cases: Ω is either a strip $\mathbb{S}_r := \{z \in \mathbb{C} : 0 < \mathsf{Re}\, z < r\}$ (where $r = \pi/\lambda$ with $\lambda > 0$ the spectral value of (ϕ_t)), or the right half plane $\mathbb{H}$, or the left half plane $\mathbb{H}^- := \{w \in \mathbb{C} : \mathsf{Re}\, w < 0\}$ or $\mathbb{C}$. The holomorphic model is universal in the sense that any other (semi)conjugation of (ϕ_t) factorizes through it (see [17, 3]). The map h is called the *Koenigs function* of (ϕ_t).

The semigroup is *hyperbolic* if Ω is a strip, it is *parabolic* otherwise. Moreover, parabolic semigroups are *of finite hyperbolic step* if Ω is a half plane, or *of zero hyperbolic step* if $\Omega = \mathbb{C}$.

This definition is equivalent to the classical one, for which a semigroup (ϕ_t) is hyperbolic provided its spectral value is > 0, it is parabolic if its spectral value is 0, and the hyperbolic step is positive if

$$\lim_{t\to\infty} \omega(\phi_t(z), \phi_{t+1}(z)) > 0$$

for some – and hence any – $z \in \mathbb{D}$. The last equivalence follows from the fact that $k_\Omega(z, w) = \lim_{t\to\infty} \omega(\phi_t(z), \phi_t(w))$ (see [3]).

5. Speeds of non-elliptic semigroups

Since the orbits of a non-elliptic semigroup converge to the Denjoy–Wolff point on $\partial\mathbb{D}$, one might study the tangential and orthogonal speed of convergence. First of all, we show that the (asymptotic behavior of) orthogonal speed and the tangential speed of an orbit of a semigroup do not depend on the starting point:

Lemma 5.1. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Let $z_1, z_2 \in \mathbb{D}$ and let $\eta_j : [0, +\infty) \to \mathbb{D}$ be the continuous curve defined by $\eta_j(t) := \phi_t(z_j)$, $j = 1, 2$. Then for every $t \geq 0$*

$$|v^o_{\mathbb{D},0}(\eta_1; t) - v^o_{\mathbb{D},0}(\eta_2; t)| \leq \omega(z_1, z_2),$$

$$|v^T_{\mathbb{D},0}(\eta_1; t) - v^T_{\mathbb{D},0}(\eta_2; t)| \leq 2\omega(z_1, z_2).$$

Proof. Let $\gamma : (-1, 1) \to \mathbb{D}$ be the geodesic of $\mathbb{D}$ defined by $\gamma(r) = r\tau$. For $z \in \mathbb{D}$ let $\pi_\gamma(z)$ be the orthogonal projection of z onto γ. Then, by the very definition of orthogonal speed of curves and Proposition 3.3, we have

$$\begin{aligned}
|v^o_{\mathbb{D},0}(\eta_1; t) - v^o_{\mathbb{D},0}(\eta_2; t)| &= |\omega(0, \pi_\gamma(\eta_1(t))) - \omega(0, \pi_\gamma(\eta_2(t)))| \\
&\leq \omega(\pi_\gamma(\eta_1(t)), \pi_\gamma(\eta_2(t))) \leq \omega(\eta_1(t), \eta_2(t)) \\
&= \omega(\phi_t(z_1), \phi_t(z_2)) \leq \omega(z_1, z_2).
\end{aligned}$$

A similar argument proves the second inequality. Namely,

$$\begin{aligned}
v^T_{\mathbb{D},0}(\eta_1; t) &= \omega(\phi_t(z_1), \pi_\gamma(\phi_t(z_1))) \\
&\leq \omega(\phi_t(z_1), \phi_t(z_2)) + \omega(\phi_t(z_2), \pi_\gamma(\phi_t(z_2))) \\
&\quad + \omega(\pi_\gamma(\phi_t(z_2)), \pi_\gamma(\phi_t(z_1))) \\
&= \omega(\phi_t(z_1), \phi_t(z_2)) + v^T_{\mathbb{D},0}(\eta_2; t) + \omega(\pi_\gamma(\phi_t(z_2)), \pi_\gamma(\phi_t(z_1))) \\
&\leq 2\omega(z_1, z_2) + v^T_{\mathbb{D},0}(\eta_2; t).
\end{aligned}$$

That is, $v^T_{\mathbb{D},0}(\eta_1;t) - v^T_{\mathbb{D},0}(\eta_2;t) \le 2\omega(z_1,z_2)$. Changing the role of z_1 and z_2, we obtain the second inequality of the statement. □

Lemmas 5.1 and 3.7 show that, in order to study the asymptotic behavior of the speed of convergence of semigroups' orbits to the Denjoy–Wolff point, it is enough to study the orbit starting at 0 and considering the speed with respect to 0. In other words, the following definition makes sense:

Definition 5.2. Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. For $t \ge 0$, we let

$$v(t) := \omega(0, \phi_t(0)),$$

and call $v(t)$ the *total speed of* (ϕ_t).

Also, let $\gamma : (-1,1) \to \mathbb{D}$ be the geodesic of $\mathbb{D}$ defined by $\gamma(r) := r\tau$ and let $\pi_\gamma : \mathbb{D} \to \gamma((-1,1))$ be the orthogonal projection. For $t \ge 0$, we let

$$v^o(t) := v^o_{\mathbb{D},0}(\phi_t(0);t) := \omega(0, \pi_\gamma(\phi_t(0))),$$

and call $v^o(t)$ the *orthogonal speed of* (ϕ_t). Finally, we let

$$v^T(t) := v^T_{\mathbb{D},0}(\phi_t(0);t) := \omega(\phi_t(0), \pi_\gamma(\phi_t(0))),$$

and call $v^T(t)$ the *tangential speed of* (ϕ_t).

Remark 5.3. It follows immediately from Remark 3.9 that the orbit $[0,+\infty) \ni t \mapsto \phi_t(z)$ converges non-tangentially to τ for some – and hence any – $z \in \mathbb{D}$ if and only if $\limsup_{t\to+\infty} v^T(t) < +\infty$.

