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Matrix Elements of the Pairing Hamiltonian for the $D(\lambda, 0)$ Representation of the R_5 Group in the (n, T, T_0) Basis

Elementy macierzowe hamiltonianu *pairing* dla reprezentacji grupy R_5 w bazie (n, T, T_0)

Использование матричных элементов гамильтониана парных взаимодействий для представления группы R_5 в базисе (n, T, T_0)

I. INTRODUCTION

It is well known [1] that the ortogonal group in the five-dimmensional quasi-spin space plays an important role in the pairing nuclear forces. Pairing Hamiltonian, which can be written in jj coupling as

$$H = -G\left\{S_+^n S_-^n + S_+^p S_-^p + \frac{1}{2} S_+^{np} S_-^{np}\right\}^* , \tag{1}$$

although noninvariant under the transformation of the R_5 group, has definite transformation property, as the operators in (1) are at the same time the generators of the transformations of the group R_5.

The question arises whether the pairing interaction is more or less real as a residual interaction between the nucleons. The answer depends also on the assumption about the pairing correlation between the same kind of nucleons or, in addition, between the two kinds of them. We will not, however, discuss this question here, stressing only the possibility of considering the pairing correlations between protons and, separately, between neutrons in the frame of quasi-spin formalism.

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* We follow the notation of [7].

The correlations between nucleons as a whole can also be considered in the quasi-spin method, but only, for the jj coupling, in the part of $T = 1$ where T is the isotopic spin for the two interacting particles.

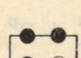
The aim of the paper is to give the matrix elements of the operators appearing in (1) by means of which one can then calculate all nonvanishing matrix elements of the pairing Hamiltonian in the mixing configurations. The calculations begin with the construction of the basis for a given irreducible representation of the R_5 group. The irreducible representations of the R_5 group have to be labelled by the two numbers, and the basis vectors are distinguished by the next four numbers.

In spite of this general rule, there are three classes of irreducible representations in R_5, the basis of which can be labelled by three numbers only [8]. They are, using the labelling of [9], $D(0, \lambda)$, $D(1, \lambda)$, $D(\lambda, 0)$. The first two representations were considered in [8] and here we present the calculations for the third one. We adopted the basis in which the isospin of the states remains diagonal, i. e. we took the basis $|st; nTT_0>$ where s (seniority number) and t (reduced isotopic spin) label the representation and n (number of particles), T, and T_0 label the states within the irreducible representation. For the representation under consideration, $D(\lambda, 0)$, we simply obtain

$$s = \lambda, \quad t = \frac{\lambda}{2}. \tag{2}$$

II. CONSTRUCTION OF THE BASIS

The graph for the state of the highest weight in the representation is

λ × 

[8] with $j = \frac{2\lambda - 1}{2}$ where j is the one-particle angular momentum. This means that we consider as many as possible n-particle configurations on the j level with given, s, t and within the irreducible representation $D(\lambda, 0)$. Let us denote the state of the highest weight (3) by the ket $|M> \equiv |S_0 = T = T_0 = \frac{\lambda}{2}>$. By applying the proper function of the infinitesimal operators R_5 to the state of the highest weight one can obtain the state of given $(S_0\ T\ T_0)$. The main part of the construction of such a function is to construct the lowering operator of T. The candidate to decrease the T by one while acting on the state $|M>$ is

$$\Omega \equiv S_{-}^{n} + c\,T_{-} S_{-}^{np} \tag{4}$$

where c is taken to fulfil the condition

$$\mathbf{T}^2 (\Omega \mid M>) = \frac{\lambda}{2}\left(\frac{\lambda}{2} - 1\right)\left(\Omega \mid M>\right). \tag{5}$$

