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Calculation of the Special Types of the O_5 Wigner Coefficients

Obliczenie szczególnych typów współczynników Wignera O_5

Вычисление специальных типов коэффициентов Вигнера O_5

I. INTRODUCTION

In recent years the quasi-spin method and the orthogonal groups O_n connected with it, have been widely applied in nuclear spectroscopy. This method is used among others for classification of the states of many nucleons [1, 2, 3, 4, 13], for the pairing interaction [4, 5, 6, 7], for the theory of nucleus splitting [8, 9], for the probabilities of Gamow-Teller transitions [10], for the energy of rotational states of nuclei with high angular momenta [11], or for collective excitations of spherical nuclei [12].

In practical application it is important to know the Wigner coefficient by means of which we can decompose the Kronecker product of the representations of the same symmetry group. In particular, in case of the configuration of protons and neutrons at the j level, some types of the Wigner coefficients for the quasi-spin group were calculated by several authors with various methods. Hecht [13] has calculated the Wigner coefficients of the O_5 group in the so-called mathematical basis by means of the recurrent formulae, and with Hemenger [15, 16], they applied in calculations the matrix elements of corresponding irreducible tensor operators in group O_5. Jahn [14] has calculated the Wigner coefficients starting from the shell d by using the relations of these coefficients with phonon fractional parentage coefficients in the seniority scheme. Finally, Wong [21] has recently used for calculations the method of adjoined tensor operators for the O_n group.

However, there is not available a general formula for the Wigner coefficients of group O_5 similar to that [e. g. 18] for the Wigner coefficients of O_3 group, which are commonly called Clebsch-Gordan coefficients. The Wigner coefficients calculated by Hecht and Hemeng e r are not only given as separate formulae for the particular representations, but also separately for admissible values So and T in given representation.

In this paper the author deals with the Wigner coefficients related to the following scheme of the Kronecker decomposition product $(0, \lambda') \times \times (0, \lambda'') \Rightarrow (0, \lambda' + \lambda'')$. Calculations given here are followed by the general formula for any coefficient of the above product.

II. FUNDAMENTAL FORMALISM

Quantum numbers labelling the states of irreducible representation of any symmetry group are usually divided into two sets. The first set includes quantum numbers distinguishing the irreducible representations of the given group, and the other — quantum numbers distinguishing the states within the given irreducible representation. For the O_5 group two quantum numbers label the irreducible representations, and additional four numbers — the particular states within the irreducible representation. In this paper the following labelling of the states is used:

$$|(\lambda_1 \lambda_2)\beta S_0 T T_0\rangle$$

λ_1, λ_2 denote numbers of the fundamental representations [19], whereby

$$\lambda_1 = 2t$$

$$\lambda_2 = j + \frac{1}{2} - \frac{v}{2} - t$$

The states in the irreducible representation (λ_1, λ_2) are distinguished by the quantum numbers of the third quasi-spin component $S_0 = {}^1/_2\,(n-2j-1)$, the total isospin and its third component T and T_0 and the β number necessary for the representations in which the same value T for the given S_0 appears more than once.

Let us take two bases of irreducible representations of O_5 group $|(\lambda_1', \lambda_2')\,\beta'\, S_0'\, T'\, T_0'\rangle$ and $|(\lambda_1'', \lambda_2'')\,\beta''\, S_0''\, T''\, T_0''\rangle$ describing two independent physical systems I and II. The compound system (I+II) is described by the bases of the irreducible representations $|(\lambda_1, \lambda_2)\,\beta\, S_0\, T\, T_0\rangle$, where (λ_1, λ_2) is obtained from the Kronecker product of the representations (λ_1', λ_2') and $(\lambda_1'', \lambda_2'')$. Any state $|(\lambda_1, \lambda_2)\,\beta\, S_0\, T\, T_0\rangle$ is a linear combination of the products of the starting states:

$$|(\lambda_1,\lambda_2)\beta S_0 T_0 T\rangle = \sum_{\substack{\beta' S_0' T' T_0' \\ \beta'' S_0'' T'' T_0''}} \langle(\lambda_1',\lambda_2')\beta' S_0' T' T_0'; (\lambda_1'',\lambda_2'')\beta'' S_0'' T'' T_0'' | (\lambda_1,\lambda_2)\beta S_0 T T_0\rangle \times |(\lambda_1',\lambda_2')\beta' S_0' T' T_0'\rangle\, |(\lambda_1'',\lambda_2'')\beta'' S_0'' T'' T_0''\rangle \qquad (1)$$