It follows from Lemma 2.1 and the previous considerations that, if (ϕ_t) is a non-elliptic semigroup in $\mathbb{D}$ with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$, and $C(z) = (\tau+z)/(\tau-z)$ (a biholomorphism from $\mathbb{D}$ to $\mathbb{H}$), setting $\rho_t e^{i\theta_t} = C(\phi_t(C^{-1}(1)))$ with $\rho_t > 0$ and $\theta_t \in (-\pi/2, \pi/2)$, then

$$v^o(t) \sim \frac{1}{2}\log \rho_t, \quad v^T(t) \sim \frac{1}{2}\log\cos\frac{1}{\theta_t}. \tag{5.1}$$

By Proposition 3.4, if (ϕ_t) is a non-elliptic semigroup, we have

$$v^o(t) + v^T(t) - \frac{1}{2}\log 2 \le v(t) \le v^o(t) + v^T(t). \tag{5.2}$$

A second less immediate relation between the orthogonal speed and the tangential speed is contained in the following proposition:

Proposition 5.4. *If (ϕ_t) is a non-elliptic semigroup in $\mathbb{D}$, then, for every $t \ge 0$,*

$$v^T(t) \le v^o(t) + 4\log 2. \tag{5.3}$$

Proof. Let $\tau \in \partial\mathbb{D}$ be the Denjoy–Wolff point of (ϕ_t) and let $\lambda \geq 0$ be its spectral value. By Julia's Lemma, for every $t \geq 0$

$$\frac{|\tau - \phi_t(0)|}{1 - |\phi_t(0)|} \leq 4\sqrt{\frac{e^{-\lambda t}}{1 - |\phi_t(0)|^2}},$$

which is equivalent to

$$e^{\lambda t}\frac{1 + |\phi_t(0)|}{1 - |\phi_t(0)|} \leq \frac{16}{|\tau - \phi_t(0)|^2}.$$

Applying the function $x \mapsto \frac{1}{2}\log x$ to the previous inequality, we obtain for every $t \geq 0$,

$$\begin{aligned}\frac{1}{2}\log\frac{1}{1 - |\phi_t(0)|} &\leq \frac{\lambda t}{2} + \frac{1}{2}\log\frac{1}{1 - |\phi_t(0)|} + \frac{1}{2}\log(1 + |\phi_t(0)|)\\ &\leq \frac{1}{2}\log 16 + \log\frac{1}{|\tau - \phi_t(0)|}.\end{aligned}$$

Therefore, by Proposition 3.8, we have for all $t \geq 0$,

$$\begin{aligned}v(t) &\leq \frac{1}{2}\log\frac{1}{1 - |\phi_t(0)|} + \frac{1}{2}\log 2\\ &\leq \frac{1}{2}\log 16 + \log\frac{1}{|\tau - \phi_t(0)|} + \frac{1}{2}\log 2\\ &\leq \frac{1}{2}\log 16 + \frac{3}{2}\log 2 + 2v^o(t) = 2v^o(t) + \frac{7}{2}\log 2.\end{aligned}$$

Hence, by (5.2), we have for all $t \geq 0$,

$$v^o(t) + v^T(t) \leq v(t) + \frac{1}{2}\log 2 \leq 2v^o(t) + \frac{7}{2}\log 2 + \frac{1}{2}\log 2.$$

Finally, the previous equation implies that $v^T(t) \leq v^o(t) + 4\log 2$ for all $t \geq 0$, and we are done. □

The speeds of convergence are essentially invariant under conjugation:

Proposition 5.5. *Let (ϕ_t) and (ψ_t) be two non-elliptic semigroups in $\mathbb{D}$. Suppose there exists $M \in \mathsf{Aut}(\mathbb{D})$ such that $\phi_t = M^{-1} \circ \psi_t \circ M$ for all $t \geq 0$. Denote by $v(t), v^o(t), v^T(t)$ (*respectively, *$\tilde{v}(t), \tilde{v}^o(t), \tilde{v}^T(t)$) the total speed, orthogonal speed and tangential speed of (ϕ_t) (*respect. *of (ψ_t)). Then there exists $C > 0$ such that for all $t \geq 0$*

$$\begin{aligned}&|v(t) - \tilde{v}(t)| < C,\\ &|v^o(t) - \tilde{v}^o(t)| < C,\\ &|v^T(t) - \tilde{v}^T(t)| < C.\end{aligned}$$

Proof. Let $\tau \in \partial\mathbb{D}$ be the Denjoy-Wolff point of (ϕ_t) and $\tilde{\tau} \in \partial\mathbb{D}$ that of (ψ_t). Let $\gamma : (0, +\infty) \to \mathbb{D}$ (respectively, $\tilde{\gamma} : (0, +\infty) \to \mathbb{D}$) be the geodesic in $\mathbb{D}$ parameterized by arc length such that $\gamma(0) = 0$ (respect., $\tilde{\gamma}(0) = 0$) and $\lim_{t\to+\infty} \gamma(t) = \tau$ (respect., $\lim_{t\to+\infty} \gamma(t) = \tilde{\tau}$).

Since M is an isometry for the hyperbolic distance, for all $t \geq 0$,

$$v(t) = \omega(0, \phi_t(0)) = \omega(0, (M^{-1} \circ \psi_t \circ M)(0)) = \omega(M(0), \psi_t(M(0)).$$

Hence, for all $t \geq 0$,

$$\begin{aligned} |v(t) - \tilde{v}(t)| &= |\omega(M(0), \psi_t(M(0)) - \omega(0, \psi_t(0))| \\ &\leq |\omega(M(0), \psi_t(M(0)) - \omega(0, \psi_t(M(0))| \\ &\quad + |\omega(0, \psi_t(M(0)) - \omega(0, \psi_t(0))| \\ &\leq \omega(M(0), 0) + \omega(\psi_t(M(0)), \psi_t(0)) \leq 2\omega(M(0), 0) =: C_0. \end{aligned}$$

Moreover, since M is an isometry for the hyperbolic distance, the curve $\gamma_1 : (0, +\infty) \to \mathbb{D}$ defined by $\gamma_1 := M^{-1} \circ \gamma$ is a geodesic in $\mathbb{D}$ parameterized by arc length. Hence, for all $t \geq 0$,

$$v^T(t) = \omega(\phi_t(0), \gamma) = \omega(M^{-1}(\phi_t(0)), \gamma_1) = \omega(\psi_t(M^{-1}(0)), \gamma_1).$$

By Lemma 3.7, $\lim_{t\to+\infty} |\tilde{v}^T(t) - \omega(\psi_t(M^{-1}(0), \gamma_1))| = 0$, thus there exists $C_1 > 0$ such that $|v^T(t) - \tilde{v}^T(t)| < C_1$ for all $t \geq 0$.