Using the commutations relations of the operators involved [7], we get $c = \frac{1}{\lambda}$. Unfortunately, the operator $(S^{n} + CT - S^{np})^2$ does not diminish the T by two. Instead of that, we have to take the operator

$$\Omega^{(2)} \equiv (S_{-}^{n})^2 + C_1 S_{-}^{n} T_{-} S_{-}^{np} + C_2 (T_{-})^2 (S_{-}^{np})^2 \tag{6}$$

with

$$C_1 = \frac{2}{\lambda - 2}, \qquad C_2 = \frac{1}{(\lambda - 1)(\lambda - 2)} \tag{7}$$

given by the condition

$$\mathbf{T}^2 (\Omega^{(2)} \mid M>) = \left(\frac{\lambda}{2} - 1\right)\left(\frac{\lambda}{2} - 2\right)(\Omega^{(2)} \mid M>). \tag{8}$$

In general, the non-normalizing state with a given T can be constructed by the operator

$$\Omega^{(k)} = \sum_{i=0}^{k} \frac{k!\,(\lambda - 2k + 1)!}{i!\,(k - i)!\,(\lambda - 2k + 1 + i)!} (S_{-}^{n})^{k-i} (T_{-})^{i} (S_{-}^{np})^{i} \tag{9}$$

which has the property

$$\Omega^{(k)} \mid S_0 = T = T_0 = \frac{\lambda}{2} > = \mid S_0 = T = T_0 = \frac{\lambda}{2} - k >. \tag{10}$$

The lowering operator of T_0 and S_0 can simply be taken as $(T_{-})^a$ and $(S_{-}^{np})^b$, and the complete operator is

$$(T_{-})^a \Omega^{(k)} (S_{-}^{np})^b = \sum_{i=0}^{k} \frac{k!\,(\lambda - 2k + 1)!}{i!\,(\lambda - i)!\,(\lambda - 2k + 1 + i)!} (S_{-}^{n})^{k-i} (T_{-})^{a+i} (S_{-}^{np})^{b+i} \tag{11}$$

because $[T_{-}, S_{-}^{n}] = 0$. The operators $(T_{-})^a$ and $(S_{-}^{np})^b$ do not affect the T, as

$$[\mathbf{T}^2, (T_{-})^a] = 0 \qquad \text{and} \qquad [\mathbf{T}^2, (S_{-}^{np})^b] \mid M> = 0. \tag{12}$$

Thus the normalizing basis vector is

$$\mid a\,b\,k> = N\,(T_{-})^a\, \Omega^{(k)}\, (S_{-}^{np})^b \mid M> \tag{13}$$

where

$$T = \frac{\lambda}{2} - k \qquad T_0 = \frac{\lambda}{2} - k - a \qquad S_0 = \frac{\lambda}{2} - k - b. \tag{14}$$

By the normalizing condition we get

$$N(a\,b\,k) = \left\{ \frac{(\lambda - k + 1)!\,(\lambda - 2k - a)!\,(\lambda - 2k - b)!}{\lambda!\,a!\,b!\,k!\,(\lambda - 2k)!\,(\lambda - 2k + 1)!} \right\}^{1/2}. \tag{15}$$

III. THE MATRIX ELEMENTS OF THE $S_{\pm}$ OPERATORS

We begin with the S^n_- operator. The physical meaning of the operator gives immediately

$$\begin{aligned} S^n_- | S_0\, T\, T_0 > \rightarrow & | S_0 - 1, T - 1, T_0 - 1 > \\ + & | S_0 - 1, \quad T, \quad T_0 - 1 > \\ + & | S_0 - 1\; T + 1, T_0 - 1 >. \end{aligned} \tag{16}$$

