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Some Applications to Briot-Bouquet Differential Subordinations

Kilka zastosowań podporządkowania różniczkowego Briota-Bouqueta

Abstract. Some applications of Briot-Bouquet differential subordination are obtained which improve and sharpen a number of classical results in the univalent functions theory. These also lead to sharp results for Libera and Bernardi transforms.

1. Introduction. Let H denote the class of functions $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $U = \{z : |z| < 1\}$. Let S, $S^*(\beta)$ and $K(\beta)$ $(0 \le \beta < 1)$ denote the subclasses of functions in H which are respectively univalent, starlike of order β and convex of order β in U. We denote $S^*(0) \equiv S^*$, $K(0) \equiv K$. For given arbitrary numbers A, B satisfying $-1 \le B < A \le 1$, we denote by $P(A, B)$, the class of functions of the form

$$p(z) = 1 + p_1 z + \cdots \tag{1}$$

which are analytic in U and satisfy the condition

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad z \in U$$

(Here "$\prec$" stands for subordination). The class $P(A, B)$ was investigated by Janowski [10]. By $S^*(A, B)$, we mean the class of functions $f \in H$ such that $zf'(z)/f(z) \in P(A, B)$.

Similarly, by $K(A, B)$ is meant the class of function $f \in H$ satisfying $(zf'(z))'/f'(z) \in P(A, B)$. It is clear that $S^*(1 - 2\beta, -1) \equiv S^*(\beta)$, $K(1 - 2\beta, -1) \equiv K(\beta)$, $(0 \le \beta < 1)$, and that for $0 < \beta < 1$, $S^*(\beta) \subset S^*$, $K(\beta) \subset K$.

In [22], Ruscheweyh introduced the class K_n of functions $f \in H$ satisfying

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > 1/2, \qquad z \in U, \quad n \in N_0 \equiv N \cup \{0\} \equiv \{0, 1, 2, \ldots\} \tag{2}$$

where $D^n f(z) = (z/(1-z)^{n+1}) * f(z)$. (Here $*$ stands for the Hadamard product or convolution of two analytic functions.) In [1], Al-Amiri investigated the classes

$K_n(\alpha)$ of functions $f \in H$, $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ satisfying,

$$\text{(3)} \qquad \mathrm{Re}\,\{J_n(f;\alpha)\} > 1/2\,, \qquad z \in U$$

for some $\alpha \geq 0$, where

$$\text{(4)} \qquad J_n(f;\alpha) = (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\,,$$

and showed that for each $n \in N_0$, and $\alpha \geq 0$

$$\text{(5)} \qquad K_{n+1}(\alpha) \subset K_n(0)$$

The case $\alpha = 1$ in (5) is due to Ruscheweyh [22].

Goel and Sohi [8] further generalized the class $K_n(\alpha)$ by introducing the class $T_{n,\beta}(\alpha)$. Thus a function $f \in H$ is said to be in $T_{n,\beta}(\alpha)$ if for $0 \leq \beta \leq 1/2$, and $\alpha \geq 0$ the condition

$$\text{(6)} \qquad \mathrm{Re}\,\{J_n(f;\alpha)\} > \beta\,, \quad z \in U$$

holds, where $J_n(f;\alpha)$ is given by (4). It was also claimed in [8, Theorem 1 and Theorem 5] that

$$\text{(7)} \qquad T_{n+1,\beta}(\alpha) \subset T_{n,\beta}(0) \quad \text{for all } n \in N_0,\ 0 \leq \beta \leq 1/2 \text{ and } \alpha \geq 0.$$

However, as shown by the authors [12], the containment relation (7) is not in general valid. In fact, a rectified version of (7) is shown to be true in [12].

Recently, many of the classical results in univalent function theory have been improved and sharpened by the powerful technique of Briot–Bouquet differential subordination (see eg. [6], [18], [19] etc.). Recall that a function $p(z)$ analytic in U with a power series of the form (1) is said to satisfy Briot–Bouquet differential subordination if

$$\text{(8)} \qquad p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)\,, \quad z \in U$$

for β and γ complex constants and $h(z)$ a convex function with $h(0) = 1$ and $\mathrm{Re}\,\{\beta h(z) + \gamma\} > 0$ in U.