Finally, by (5.2), we have for all $t \geq 0$,

$$v^o(t) - \tilde{v}^o(t) \leq v(t) - v^T(t) + \frac{1}{2}\log 2 - \tilde{v}(t) + \tilde{v}^T(t) \leq C_0 + C_1 + \frac{1}{2}\log 2.$$

The same argument proves that $\tilde{v}^o(t) - v^o(t) \leq C_0 + C_1 + \frac{1}{2}\log 2$, and we are done. □

If Ω is a domain starlike at infinity, and $p \in \Omega$, we let

$$\Omega^+ := \Omega \cup \{w \in \mathbb{C} : \mathsf{Re}\, w > \mathsf{Re}\, p\}, \quad \Omega^- := \Omega \cup \{w \in \mathbb{C} : \mathsf{Re}\, w < \mathsf{Re}\, p\}.$$

Note that $\Omega^\pm$ is a domain starlike at infinity. Moreover, for any open set $D \subset \mathbb{C}$ and $p \in D$, we let

$$\delta_D(p) = \inf\{|z - p| : z \in \mathbb{C} \setminus D\}.$$

The following result is a consequence of [12] and Remark 5.3:

Theorem 5.6. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$, with the Koenigs function h. Let $p \in h(\mathbb{D})$. Then* $\limsup_{t\to\infty} v^T(t) < +\infty$ *if and only if there exists $C > 0$ such that*

$$\frac{1}{C}\min\{t, \delta_{h(\mathbb{D})^+}(p + it)\} \leq \min\{t, \delta_{h(\mathbb{D})^-}(p + it)\} \leq C\min\{t, \delta_{h(\mathbb{D})^+}(p + it)\}$$

for all $t \geq 0$.

In particular, if (ϕ_t) is hyperbolic, there exists $C > 0$ such that $v^T(t) \leq C$ for all $t \geq 0$. Hence, for hyperbolic semigroups, $v^o(t) \sim v(t)$.

Note that this implies that, in particular, for hyperbolic semigroup the orthogonal speed is *essentially monotone*, in the sense that, if (ϕ_t) is a hyperbolic semigroup with the Koenigs function h, total speed $v(t)$ and orthogonal speed $v^o(t)$ and $(\tilde{\phi}_t)$ is a hyperbolic semigroup with the Koenigs function $\tilde{h}$ and $h(\mathbb{D}) \subset \tilde{h}(\mathbb{D})$, total speed $\tilde{v}(t)$ and orthogonal speed $\tilde{v}^o(t)$, then by (5.2),

$$v^o(t) \geq \tilde{v}^o(t) + C$$

for all $t \geq 0$ and some $C > 0$, since in the previous case, $v(t) \geq \tilde{v}(t)$ for all $t \geq 0$ by the monotonicity of the hyperbolic distance.

6. Total speed of convergence

In this section we consider the total speed of convergence of orbits of hyperbolic and parabolic semigroups to the Denjoy–Wolff point.

Proposition 6.1. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$, with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and $\phi_t'(\tau) = e^{-\lambda t}$ for $\lambda \geq 0$ and $t \geq 0$ (in particular, (ϕ_t) is hyperbolic if $\lambda > 0$, parabolic otherwise). Then*

$$\lim_{t\to+\infty} \frac{v(t)}{t} = \lim_{t\to+\infty} \frac{v^o(t)}{t} = \frac{\lambda}{2}, \tag{6.1}$$

and

$$\lim_{t\to+\infty} \frac{v^T(t)}{t} = 0.$$

Proof. By [3],

$$\frac{\lambda}{2} = \lim_{t\to+\infty} \frac{\omega(0, \phi_t(0))}{t} = \lim_{t\to+\infty} \frac{v(t)}{t}.$$

In case $\lambda = 0$, that is, (ϕ_t) is parabolic, it follows immediately from (5.2) that

$$\lim_{t\to+\infty} \frac{v^o(t)}{t} = \lim_{t\to+\infty} \frac{v^T(t)}{t} = 0.$$

In case $\lambda > 0$, that is, (ϕ_t) is hyperbolic, we have already noticed that $\limsup_{t\to+\infty} v^T(t) < +\infty$. Thus from (5.2) we have the result. □

According to the type of the semigroup, we have also a simple lower bound on the total speed:

Proposition 6.2. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$, with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$.*

- *If (ϕ_t) is hyperbolic with spectral value $\lambda > 0$, then*

$$\liminf_{t\to+\infty} \left[v(t) - \frac{\lambda}{2} t \right] > -\infty,$$

- *if (ϕ_t) is parabolic of positive hyperbolic step, then*

$$\liminf_{t\to+\infty} [v(t) - \log t] > -\infty,$$

- *if (ϕ_t) is parabolic of zero hyperbolic step, then*

$$\liminf_{t\to+\infty} \left[v(t) - \frac{1}{4} \log t \right] > -\infty.$$

Proof. Let (ϕ_t) be hyperbolic with spectral value $\lambda > 0$. The canonical model of (ϕ_t) is $(\mathbb{S}_{\frac{\pi}{\lambda}}, h, z+it)$. Hence, for every $t \geq 0$,

$$\begin{aligned} v(t) &= \omega(0, \phi_t(0)) = k_{h(\mathbb{D})}(h(0), h(\phi_t(0))) \\ &= k_{h(\mathbb{D})}(h(0), h(0)+it) \geq k_{\mathbb{S}_{\pi/\lambda}}(h(0), h(0)+it) \\ &\geq k_{\mathbb{S}_{\pi/\lambda}}\left(\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}+it\right) - k_{\mathbb{S}_{\pi/\lambda}}\left(\frac{\pi}{2\lambda}, h(0)\right) - k_{\mathbb{S}_{\pi/\lambda}}\left(h(0)+it, \frac{\pi}{2\lambda}+it\right) \\ &= \frac{\lambda}{2}t - 2k_{\mathbb{S}_{\pi/\lambda}}\left(\frac{\pi}{2\lambda}, h(0)\right), \end{aligned}$$

where the last equality follows from a direct computation and taking into account that $k_{\mathbb{S}_{\pi/\lambda}}(h(0)+it, \frac{\pi}{2\lambda}+it) = k_{\mathbb{S}_{\pi/\lambda}}(h(0), \frac{\pi}{2\lambda})$ for all $t \in \mathbb{R}$ since $z \mapsto z+it$ is an automorphism of $\mathbb{S}_{\frac{\pi}{\lambda}}$. From this, the result for hyperbolic semigroups follows at once.