According to the Wigner-Eckart theorem, the Wigner coefficients appearing in (1) can be reduced in the isospin space:

$$\begin{aligned}\langle(\lambda'_1, \lambda'_2)\beta' S'_0 T' T'_0; (\lambda''_1, \lambda''_2)\beta'' S''_0 T'' T''_0 | (\lambda_1, \lambda_2)\beta S_0 T T_0\rangle = \\ = (T' T'_0; T'' T''_0 \; T T_0)\langle(\lambda'_1, \lambda'_2)\beta' S'_0 T'; (\lambda''_1, \lambda''_2)\beta'' S''_0 T'' \| (\lambda_1, \lambda_2)\beta S_0 T\rangle\end{aligned} \tag{2}$$

The most interesting representations are those of a small seniority number v and of a small reduced isospin t. These requirements can be met by the representations $(0, \lambda_2)$, $(\lambda_1, 0)$ and $(1, \lambda_2)$. Only five quantum numbers are necessary for labelling the states of each of these representations. The number β is superfluous, because for every S_0, each of the admissible values T occurs only once.

In the paper [1] general formulae for the Clebsch-Gordan series for the Kronecker products are given

$$(\lambda_1', 0)\times(\lambda_1'', 0)$$
$$(0, \lambda_2')\times(0, \lambda_2'')$$
$$\text{and } (\lambda_1', 0)\times(0, \lambda_2'')$$

and a number of series of the type $(\lambda'_1, \lambda'_2)\times(\lambda''_1, \lambda''_2)$ for the particular values $\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2$. It can be easily noticed that in each of the Kronecker products $(\lambda'_1, \lambda'_2)\times(\lambda''_1, \lambda''_2)$ the representation

$$(\lambda'_1+\lambda''_1, \lambda'_2+\lambda''_2)$$

occurs only once. The state of the highest weight in this representation is the product of the states of the highest weights of starting representations:

$$\begin{aligned}|(\lambda'_1+\lambda''_1, \lambda'_2+\lambda''_2)(S'_0+S''_0)_{max}(T'+T'')_{max}(T'+T'')_{max}\rangle = \\ = |(\lambda'_1\lambda'_2)S_{0max}T'_{max}T'_{max}\rangle|(\lambda''_1\lambda''_2)S''_{0max}T''_{max}\;T''_{max}\rangle\end{aligned} \tag{3}$$

or, in other words, the sum in the formula (1) is reduced to one term, and the corresponding Wigner coefficient is equal to one.

For the representation of the type $(0, \lambda)$ we have $S_{0max}=\lambda$, $(T_{0max})\, S_{0max}= 0$, thus from (3) we get

$$|(0,\lambda'+\lambda''), \lambda'+\lambda'', 00\rangle = |(0,\lambda')\lambda' 00\rangle\,|(0,\lambda'')\lambda'' 00\rangle \tag{4}$$

In the papers [1] and [20] there were constructed the states within the representations $(0, \lambda_2)$, $(1, \lambda_2)$ and $(\lambda_1, 0)$. For the representation $(0, \lambda_2)$ any allowed state is expressed by the formula

$$|(0,\lambda_2)S_0 T T_0\rangle = N(a,b,k)(T_-)^a(S_-^p)^b\{(S_-^{np})^2-4S_-^n S_-^p\}^k|(0,\lambda_2)\lambda_2 00\rangle \tag{5}$$

where:

$$a = T - T_0$$

$$b = T$$

$$k = \frac{1}{2}(\lambda_2 - T - S_0)$$

$N(a, b, k)$ — normalization coefficient.

III. THE CALCULATION OF THE WIGNER COEFFICIENTS

According to (2) only the reduced Wigner coefficients should be calculated. Therefore in (5) we can put $T_0 = T$, to simplify the calculation, without narrowing the results. Thus from (4) and (5) we get the relation

$$(S^p_-)^b \{(S''^p_-)^2 - 4S''_- S^p_-\}^k |(0, \lambda' + \lambda'')\lambda' + \lambda'', 00\rangle =$$
$$= (S^p_-)^b \{(S''^p_-)^2 - 4S''_- S^p_-\}^k \{|(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' 00\rangle\} \tag{6}$$

Then, we consider the special cases.