The univalent function $q(z)$ is said to be a dominant of the Briot–Bouquet differential subordination (8) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (8). If $\tilde{q}(z)$ is a dominant of (8) and $\tilde{q}(z) \prec q(z)$ for all other dominants $q(z)$ of (8), then $\tilde{q}(z)$ is said to be the best dominant.

In the present paper, we propose to give some applications of Briot–Bouquet differential subordination which would not only improve and sharpen many of the earlier results for the classes $S^*(A,B)$, $K(A,B)$, $K_n(\alpha)$, $T_{n,\beta}(\alpha)$ etc., but would also give rise to a number of new results for other subclasses as well. This is accomplished by introducing and studying a very wide class $T_{\delta,\alpha}(A,B)$. Further use of Briot–Bouquet differential subordination to the investigation of Libera and Bernardi transforms of this class leads, perhaps for the first time, to sharp results in this direction. Finally, using differential subordination, we improve and generalize results of Singh and Singh [25], and Mocanu [20].

2. We introduce the class $T_{\delta,\alpha}(A,B)$ as follows.

Definition 1. Let A, B, α and δ be arbitrary fixed real numbers such that $-1 \le B < A \le 1$, $\alpha \ge 0$ and $\delta \ge -1$. A function $f \in H$ is said to be in the class $T_{\delta,\alpha}(A,B)$ if it satisfies

$$J_\delta(f;\alpha) \prec \frac{1+Az}{1+Bz}, \quad z \in U \tag{9}$$

where

$$J_\delta(f;\alpha) = (1-\alpha)\frac{D^{\delta+1}f(z)}{D^\delta f(z)} + \alpha\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}, \text{ and } D^\delta f(z) = (z/(1-z)^{\delta+1}) * f(z).$$

It is readily seen that $T_{0,0}(A,B) \equiv S^*(A,B)$, $T_{0,1}(\frac{A+B}{2},B) \equiv K(A,B)$. Further it is clear that $T_{n,0}(0,-1)$, $n \in N_0$ is the class K_n defined by Ruscheveyh [22], whereas the class $T_{n,0}(\frac{1-n}{1+n},-1) \equiv R_n$ has been studied by Singh and Singh [24], the class $T_{n,\alpha}(1-2\beta,-1)$ $(0 \le \beta \le 1/2)$ is the class considered by Goel and Sohi [8]. The classes $T_{n,\alpha}(0,-1)$ and $T_{\delta,0}(0,-1)$ were considered by Al-Amiri [1, 2] and $T_{\delta,0}(1-2\beta,-1)$ $(\beta < 1)$ has been recently studied by the authors [12]. Further taking $\delta = 0$, $\alpha = 2\mu/(\mu+1)$ $(\mu \ge 0)$, $A = 1-2(\mu/(\mu+1))$, $B = -1$, it is seen that the class $T_{\delta,\alpha}(A,B)$ reduces to the well known class of μ-convex functions [16] which is a subclass of S^* if $\mu \ge 0$ and of K if $\mu \ge 1$.

In order to prove the main theorems we will need the following lemmas.

Lemma 1. [18, Corollary 3.2] *If $-1 \le B < A \le 1$, $\beta > 0$ and complex number γ satisfy $\operatorname{Re}\gamma \ge -(1-A)\beta/(1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz}$$

has a univalent solution in U given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta((A-B)/B)}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta((A-B)/B)}dt} - \dfrac{\gamma}{\beta} & \text{if } B \ne 0 \\[2ex] \dfrac{z^{\beta+\gamma}\exp\{\beta Az\}}{\beta\int_0^z t^{\beta+\gamma-1}\cdot\exp\{\beta At\}dt} - \dfrac{\gamma}{\beta} & \text{if } B = 0 \end{cases} \tag{10}$$

If $p(z)$ is analytic in U and satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz}$$

then $p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}$ *and* $q(z)$ *is the best dominant.*

Lemma 2. [28, Lemma 2] *Let* μ *be a positive measure on the unit interval* $I=[0,1]$. *Let* $g(t,z)$ *be a function analytic in* U *for each* $t\in[0,1]$, *and integrable in* t *for each* $z\in U$ *and for almost all* $t\in[0,1]$, *and suppose that* $\operatorname{Re}\{g(t,z)\}>0$ *on* U $g(t,-r)$ *is real for* r *real and* $\operatorname{Re}\left\{\frac{1}{g(t,z)}\right\}\geq\frac{1}{g(t,-r)}$ *for* $|z|\leq r$ *and* $t\in[0,1]$. *If* $g(z)=\int_I g(t,z)\,d\mu(t)$, *then* $\operatorname{Re}\left\{\frac{1}{g(z)}\right\}\geq\frac{1}{g(-r)}$ *for* $|z|\leq r$.