Now, assume that (ϕ_t) is parabolic of positive hyperbolic step. We can assume that its canonical model is $(\mathbb{H}, h, z+it)$ (in case the canonical model is $(\mathbb{H}^-, h, z+it)$ the argument is similar). Arguing as in the hyperbolic case, we see that

$$v(t) \geq k_{\mathbb{H}}(1, 1+it) + C,$$

for some constant $C \in \mathbb{R}$ and every $t \geq 0$. Now, write $1+it = \rho_t e^{i\theta_t}$ for $\rho_t > 0$ and $\theta_t \in [0, \pi/2)$. A simple computation shows that $\rho_t = \sqrt{1+t^2}$ and $\cos\theta_t = \frac{1}{\sqrt{1+t^2}}$. Therefore, by Lemma 2.1(1) and (2), we have

$$k_{\mathbb{H}}(1, 1+it) \geq k_{\mathbb{H}}\left(1, \sqrt{1+t^2}\right) + \frac{1}{2}\log\sqrt{1+t^2} = \log\sqrt{1+t^2} \geq \log t,$$

and the result follows in this case as well.

Finally, in case (ϕ_t) is parabolic of zero hyperbolic step, the canonical model is $(\mathbb{C}, h, z+it)$. Since $h(\mathbb{D})$ is starlike at infinity and is different from $\mathbb{C}$, there exists $p \in \mathbb{C}$ such that $p - it \notin h(\mathbb{D})$ for all $t \geq 0$ and $p + it \in h(\mathbb{D})$ for all $t > 0$. Hence, $h(\mathbb{D}) \subseteq \mathcal{K}_p$, where $\mathcal{K}_p$ is the Koebe domain $\mathbb{C} \setminus \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = \operatorname{Re} p, \operatorname{Im}\zeta \leq \operatorname{Im} p\}$. Therefore, arguing as in the previous cases, we find $C \in \mathbb{R}$ such that for every $t \geq 0$,

$$v(t) \geq k_{\mathcal{K}_p}(p+i, p+ti) + C = k_{\mathcal{K}_0}(i, ti) + C.$$

Taking into account that the map $\mathcal{K}_0 \ni z \mapsto \sqrt{-iz} \in \mathbb{H}$ is a biholomorphism, where the branch of the square root is chosen so that $\sqrt{1} = 1$, we have by Lemma 2.1(1)

$$k_{\mathcal{K}_0}(i, ti) = k_{\mathbb{H}}\left(1, \sqrt{t}\right) = \frac{1}{4}\log t,$$

and we are done. □

Remark 6.3. The bound given by Proposition 6.2 is sharp. Indeed, as it is clear from the proof, if (ϕ_t) is a hyperbolic group in $\mathbb{D}$ with spectral value $\lambda > 0$, then there exists $C > 0$ such that $|v(t) - \frac{\lambda}{2}t| < C$ for every $t \geq 0$, while, if (ϕ_t) is a parabolic group, then there exists $C > 0$ such that

$|v(t) - \log t| < C$ for every $t \geq 0$ – so that, in this sense, non-elliptic groups in $\mathbb{D}$ have the lowest total speed. Moreover, the semigroup (ϕ_t) in $\mathbb{D}$ defined as $\phi_t(z) := h^{-1}(h(z) + it)$, $z \in \mathbb{D}$, where $h : \mathbb{D} \to \mathcal{K}_0$ is a Riemann map for the Koebe domain $\mathcal{K}_0$, has the property that there exists $C > 0$ such that $|v(t) - \frac{1}{4}\log t| < C$ for all $t \geq 0$.

A direct consequence of Proposition 6.1 and Proposition 6.2 is the following:

Corollary 6.4. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$. Then*

$$\liminf_{t\to+\infty} \frac{v(t)}{\log t} > 0, \quad \limsup_{t\to+\infty} \frac{v(t)}{t} < +\infty.$$

As it is clear from the proof of the previous proposition, one can get lower or upper estimates on the total speed of convergence according to the geometry of the image of the Koenigs function using the domain monotonicity of the hyperbolic distance. We provide here an example of such situation by studying a particular case.

For $\alpha, \beta \in [0, \pi]$, with $\alpha + \beta > 0$, we denote

$$V(\alpha, \beta) := \left\{ re^{i\theta} : r > 0, \ -\alpha < \theta < \beta \right\}.$$

Proposition 6.5. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with the Koenigs function h. Suppose $h(\mathbb{D}) = p + iV(\alpha, \beta)$ for some $\alpha, \beta \in (0, \pi]$ with $\alpha + \beta > 0$.*

(1) *If $\alpha > 0$, $\beta > 0$, then there exists $C > 0$ such that $v^T(t) \leq C$ and*

$$\left| v^o(t) - \frac{\pi}{2(\alpha+\beta)} \log t \right| \leq C, \quad \left| v(t) - \frac{\pi}{2(\alpha+\beta)} \log t \right| \leq C,$$

for all $t \geq 0$.