1. For $k = 0$, we get

$$T = \lambda - S_0 \tag{7}$$

We act with the operator S^p_- successively b times on the right and left side (4) and, using the matrix elements [1], we obtain:

$$S^p_- \{|(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' 00\rangle\} = \sqrt{\lambda'}\, |(0, \lambda')\lambda' - 1, 11\rangle |(0, \lambda'')\lambda'' 00\rangle +$$
$$+ \sqrt{\lambda''}\, |(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' - 1, 11\rangle$$

$$(S^p_-)^2 \{|(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' 00\rangle\} = \sqrt{2\lambda'(\lambda' - 1)}\, |(0, \lambda')\lambda' - 2, 22\rangle |(0, \lambda'')\lambda'' 00\rangle +$$
$$+ 2\sqrt{\lambda' \lambda''}\, |(0, \lambda')\lambda' - 1, 11\rangle |(0, \lambda'')\lambda'' - 1, 11\rangle +$$
$$+ \sqrt{2\lambda''(\lambda'' - 1)}\, |(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' - 2, 22\rangle$$

and generally

$$(S^p_-)^b \{|(0, \lambda')\lambda' 00\rangle |(0, \lambda'')\lambda'' 00\rangle\} = \sum_{l=0}^{b} \binom{b}{l} \left[(b-l)!\, l!\, \frac{(\lambda')!\,(\lambda'')!}{(\lambda' - b + l)!\,(\lambda'' - l)!}\right]^{\frac{1}{2}} \times$$
$$\times |(0, \lambda')\lambda' - b + l, b - l, b - l\rangle |(0, \lambda'')\lambda'' - l, ll\rangle$$

and, on the other hand

$$(S^p_-)^b |(0, \lambda' + \lambda'')\lambda' + \lambda'', 00\rangle = \left[\frac{b!\,(\lambda' + \lambda'')!}{(\lambda' + \lambda'' - b)!}\right]^{\frac{1}{2}} |(0, \lambda' + \lambda'')\lambda' + \lambda'' - b, bb\rangle$$

Therefore, for $T=\lambda-S_0$, the reduced Wigner coefficient is given by general formula

$$\langle (0,\lambda')\,\lambda'-b+l,b-l;(0,\lambda'')\,\lambda''-l,l\,\|\,(0,\lambda'+\lambda'')\,\lambda'+\lambda''-b,b\rangle=$$
$$=\binom{b}{l}\left[\frac{(b-l)!\,l!\,(\lambda')!\,(\lambda'')!\,(\lambda'+\lambda''-b)!}{(\lambda'-b+l)!\,(\lambda''-l)!\,b!\,(\lambda'+\lambda'')!}\right]^{\frac{1}{2}} \tag{8}$$

where:

$$b=0,1,\dots \min(\lambda',\lambda'')$$
$$l=0,1,\dots b$$

Substituting the physical values for b and l (5), we get:

$$\langle (0,\lambda')\,\lambda'-T+T'',T-T'';(0,\lambda'')\,\lambda''-T'',T''\,\|\,(0,\lambda'+\lambda'')\,\lambda'+\lambda''-T,T\rangle=$$
$$=\binom{T}{T''}\left[\frac{(T-T'')!\,(T'')!\,(\lambda')!\,(\lambda'')!\,(\lambda'+\lambda''-T)!}{(\lambda'-T+T'')!\,(\lambda''-T'')!\,(T)!\,(\lambda'+\lambda'')!}\right]^{\frac{1}{2}} \tag{9}$$