In the (abk) representation, taking the normalizing kets (13), we obtain the identity

$$\begin{aligned} S^n_- | a\,b\,k > = & \frac{N(a\,b\,k)}{N(a, b, k+1)} | a, b, k+1 > \\ & + k_1 \frac{N(a\,b\,k)}{N(a+1, b+1, k)} | a+1, b+1, k > \\ & + k_2 \frac{N(a\,b\,k)}{N(a+2, b+2, k-1)} | a+2, b+2, k-1 > \end{aligned} \tag{17}$$

where

$$\begin{aligned} k_1 &= -\frac{\lambda + 2}{(\lambda - 2k)(\lambda - 2k + 2)} \\ k_2 &= \frac{k(\lambda - k + 2)}{(\lambda - 2k + 1)(\lambda - 2k + 2)^2(\lambda - 2k + 3)} \end{aligned} \tag{18}$$

Using the relations (15), (17), (18) we can obtain all the nonvanishing matrix elements of S^n_- (Appendix). In the similar way we get for the operators S^p_- and S^{np}_-:

$$\begin{aligned} S^p_- | a\,b\,k > = & k_1 \frac{N(a\,b\,k)}{N(a-2, b, k+1)} | a-2, b, k+1 > \\ & + k_2 \frac{N(a\,b\,k)}{N(a-1, b+1, k)} | a-1, b+1, k > \\ & + k_3 \frac{N(a\,b\,k)}{N(a, b+2, k-1)} | a, b+2, k-1 > \end{aligned} \tag{19}$$

with

$$\begin{aligned} k_1 &= a(a-1) \\ k_2 &= \frac{a(\lambda + 2)(\lambda - 2k - a + 1)}{(\lambda - 2k)(\lambda - 2k + 2)} \\ k_3 &= \frac{k(\lambda - k + 2)(\lambda - 2k - a + 1)(\lambda - 2k - a + 2)}{(\lambda - 2k + 1)(\lambda - 2k + 2)^2(\lambda - 2k + 3)} \end{aligned} \tag{20}$$

$$S_{-}^{np} \mid a b k > = k_1 \frac{N(a b k)}{N(a-1, b, k+1)} \mid a-1, b, k+1 >$$
$$+ k_2 \frac{N(a b k)}{N(a, b+1, k)} \mid a, b+1, k >$$
$$+ k_3 \frac{N(a b k)}{N(a+1, b+2, k-1)} \mid a+1, b+2, k-1 > \qquad (21)$$

with

$$k_1 = 2a \qquad k_2 = \frac{(\lambda+2)(\lambda-2k-2a)}{(\lambda-2k)(\lambda-2k+2)}$$
$$k_3 = -\frac{2k(\lambda-k+2)(\lambda-2k+1-a)}{(\lambda-2k+1)(\lambda-2k+2)^2(\lambda-2k+3)} \qquad (22)$$

The matrix elements of the S_{+} operators can be given by the relations

$$< T_0 T S_0 \mid S_{+} \mid S_0' T' T_0' > = < T_0' T' S_0' \mid S_{-} \mid S_0 T T_0 > \qquad (23)$$

All of the matrix elements of the operators S_{+} are completed in the Appendix.

With the representation $D(\lambda, 0)$ of the R_5 group, considered here, we finished the consideration of the simple representations that demand to be labelled by only three numbers to distinguish the states under the irreducible representation. To consider the general irreducible representation, we have to construct the fourth commuting operator, the eigenvalue of which will label the states. It has been recently constructed [10] as

$$\beta = \left\{ \frac{1}{4} (S_{+}^{np})^2 - S_{+}^{p} S_{+}^{n} \right\} \left\{ \frac{1}{4} (S_{-}^{np})^2 - S_{-}^{n} S_{-}^{p} \right\}, \qquad (24)$$

having the property to count all the possible four-particle states coupled to the total $T = 0$ and $J = 0$ within a given basis vector. The commuting operators β, S_0, $\mathbf{T}^2$, T_0 form a complete set to classify the general irreducible representation $D(\lambda_1 \lambda_2)$ of R_5. However, the most interesting physical cases (seniority equals one or zero) are included in the irreducible representations $D(0, \lambda)$, $D(1, \lambda)$. Using the calculated matrix elements one can describe some of the excited levels of nuclei under the assumption of pairing correlations. The physical representation of the states by the numbers (nTT_0) allows to assign with each calculated level the T number.