(2) *If either $\alpha = 0$ or $\beta = 0$, then there exists $C > 0$ such that for all $t \geq 0$*

$$\left| v(t) - \frac{\pi + \alpha + \beta}{2(\alpha+\beta)} \log t \right| \leq C$$

$$\left| v^o(t) - \frac{\pi}{2(\alpha+\beta)} \log t \right| \leq C$$

$$\left| v^T(t) - \frac{1}{2} \log t \right| \leq C.$$

Proof. Without loss of generality, up to a translation, we can assume that $p = 0$. Moreover, by Lemma 5.1, in order to get asymptotic estimates of $v(t)$ and $v^o(t)$, it is enough to estimate $\omega(z_0, \phi_t(z_0))$ for any suitably chosen $z_0 \in \mathbb{D}$. Note that $\omega(z_0, \phi_t(z_0)) = k_V(h(z_0), h(z_0) + it)$, where $V := V(\alpha, \beta)$.

In case $\alpha, \beta > 0$, we choose $h(z_0) = i$. Note that $V = R(W)$, where $R(z) = ie^{i(\beta-\alpha)/2}z$ and

$$W := \{\rho e^{i\theta} : \rho > 0, |\theta| < (\alpha+\beta)/2\}.$$

Hence, taking into account that $h(z_0) = i$, we have

$$k_V(h(z_0), h(z_0) + it) = K_W\left(e^{i(\alpha-\beta)/2}, e^{i(\alpha-\beta)/2}(1+t)\right).$$

The map $f : W \to \mathbb{H}$ given by $f(w) := w^{\pi/(\alpha+\beta)}$ is a biholomorphism. Therefore, if we set $\theta_0 := \frac{\pi(\alpha-\beta)}{2(\alpha+\beta)}$, we have

$$k_V(h(z_0), h(z_0) + it) = k_{\mathbb{H}}\left(e^{i\theta_0}, e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}\right).$$

Now, by Lemma 2.1(6),

$$\begin{aligned}
&\left|k_{\mathbb{H}}\left(e^{i\theta_0}, e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}\right) - k_{\mathbb{H}}\left(1, (1+t)^{\pi/(\alpha+\beta)}\right)\right| \\
&\leq \left|k_{\mathbb{H}}\left(e^{i\theta_0}, e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}\right) - k_{\mathbb{H}}\left(1, e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}\right)\right| \\
&\quad + \left|k_{\mathbb{H}}\left(1, e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}\right) - k_{\mathbb{H}}\left(1, (1+t)^{\pi/(\alpha+\beta)}\right)\right| \\
&\leq k_{\mathbb{H}}\left(1, e^{i\theta_0}\right) + k_{\mathbb{H}}\left(e^{i\theta_0}(1+t)^{\pi/(\alpha+\beta)}, (1+t)^{\pi/(\alpha+\beta)}\right) \\
&\leq k_{\mathbb{H}}\left(1, e^{i\theta_0}\right) + \frac{1}{2}\log\frac{2}{\cos\theta_0}.
\end{aligned}$$

Since $k_{\mathbb{H}}(1, (1+t)^{\pi/(\alpha+\beta)}) = \frac{1}{2}\log(1+t)^{\pi/(\alpha+\beta)}$, the previous considerations show that there exists $C > 0$ such that

$$\left|k_V(h(z_0), h(z_0) + it) - \frac{\pi}{2(\alpha+\beta)}\log t\right| < C$$

for all $t \geq 0$, and we are done in case $\alpha, \beta > 0$.

Now we assume that $\beta = 0$ (the case $\alpha = 0$ being similar). In this case, we choose $h(z_0) = e^{i(\pi-\alpha)/2}$ (note that $(0, +\infty) \ni t \mapsto te^{i(\pi-\alpha)/2}$ is the symmetry axis of V). Arguing as before, one can see that

$$k_V(h(z_0), h(z_0) + it) = k_W(1, 1 + te^{i\alpha/2}).$$

We write $1 + te^{i\alpha/2} = \rho_t e^{i\theta_t}$. Since $f : W \to \mathbb{H}$ defined as $f(w) = w^{\pi/\alpha}$ is a biholomorphism, we have

$$k_W(1, 1 + te^{i\alpha/2}) = k_{\mathbb{H}}(1, \rho_t^{\pi/\alpha} e^{i(\theta_t\pi)/\alpha}).$$

By Proposition 3.4,

$$|k_{\mathbb{H}}(1, \rho_t^{\pi/\alpha} e^{i(\theta_t\pi)/\alpha}) - k_{\mathbb{H}}(1, \rho_t^{\pi/\alpha}) - k_{\mathbb{H}}(\rho_t^{\pi/\alpha}, \rho_t^{\pi/\alpha} e^{i(\theta_t\pi)/\alpha})| \leq \frac{1}{2}\log 2.$$

Hence, we are left to compute $k_{\mathbb{H}}(1,\rho_t^{\pi/\alpha}) + k_{\mathbb{H}}(\rho_t^{\pi/\alpha}, \rho_t^{\pi/\alpha} e^{i(\theta_t\pi)/\alpha})$. By Lemma 2.1, we have

$$k_{\mathbb{H}}(1,\rho_t^{\pi/\alpha}) = \frac{\pi}{2\alpha}\log\rho_t, \quad k_{\mathbb{H}}(\rho_t^{\pi/\alpha}, \rho_t^{\pi/\alpha} e^{i(\theta_t\pi)/\alpha}) = k_{\mathbb{H}}(1, e^{i(\theta_t\pi)/\alpha}),$$

and

$$\left| k_{\mathbb{H}}(1, e^{i(\theta_t\pi)/\alpha}) - \frac{1}{2}\log\frac{1}{\cos(\frac{\theta_t\pi}{\alpha})}\right| < \frac{1}{2}\log 2.$$

Therefore, there exists $C>0$ such that

$$\left| k_V(h(z_0), h(z_0)+it) - \frac{\pi}{2\alpha}\log\rho_t - \frac{1}{2}\log\frac{1}{\cos(\frac{\theta_t\pi}{\alpha})}\right| < C.$$

Now,

$$\rho_t = \sqrt{t^2+2\cos(\alpha/2)t+1}, \quad \cos\theta_t = \frac{1+\cos(\alpha/2)t}{\rho_t}.$$