All coefficients of this type are positive. Then, some of the symmetry properties of these coefficients are given. In cases in which the multiplicity of T is not higher than 2 [17], we get the relations

$$\langle (\lambda_1',\lambda_2')\,S_0'T';(\lambda_1'',\lambda_2'')\,S_0''T''\,\|\,(\lambda_1,\lambda_2)\,S_0T\rangle_\rho=$$
$$=(-1)^{\xi+\nu'+\nu''-\nu}\langle (\lambda_1',\lambda_2'),-S_0',T';(\lambda_1'',\lambda_2''),-S_0'',T''\,\|\,(\lambda_1,\lambda_2),-S_0,T\rangle_\rho \tag{10}$$

where:

$$\xi=\lambda'_2+\lambda''_2-\lambda_2$$

$$\nu=\begin{cases}0 & \text{for the representations } (0,\lambda_2),\ (1,\lambda_2)\\ t-T & \text{for } (\lambda_1,0)\end{cases}$$

and

$$\langle (\lambda_1',\lambda_2')\,S_0'T';(\lambda_1'',\lambda_2'')\,S_0''T''\,\|\,(\lambda_1,\lambda_2)\,S_0T\rangle_\rho=$$
$$=(-1)^{t-t'-T+T'+T''-\tilde{T}''+\nu''-\tilde{\nu}''}\left[\frac{(2T'+1)\cdot \dim(\lambda_1,\lambda_2)}{(2T+1)\cdot \dim(\lambda_1',\lambda_2')}\right]^{\frac{1}{2}}\times$$
$$\times\langle (\lambda_1,\lambda_2)\,S_0T;(\lambda_1'',\lambda_2'')-S_0''T''\,\|\,(\lambda_1',\lambda_2')\,S_0'T'\rangle_\rho \tag{11}$$

The dimension $\dim(\lambda_1,\lambda_2)$ of the irreducible representation is given by

$$\dim(\lambda_1,\lambda_2)=\frac{1}{6}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\lambda_1+2\lambda_2+3)$$

For several admissible values of the isospin T'', with the given S_0'', $\tilde{T}''$ is the highest value of T'', so that when combined with T' it gives T; and $\tilde{\nu}''=t-\tilde{T}''$. The label ϱ from the formulae (10) and (11) distinguishes the representations (λ_1,λ_2) appearing more than once in the decomposition of Kronecker product $(\lambda'_1,\lambda'_2)\times(\lambda''_1,\lambda''_2)$.

It appears from the formulae (9), (10) and (11) that

$$\langle (0,\lambda')\lambda'-T+T'',T-T'';(0,\lambda'')\lambda''-T'',T''\|(0,\lambda'+\lambda'')\lambda'+\lambda''-T,T\rangle = \tag{12}$$

$$=\langle (0,\lambda')-\lambda'+T-T'',T-T'';(0,\lambda'')T''-\lambda'',T''\|(0,\lambda'+\lambda'')T''-\lambda'-\lambda'',T\rangle =$$

$$=\left[\frac{(2T-2T''+1)\,dim(0,\lambda'+\lambda'')}{(2T+1)\,dim(0,\lambda')}\right]^{\frac{1}{2}}\langle (0,\lambda'+\lambda'')\lambda'+\lambda''-T,T,(0,\lambda'')T''-\lambda'',T''\|(0,\lambda')\lambda'-T+T'',T-T''\rangle$$

2. Setting in (6) $b=0$ with $k\neq 0$ and taking the operator $[(S^{np}_-)^2 - 4S^n_-\,S^p_-]^k$ instead of $(S^p_-)^b$ as in previous case, we obtain in the similar manner

$$\{(S^{np}_-)^2-4S^n_-S^p_-\}^k|(0,\lambda'+\lambda'')\lambda'+\lambda''00\rangle = \left[\frac{(2k+1)!\,(2\lambda'+2\lambda''+1)!}{(2\lambda'+2\lambda''-2k+1)!}\right]^{\frac{1}{2}}|(0,\lambda'+\lambda'')\,\lambda'+\lambda''-2k,00\rangle \tag{13}$$

and

$$\{(S^{np}_-)^2-4S^n_-S^p_-\}^k\left(|(0,\lambda')\lambda'00\rangle|(0,\lambda'')\lambda''00\rangle\right)=\sum_{n,T'T'_0}\frac{(1)^{T'-T'_0}}{\sqrt{2T'+1}}f(k,n,T')\times$$

$$\times\left[\frac{2^k(2k+1)!\,(\lambda')!\,(\lambda'')!\,(2\lambda'+1)!!\,(2\lambda''+1)!!}{(\lambda'-k+\frac{n}{2}-\frac{T'}{2})!\,(2\lambda'-2k+T'+n+1)!!\,(\lambda''-\frac{n}{2}-\frac{T''}{2})!\,(2\lambda''-n+T'+1)!!}\right]^{\frac{1}{2}}\times \tag{14}$$

$$\times|(0,\lambda')\lambda'-2k+n,T'T'_0\rangle|(0,\lambda'')\lambda''-n,T'-T'_0\rangle$$