APPENDIX

By means of the relations (13—23) with the eigenvalue

$S_0 = \frac{1}{2}(n\text{-}2j\text{-}1)$ [7] we obtain the following formulas giving all the matrix elements of the $_{\pm}$ operators:

$$<n-2, T-1, T_o-1 \,|\, S_-^n \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(T+T_o-1)(T+T_o)(2j-4T+5)(2j+4T+5)(2T+n-2j-3)(2T+n-2j-1)}{256\,T^2(2T-1)(2T+1)}\right\}^{1/2}$$

$$<n-2, T, T_o-1 \,|\, S_-^n \,|\, n\,T\,T_o> =$$

$$=-\left\{\frac{(2j+5)^2(T-T_o+1)(T+T_o)(2T-n+2j+3)(2T+n-2j-1)}{256\,T^2(T+1)^2}\right\}^{1/2}$$

$$<n-2, T+1, T_o-1 \,|\, S_-^n \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(T-T_o+1)(T-T_o+2)(2j-4T+1)(2j+4T+9)(2T-n+2j+3)}{256\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times(2T-n+2j+5)^{1/2}$$

$$<n+2, T+1, T_o+1 \,|\, S_+^n \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(T+T_o+1)(T+T_o+2)(2j-4T+1)(2j+4T+9)(2T+n-2j+1)}{256\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times(2T+n-2j+3)^{1/2}$$

$$n<+2, T, T_o+1 \,|\, S_+^n \,|\, n\,T\,T_o> =$$

$$=-\left\{\frac{(2j+5)^2(T-T_o)(T+T_o+1)(2T-n+2j+1)(2T+n-2j+1}{256\,T^2(T+1)^2}\right\}^{1/2}$$

$$<n+2, T-1, T_o+1 \,|\, S_+^n \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(T-T_o+1)(T-T_o+2)(2j-4T+5)(2j+4T+5)(2T-n+2j-1)}{256\,T^2(2T-1)(2T+1)}\right\}^{1/2} \times(2T-n+2j+1)^{1/2}$$

$$<n-2, T-1, T_o+1 \,|\, S_-^p \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(T-T_o-1)(T-T_o)(2j-4T+5)(2j+4T+5)(2T+n-2j-3)}{256\,T^2(2T-1)(2T+1)}\right\}^{1/2} \times(2T+n-2j-1)^{1/2}$$

$$<n-2, T, T_o+1 \,|\, S_-^p \,|\, n\,T\,T_o> =$$

$$=+\left\{\frac{(2j+5)^2(T-T_o)(T+T_o+1)(2T-n+2j+3)(2T+n-2j-1)}{256\,T^2(T+1)^2}\right\}^{1/2}$$

$$<n-2, T+1, T_o+1 \,|\, S_-^p \,|\, n\,T\,T_o> +$$

$$=+\left\{\frac{(T+T_o+1)(T+T_o+2)(2j-4T+1)(2j+4T+9)(2T-n+2j+3)}{256\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times(2T-n+2j+5)^{1/2}$$

$$<n+2, T+1, T_0-1\,|\,S_+^p\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(T-T_0-1)(T-T_0)(2j-4T+1)(2j+4T+9)(2T+n-2j+1)}{256\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times (2T+n-2j+3)^{1/2}$$

$$<n+2, T, T_0-1\,|\,S_+^p\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(2j+5)^2(T-T_0+1)(T+T_0+2)(2T-n+2j+1)(2T+n-2j+1)}{256\,T^2(T+1)^2}\right\}^{1/2}$$

$$<n+2, T-1, T_0-1\,|\,S_+^p\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(T+T_0-1)(T+T_0)(2j-4T+5)(2j+4T+5)(2T-n+2j-1)}{256\,T^2(2T-1)(2T+1)}\right\}^{1/2} \times (2T-n+2j+1)^{1/2}$$