Clearly, $\lim_{t\to+\infty}\frac{\rho_t}{t} = 1$, which implies that $\frac{\pi}{2\alpha}\log\rho_t$ goes like $\frac{\pi}{2\alpha}\log t$ as $t\to+\infty$. Let us analyze the asymptotic behavior of the term $\frac{1}{2}\log\frac{1}{\cos(\frac{\theta_t\pi}{\alpha})}$. Notice that $\lim_{t\to+\infty}\cos\theta_t = \cos(\alpha/2)$ and $\lim_{t\to+\infty}(\rho_t - t) = \cos(\alpha/2)$. Applying the Mean Value Theorem to the function $g(x) = \arccos(x)$, we deduce that for each $x\in[0,1]$ there is a point ξ in the interval of extremes points x and $\cos(\alpha/2)$ such that

$$g(x) - \frac{\alpha}{2} = g'(\xi)(x-\cos(\alpha/2)).$$

Taking $x=\cos(\theta_t)$ we deduce that there is ξ_t in the interval of extremes points $\cos\theta_t$ and $\cos(\alpha/2)$ such that

$$\theta_t - \frac{\alpha}{2} = -\frac{1}{\sqrt{1-\xi_t^2}}(\cos(\theta_t)-\cos(\alpha/2)).$$

Clearly, we have that $\lim_{t\to+\infty}\xi_t = \cos(\alpha/2)$. Thus,

$$\lim_{t\to+\infty}\frac{\cos(\theta_t)-\cos(\alpha/2)}{\theta_t-\frac{\alpha}{2}} = -\lim_{t\to+\infty}\sqrt{1-\xi_t^2} = -\sin(\alpha/2).$$

Therefore

$$\begin{aligned}\lim_{t\to+\infty} t\cos\left(\frac{\theta_t\pi}{\alpha}\right) &= \frac{\pi}{\alpha}\lim_{t\to+\infty} t\left(\theta_t-\frac{\alpha}{2}\right)\frac{\cos(\frac{\theta_t\pi}{\alpha})}{\theta_t\frac{\pi}{\alpha}-\frac{\pi}{2}} = -\frac{\pi}{\alpha}\lim_{t\to+\infty} t\left(\theta_t-\frac{\alpha}{2}\right)\\ &= \frac{\pi}{\alpha\sin(\alpha/2)}\lim_{t\to+\infty} t\,(\cos\theta_t-\cos(\alpha/2))\\ &= \frac{\pi}{\alpha\sin(\alpha/2)}\lim_{t\to+\infty}\frac{t}{\rho_t}\,(1+\cos(\alpha/2)(t-\rho_t))\\ &= \frac{\pi}{\alpha\sin(\alpha/2)}\left(1-\cos^2(\alpha/2)\right) = \frac{\pi}{\alpha}\sin(\alpha/2)\in(0,+\infty).\end{aligned}$$

Thus, $\frac{1}{2}\log\frac{1}{\cos(\frac{\theta_t\pi}{\alpha})}$ goes like $\frac{1}{2}\log t$ as $t\to+\infty$ and the result follows. □

In Proposition 6.1, we showed that if (ϕ_t) is a parabolic semigroup in $\mathbb{D}$, then $v(t)/t\to 0$ as $t\to+\infty$. This is essentially the only possible upper bound, as the following proposition shows:

Proposition 6.6. *Let $g:[0,+\infty)\to[0,+\infty)$ be a function such that $\lim_{t\to+\infty} g(t)=+\infty$ and $\lim_{t\to+\infty}\frac{g(t)}{t}=0$. Then there exists a parabolic semigroup (ϕ_t) in $\mathbb{D}$ of zero hyperbolic step such that*

$$\limsup_{t\to+\infty}\frac{v(t)}{g(t)}=+\infty.$$

Proof. Let $\{a_j\}$ be a strictly increasing sequence of positive real numbers, $a_1>0$, $\lim_{j\to+\infty}a_j=+\infty$. Let $\{b_j\}$ be a strictly increasing sequence of positive real numbers to be chosen later on. Let

$$\Omega:=\mathbb{C}\setminus\left(\bigcup_{j=1}^{\infty}\{z\in\mathbb{C}:\mathsf{Re}\,z=\pm a_j,\mathsf{Im}\,z\le b_j\}\right).$$

Note that Ω is simply connected and starlike at infinity. Let $h:\mathbb{D}\to\Omega$ be a Riemann map such that $h(0)=0$, and let $\phi_t(z):=h^{-1}(h(z)+it)$ for $z\in\mathbb{D}$ and $t\ge 0$. Then (ϕ_t) is a semigroup in $\mathbb{D}$ and, since $\bigcup_{t\ge0}(\Omega-it)=\mathbb{C}$, it follows that (ϕ_t) is parabolic of zero hyperbolic step.

In order to estimate the total speed $v(t)$ of (ϕ_t), note that Ω is symmetric with respect to the imaginary axis $i\mathbb{R}$, hence the orbit $[0,+\infty)\ni t\mapsto it$ is a geodesic in Ω, and so is $[0,+\infty)\ni t\mapsto\phi_t(0)$ in $\mathbb{D}$.

In particular, if we set $\gamma(t)=it$, we have

$$\begin{aligned} v(t)=\omega(0,\phi_t(0))=k_\Omega(0,it)&=\int_0^t\kappa_\Omega(\gamma(r);\gamma'(r))dr\\ &\ge\frac{1}{4}\int_0^t\frac{dr}{\delta_\Omega(ir)}, \end{aligned}\tag{6.2}$$

where the last inequality follows from the classical estimates on the hyperbolic metric (see, *e.g.*, [10])

Now, we claim that we can choose the b_j's in such a way that for every $j\ge1$ there exists $x_j\in(b_j,b_{j+1})$ such that $\delta_\Omega(it)=a_{j+1}$ for every $t\in[x_j,b_{j+1}]$ and such that

$$b_{j+1}-x_j\ge ja_{j+1}g(b_{j+1}).\tag{6.3}$$

Indeed, set $b_1=1$. Let $x_1>1$ be such that $|ix_1-(a_1+ib_1)|=a_2$. Simple geometric consideration shows that, if we take $b_2>x_1$ then $\delta_\Omega(it)=a_2$ for every $t\in[x_1,b_2]$. Moreover, since $g(t)/t\to0$ as $t\to+\infty$, we can find $b_2>x_1$ such that

$$\frac{a_2g(b_2)+x_1}{b_2}<1.$$

Therefore, there exist x_1, b_2 such that (6.3) is satisfied for $j = 1$. Now, we can argue by induction is a similar way. Suppose we constructed $b_1, \ldots, b_j$ and $x_1, \ldots, x_{j-1}$ for $j > 1$. Then we select x_j in such a way that $|ix_j - (a_j + ib_j)| = a_{j+1}$ and, again since $g(t)/t \to 0$ as $t \to +\infty$, we choose $b_{j+1} > x_j$ such that $\frac{ja_{j+1}g(b_{j+1})+x_j}{b_{j+1}} < 1$.