For a, b, c real numbers other than $0,-1,-2,\dots$, the hypergeometric series

$$F(a,b;c;z)=1+\frac{a\cdot c}{1\cdot c}\,z+\frac{a(a+1)b(b+1)}{1\cdot 2\,c(c+1)}\,z^2+\cdots \tag{11}$$

represents an analytic function in U [27, p.281]. The following identities are well known.

Lemma 3. [27, Chapter XIV]. *For* a, b, c *real numbers other than* $0,-1,-2,\dots$, *and* $c>b>0$ *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t-z)^{-a}\,dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z) \tag{12}$$

$$F(a,b;c;z)=F(b,a;c;z) \tag{13}$$

$$F(a,b;c;z)=(1-z)^{-a}F\big(a,c-b;c;z/(1-z)\big) \tag{14}$$

$$F\left(a,b;\tfrac{a+b+1}{2};1/2\right)=\frac{\Gamma(1/2)\Gamma((a+b+1)/2)}{\Gamma\left(\frac{a+1}{2}\right)\cdot\Gamma\left(\frac{b+1}{2}\right)} \tag{15}$$

3. Containment relations.

Theorem 1. *Let* $-1\leq B<A\leq 1$, $\delta>-1$ *and* $0<\alpha<\delta+2$ *satisfy*

$$(\delta+2)(1-A)-\alpha(1-B)\geq 0. \tag{16}$$

(a) *Then*

$$T_{\delta,\alpha}(A,B)\subset T_{\delta,0}(A',B) \tag{17}$$

where

$$A'=1-\frac{1}{\delta+2-\alpha}\left((\delta+2)(1-A)-\alpha(1-B)\right). \tag{18}$$

Further for $f(z) \in T_{\delta,\alpha}(A,B)$ we also have

(19) $$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \frac{\alpha}{\delta+2-\alpha}\left(\frac{1}{Q(z)}\right) \equiv \tilde{q}(z)\,, \quad z \in U$$

where

(20) $$Q(z) = \begin{cases} \int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\frac{\delta+2}{\alpha}\left(\frac{A-B}{B}\right)} t^{\frac{\delta+2}{\alpha}-2}\,dt & \text{if } B \neq 0 \\ \int_0^1 \exp\left\{\frac{\delta+2}{\alpha}(t-1)Az\right\} t^{\frac{\delta+2}{\alpha}-2}\,dt & \text{if } B = 0 \end{cases}$$

(b) *If in addition to* (16) *one has $-1 \leq B < A < 0$, then*

(21) $$T_{\delta,\alpha}(A,B) \subset T_{\delta,0}(1-2\rho',-1)$$

where $\rho' = \left[\,F\left(1, \frac{\delta+2}{\alpha}\left(\frac{B-A}{B}\right), \frac{\delta+2}{\alpha}; \frac{-B}{1-B}\right)\right]^{-1}$.
The result is sharp.

Proof. We follow the method similar to that of Mocanu et al. [19]. Since, for $\delta > -1$, $D^{\delta}f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)a_n z^n}{(n-1)!\Gamma(\delta+1)}$, it can be easily verified that

(22) $$z(D^{\delta}f(z))' = (\delta+1)D^{\delta+1}f(z) - \delta D^{\delta}f(z)\,.$$

Let $f(z) \in T_{\delta,\alpha}(A,B)$ where $\delta > -1$, $\alpha > 0$ and $-1 \leq B < A \leq 1$. Set $g(z) = z[D^{\delta}f(z)/z]^{1/(\delta+1)}$ and $r_1 = \sup\{r : g(z) \neq 0,\ 0 < |z| < r < 1\}$. Using (22) it follows that

(23) $$p(z) = \frac{zg'(z)}{g(z)} = \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}$$

is analytic in $|z| < r_1$ and $p(0) = 1$. Since $f(z) \in T_{\delta,\alpha}(A,B)$, (9) coupled with (22) easily leads to