$$<n-2, T-1, T_0\,|\,S_-^{np}\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(T-T_0)(T+T_0)(2j-4T+5)(2j+4T+5)(2T+n-2j-3)}{64T^2(2T-1)(2T+1)}\right\}^{1/2} \times (2T+n-2j-1)^{1/2}$$

$$<n-2, T, T_0\,|\,S_-^{np}\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(2j+5)^2\,T_0^2\,(2T-n+2j+3)(2T+n-2j-1)}{64T^2(T+1)^2}\right\}^{1/2}$$

$$<n-2, T+1, T_0\,|\,S_-^{np}\,|\,n\,T\,T_0> =$$

$$= -\left\{\frac{(T-T_0+1)(T+T_0+1)(2j-4T+1)(2j+4T+9)(2T-n+2j+3)}{64\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times (2T-n+2j+5)^{1/2}$$

$$<n+2, T+1, T_0\,|\,S_+^{np}\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(T-T_0+1)(T+T_0+1)(2j-4T+1)(2j+4T+9)(2T+n-2j+1)}{64\,(T+1)^2(2T+1)(2T+3)}\right\}^{1/2} \times (2T+n-2j+3)^{1/2}$$

$$<n+2, T, T_0\,|\,S_+^{np}\,|\,n\,T\,T_0> =$$

$$= +\left\{\frac{(2j+5)^2\,T_0^2\,(2T-n+2j+1)(2T+n-2j+1)}{64T^2(T+1)^2}\right\}^{1/2}$$

$$<n+2, T-1, T_0\,|\,S_+^{np}\,|\,n\,T\,T_0> -$$

$$= -\left\{\frac{(T-T_0)(T+T_0)(2j-4T+5)(2j+4T+5)(2T-n+2j-1)}{64\,T^2(2T-1)(2T+1)}\right\}^{1/2} \times (2T-n+2j+1)^{1/2}$$

With these matrix elements we are in position to calculate the mean value of the Hamiltonian (1). After the straightforward calculations we get

$$<-\frac{H}{G}>=\frac{1}{4}\left\{\left(n-j-\frac{1}{2}\right)\left(2j+\frac{7}{2}-\frac{n}{2}-\frac{j}{2}\right)-2T(T+1)+\left(j+\frac{1}{2}\right)\left(\frac{j}{2}+\frac{5}{4}\right)\right\}$$

and that is the formula (18) of [7] for $s=\lambda=j+\frac{1}{2}$ and $t=\frac{\lambda}{2}=\frac{j}{2}+\frac{1}{4}$.

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STRESZCZENIE

W sposób analogiczny, jak w pracy [7], skonstruowano bazę znakowaną liczbami n, T, T_0 dla nieprzywiedlnej reprezentacji $D(\lambda, 0)$ grupy R_5 w przestrzeni quasi-spinowej. Mając ogólną funkcję bazy, wyliczono następnie w tej bazie elementy macierzowe operatorów infinitezymalnych grupy, które jednocześnie służą do konstrukcji hamiltonianu oddziaływania *pairing*. Dzięki temu, można wyliczyć wszystkie niezerowe elementy macierzowe hamiltonianu i to zarówno dla konfiguracji czystych, jak i mieszanych w przybliżeniu sprzężania *j-j*.

РЕЗЮМЕ

Аналогично как и в работе [7], построенный базис обозначен n, T, T_0 для неприводимого представления $D(\lambda, 0)$ группы R_5 в квазиспиновом пространстве. Исходя из общей функции базиса, вычислены матричные элементы инфинитезимальных операторов группы, которые используются для составления гамильтониана парных взаимодействий. Благодаря этому можно вычислить все отличные от нуля матричные элементы гамильтониана как для чистых конфигураций, так и для смешанных в приближении связи *j-j*.

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