Thus, by (6.2) and (6.3), we have

$$v(b_{j+1}) \geq \frac{1}{4}\int_0^{b_{j+1}} \frac{dr}{\delta_\Omega(ir)} \geq \frac{1}{4}\int_{x_j}^{b_{j+1}} \frac{dr}{a_{j+1}} = \frac{b_{j+1} - x_j}{4a_{j+1}} \geq \frac{jg(b_{j+1})}{4}.$$

Therefore,

$$\frac{v(b_{j+1})}{g(b_{j+1})} \geq \frac{j}{4},$$

hence $\limsup_{t\to+\infty} \frac{v(t)}{g(t)} = +\infty$, and we are done. □

7. Orthogonal speed of convergence of parabolic semigroups

In this section we give estimates on the orthogonal speed of convergence of semigroups. Since the orbits of hyperbolic semigroups converge non-tangentially to the Denjoy–Wolff point, it follows from (5.2) that the total and the orthogonal speeds of hyperbolic semigroups have the same asymptotic behavior. Therefore, we concentrate on parabolic semigroups.

In order to simplify the notation, for any $\alpha \in (0, \pi]$, we write

$$V(\alpha) := V(\alpha, 0) = \left\{w = \rho e^{i\theta} : \rho > 0, |\theta| < \alpha\right\}.$$

The first part of the following result follows immediately from the fact that $h(\mathbb{D})$ is contained in the Koebe domain $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathsf{Re}\, z = \mathsf{Re}\, p, \mathsf{Im}\, z \leq \mathsf{Im}\, p\}$, where $p \in \mathbb{C} \setminus h(\mathbb{D})$ and Proposition 6.5. Whereas, the second part is a deep result in [6], where the analogue Euclidean expression is estimated using harmonic measure theory (and then the result in terms of speed follows from Proposition 3.8).

Theorem 7.1. *Let (ϕ_t) be a parabolic semigroup, not a group, in $\mathbb{D}$ with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and the Koenigs function h. Suppose that $h(\mathbb{D})$ is contained in a sector $p + iV(\alpha)$, $p \in \mathbb{C}$, $\alpha \in (0, \pi]$. Then*

$$\liminf_{t\to+\infty} \left[v(t) - \frac{\pi}{4\alpha}\log t\right] > -\infty,$$

and,

$$\liminf_{t\to+\infty} \left[v^o(t) - \frac{\pi}{4\alpha}\log t\right] > -\infty.$$

Remark 7.2. The previous bounds are sharp, as shown by Proposition 6.5.

In general, we have the following bounds (which was proved in its Euclidean counterpart by D. Betsakos [5]):

Theorem 7.3. *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$.*

(1) $\liminf_{t\to+\infty}[v^o(t) - \frac{1}{4}\log t] > -\infty$.

(2) *If, in addition, the semigroup is of positive hyperbolic step, then* $\liminf_{t\to+\infty}[v^o(t) - \frac{1}{2}\log t] > -\infty$.

Proof. Let $(\Omega, h, z \mapsto z + it)$ be the canonical model of the semigroup.

(1) Take a point $p \in \mathbb{C} \setminus h(\mathbb{D})$. Since h is starlike at infinity, $h(\mathbb{D}) \subset p + iV(\pi)$ and the result follows immediately from Theorem 7.1.

(2) By (5.2) and (5.3), we have

$$v(t) \leq 2v^o(t) + 4\log 2.$$

Hence, by Proposition 6.2,

$$\liminf_{t\to+\infty}\left[v^o(t) - \frac{1}{2}\log t\right] \geq \frac{1}{2}\liminf_{t\to+\infty}[v(t) - \log t - 2\log 2] > -\infty.$$

□

Remark 7.4. The bounds given by Theorem 7.3 are sharp (see Proposition 6.5).

Remark 7.5. Proposition 6.2, Theorem 7.3 and (5.2) imply at once that if (ϕ_t) is a non-elliptic semigroup in $\mathbb{D}$ and there exists a constant $C > 0$ such that for all $t \geq 0$

$$|v^o(t) - \frac{1}{4}\log t| < C,$$

then $\limsup_{t\to+\infty} v^T(t) < +\infty$ and hence $[0, +\infty) \ni t \mapsto \phi_t(z)$ converges non-tangentially to the Denjoy–Wolff point for every $z \in \mathbb{D}$.

8. Open Questions

The previous results give rise to the following questions:

Question 1: Suppose (ϕ_t) is a non-elliptic semigroup in $\mathbb{D}$. Is it true that $\limsup_{t\to\infty}[v^T(t) - \frac{1}{2}\log t] < +\infty$?

Question 2: Suppose (ϕ_t) is a parabolic semigroup in $\mathbb{D}$ of positive hyperbolic step. Is it true that $|v^T(t) - \frac{1}{2}\log t| < C$ for some constant $C > 0$? If so, does this condition characterize parabolic semigroups of positive hyperbolic step?

Question 3: Suppose (ϕ_t) is a parabolic semigroup in $\mathbb{D}$. Is it possible to characterize, in dynamical terms, when $\lim_{t\to\infty} v^T(t) = \infty$ and $\lim_{t\to\infty}\frac{v^T(t)}{v^o(t)} = 0$?

Question 4: Is it true that the orthogonal speed is essentially monotone? Namely, suppose (ϕ_t), $(\tilde{\phi}_t)$ are a parabolic semigroup in $\mathbb{D}$ with Koenigs' functions h and $\tilde{h}$ and orthogonal speeds $v^o(t)$ and $\tilde{v}^0(t)$, respectively. Suppose $h(\mathbb{D}) \subset \tilde{h}(D)$. Is it true that $\liminf_{t\to\infty}[v^o(t) - \tilde{v}^o(t)] > -\infty$?