From these formulae we immediately get

$$\langle (0,\lambda')\lambda-2k+n,T';(0,\lambda'')\lambda''-n,T'\|(0,\lambda'+\lambda'')\lambda'+\lambda''-2k,0\rangle = f(k,n,T)\times \tag{15}$$

$$\times\left[\frac{(\lambda')!\,(\lambda'')!\,(\lambda'+\lambda''-k)!\,(2\lambda'+1)!!\,(2\lambda''+1)!!\,(2\lambda'+2\lambda''-2k+1)!!}{(\lambda'-k+\frac{n}{2}-\frac{T'}{2})!\,(\lambda''-\frac{n}{2}-\frac{T'}{2})!\,(\lambda'+\lambda'')!\,(2\lambda'-2k+T'+n-1)!!\,(2\lambda''-n+T'+1)!!\,(2\lambda'+2\lambda''+1)!!}\right]^{\frac{1}{2}}$$

where $f(k,n,T')$ is given in Table II for several values of k, n, T'. The quantum numbers k, n, T' take on the values:

$$k=0,1,\ldots\lambda'$$
$$n=0,1,\ldots min.\,(\lambda',\lambda'')$$

The admissible isospin values T for a given S_0 can be read from Table 1.

The above method can be also applied to calculation of the Wigner coefficients for the representation $(\lambda'+\lambda'',0)$ obtained from Kronecker product $(\lambda',0)\times(\lambda'',0)$ and for the representation $(1,\lambda'+\lambda'')$ from Kronecker produkt $(1,\lambda')\times(0,\lambda'')$.

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Table 1. Isospins T for the representation $(0,\lambda)$

S_0	T
$\pm(\lambda)$	0
$\pm(\lambda-1)$	1
$\pm(\lambda-2)$	0 2
$\pm(\lambda-3)$	1 3
.	. .
.	. .
.	. .
0	... $\lambda-2,\ \lambda$

Table 2. The value of the function $f(k,n,T')$ appearing in the reduced Wigner coefficients (formula (15)

k	n	T'	$f(k,n,T')$	k	n	T'	$f(k,n,T')$
1	0	0	1	3	0	0	1
1	1	1	-2	3	1	1	$-2\sqrt{3}$
1	2	0	1	3	2	0	$\sqrt{7}$
2	1	1	$-2\sqrt{2}$	3	2	2	$4\sqrt{2}$
2	2	0	$\sqrt{10/3}$	3	3	1	$-2\sqrt{42/5}$
2	2	2	$4\sqrt{2/3}$	3	3	3	$-8\sqrt{2/5}$
2	3	1	$-2\sqrt{2}$	3	4	0	$\sqrt{7}$
2	4	0	1	3	4	2	$4\sqrt{2}$
2	0	0	1	3	5	1	$-2\sqrt{3}$
4	0	0	1	3	6	0	1

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STRESZCZENIE

Wykorzystując elementy macierzowe generatorów grupy O_5 [1], wyliczono następujące typy współczynników Wignera O_5 w bazie $|(0, \lambda)\, S_0\, T\, T_0)$:

1. $\langle (0,\lambda')S_0'T'; (0,\lambda'')S_0''T''=T' \| (0,\lambda'+\lambda'')S_0 T = 0 \rangle$

2. $\langle (0,\lambda')S_0'T'; (0,\lambda'')S_0''T'' \| (0,\lambda'+\lambda'')S_0 T = \lambda'+\lambda''-S_0 \rangle$

РЕЗЮМЕ

Используя матричные элементы генераторов группы O_5 [1], были вычислены следующие типы коэффициентов Вигнера O_5 в базисе

1. $\langle (0,\lambda')S_0'T'; (0,\lambda'')S_0''T''=T' \| (0,\lambda'+\lambda'')S_0 T = 0 \rangle$

2. $\langle (0,\lambda')S_0'T'; (0,\lambda'')S_0''T'' \| (0,\lambda'+\lambda'')S_0 T = \lambda'+\lambda''-S_0 \rangle$