Question 5: Does there exist a non-elliptic semigroup whose total speed (respect. orthogonal speed) does not have a precise asymptotic value? Namely, does there exist a parabolic semigroup such that $\limsup_{t\to\infty} \frac{v(t)}{g(t)} = \infty$ and $\liminf_{t\to\infty} \frac{v(t)}{g(t)} = 0$ for some function $g : [0, +\infty) \to [0, +\infty)$ such that $\lim_{t\to\infty} g(t) = \infty$?

References

[1] Abate, M., *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, Rende, 1989.

[2] Aharonov, D., Elin, M., Reich, S., Shoikhet, D., *Parametric representation of semi-complete vector fields on the unit balls of $\mathbb{C}^n$ and in Hilbert space*, Atti Accad. Naz. Lincei **10** (1999), 229–253.

[3] Arosio, L., Bracci, F., *Canonical models for holomorphic iteration*, Trans. Amer. Math. Soc. **368** (5) (2016), 3305–3339.

[4] Berkson, E., Porta, H., *Semigroups of holomorphic functions and composition operators*, Michigan Math. J. **25** (1978), 101–115.

[5] Betsakos, D., *On the rate of convergence of parabolic semigroups of holomorphic functions*, Anal. Math. Phys. **5** (2015), 207–216.

[6] Betsakos, D., Contreras, M. D., Díaz-Madrigal, S., *On the rate of convergence of semigroups of holomorphic functions at the Denjoy–Wolff point*, to appear in Rev. Mat. Iberoamericana.

[7] Bracci, F., Gumenyuk, P., *Contact points and fractional singularities for semigroups of holomorphic self-maps in the unit disc*, J. Anal. Math. **130** (1) (2016), 185–217.

[8] Bracci, F., Contreras, M. D., Díaz-Madrigal, S., *Topological invariants for semigroups of holomorphic self-maps of the unit disc*, J. Math. Pures Appl. **107** (1) (2017), 78–99.

[9] Bracci, F., Contreras, M. D., Díaz-Madrigal, S., *On the Koenigs function of semigroups of holomorphic self-maps of the unit disc*, Anal. Math. Phys. **8** (4) (2018), 521–540.

[10] Bracci, F., Contreras, M. D., Díaz-Madrigal, S., Gaussier, H., *Backward orbits and petals of semigroups of holomorphic self-maps of the unit disc,*. Ann. Math. Pura Appl. **198** (2) (2019), 411–441.

[11] Bracci, F., Contreras, M. D., Díaz-Madrigal, S., Gaussier, H., *Non-tangential limits and the slope of trajectories of holomorphic semigroups of the unit disc*, Trans. Amer. Math. Soc. **373** (2) (2020), 939–969.

[12] Bracci, F., Contreras, M. D., Díaz-Madrigal, S., Gaussier, H., Zimmer, A., *Asymptotic behavior of orbits of holomorphic semigroups*, J. Math. Pures Appl. doi:10.1016/j.matpur.2019.05.005 online print.

[13] Collingwood, E. F., Lohwater, A. J., *The theory of Cluster Sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56 Cambridge Univ. Press, Cambridge, 1966.

[14] Contreras, M. D., Díaz-Madrigal, S., *Analytic flows on the unit disk: angular derivatives and boundary fixed points*, Pacific J. Math., **222** (2005), 253–286.

[15] Contreras, M. D., Díaz-Madrigal, S., Pommerenke, Ch., *Fixed points and boundary behavior of the Koenigs function*, Ann. Acad. Sci. Fenn. Math. **29** (2004), 471–488.

[16] Contreras, M. D., Díaz-Madrigal, S., Pommerenke, Ch., *On boundary critical points for semigroups of analytic functions*, Math. Scand. **98** (2006), 125–142.

[17] Cowen, C. C., *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), 69–95.

[18] Elin, M., Jacobzon, F., *Parabolic type semigroups: asymptotics and order of contact*, Anal. Math. Phys. **4** (2014), 157–185.

[19] Jacobzon, F., Reich, S., Shoikhet, D., *Linear fractional mappings, invariant sets, semigroups and commutativity*, J. Fixed Point Theory Appl. **5** (2009), 63–91.

[20] Jacobzon, F., Levenshtein, M., Reich, S., *Convergence characteristics of one-parameter continuous semigroups*, Anal. Math. Phys. **1** (2011), 311–335.

[21] Elin, M., Khavinson, D., Reich, S., Shoikhet, D., *Linearization models for parabolic dynamical systems via Abel's functional equation*, Ann. Acad. Sci. Fen. Math. **35** (2010), 439–472.

[22] Elin, M., Levenshtein, M., Reich, S., Shoikhet, D., *Commuting semigroups of holomorphic mappings*, Math. Scand. **103** (2008), 295–319.

[23] Elin, M., Shoikhet, D., *Linearization Models for Complex Dynamical Systems. Topics in Univalent Functions, Functional Equations and Semigroup Theory*, Birkhäuser, Basel, 2010.

[24] Elin, M., Reich, S., Shoikhet, D., Yacobzon, F., *Rates of convergence of one-parameter semigroups with boundary Denjoy–Wolff fixed points*, in: *Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2008, 43–58.

[25] Elin, M., Shoikhet, D., Zalcman, L., *A flower structure of backward flow invariant domains for semigroups*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 3–34.

[26] Shoikhet, D., *Semigroups in Geometrical Function Theory*, Kluwer Academic Publishers, Dordrecht, 2001.

[27] Siskakis, A. G., *Semigroups of Composition Operators and the Cesàro Operator on* $H^p(D)$, Ph. D. Thesis, University of Illinois, 1985.

[28] Siskakis, A. G., *Semigroups of composition operators on spaces of analytic functions, a review*, Contemp. Math. **213**, Amer. Math. Soc., Providence, RI, 1998, 229–252.

Filippo Bracci
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133, Roma
Italia
e-mail: `fbracci@mat.uniroma2.it`

Received July 16, 2019