(24) $$P(z) + \frac{zP'(z)}{\beta P(z) + \gamma} \prec \frac{1+Az}{1+Bz}\,, \quad |z| < r_1$$

where

(25) $$P(z) = \left(1 - \frac{1}{\beta}\right)p(z) + \frac{1}{\beta}\,, \quad \text{with } \beta = \frac{\delta+2}{\alpha} \text{ and } \gamma = -1.$$

Using Lemma 1 we deduce that

(26) $$P(z) \prec q(z) \prec \frac{1+Az}{1+Bz}\,, \quad |z| < r\,,$$

where $q(z)$ is the best dominant of (24) and is given by (10). Again by (19) we get

$$p(z) \prec \frac{\alpha}{\delta+2-\alpha}\left[\frac{1}{Q(z)}\right] \equiv \tilde{q}(z)\,, \quad |z| < r_1 \tag{27}$$

where $Q(z)$ is given by (20).

By (25) and (26), we see from (23) that $g(z)$ is starlike (univalent) in $|z| < r_1$. Thus it is not possible that $g(z)$ vanishes in $|z| < r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore $p(z)$ is analytic in U. However (27) implies that $p(z) \prec \tilde{q}(z)$ in U. Hence by (23), $f(z) \in T_{\delta,\alpha}(A,B)$ implies $\dfrac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \tilde{q}(z)$ provided δ, α, A and B satisfy (16). This proves (17) and (19).

(b) Next we show that

$$\operatorname*{Inf}_{|z|<1}\{\operatorname{Re} q(z)\} = \tilde{q}(-1)\,. \tag{28}$$

If we set $a = \beta\left(\frac{B-A}{B}\right)$, $b = \beta+\gamma$, $c = \beta+\gamma+1$ $\left(\beta = \frac{\alpha+2}{\alpha},\ \gamma = -1\right)$ then $c > b > 0$. From (20) by using (12), (13) and (14) we see that for $B \neq 0$

$$\begin{aligned} Q(z) &= (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1}\,dt = \\ &= (1+Bz)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}(1+Bz)^{-a} F\left(a, c-b; c; \frac{Bz}{Bz+1}\right) = \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(1, a; c; Bz/(Bz+1))\,. \end{aligned} \tag{29}$$

To prove (28) we show that $\operatorname{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)}$, $z \in U$. Again (20), by (29) for $-1 \leq B < A < 0$ (so that $c > a > 0$), can be rewritten as

$$Q(z) = \int_0^1 g(t,z)\,d\mu(t)$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz}$$

and

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\,dt$$

is a positive measure on $[0,1]$.

For $-1 \leq B < A < 0$ it may be noted that $\operatorname{Re}\{g(t,z)\} > 0$, $g(t,-r)$ is real for $0 \leq r < 1$, $t \in [0,1]$ and

$$\operatorname{Re}\left\{\frac{1}{g(t,z)}\right\} = \operatorname{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t,r)}$$

for $|z| \le r < 1$ and $t \in [0,1]$. Therefore by using Lemma 2 we deduce that $\{\mathrm{Re}\,\frac{1}{Q(z)}\} \ge \frac{1}{Q(-r)}$, $|z| \le r < 1$ and by letting $r \to 1^-$ we obtain $\mathrm{Re}\,\{\frac{1}{Q(z)}\} \ge \frac{1}{Q(-1)}$, $z \in U$. This by (27) leads to (21). Hence the theorem.

Putting $A = 0$ in the above theorem and using (23), (13) and (14) we obtain

Corollary 1. *For $f \in H$ and $1 - (\frac{\delta+2}{\alpha}) \le B < 0$, $(|B| \le 1)$ we have the sharp result*

$$(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} + \alpha\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} \prec \frac{1}{1+Bz}$$

implies

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \frac{1}{1+Bz}, \quad z \in U .$$

The case $\delta = n \in N_0$, $B = -1$ was obtained by Al-Amiri [1] and by Ruscheweyh [22] for $\alpha = 1$. Taking $\alpha = 1$, $A = 1 - 2^{\rho}$, $B = -1$ in the above theorem we obtain

Corollary 2. *For all $\delta > -1$ and $\max\{\frac{1}{\delta+2}, \frac{1}{2}\} \le \rho < 1$ we have $T_{\delta,1}(1-2\rho,-1) \subset T_{\delta,0}(1-2\rho'',-1)$ where $\rho'' = [\,F(1, 2(\delta+2)(1-\rho); (\delta+2); 1/2)]^{-1}$.*

If we take $\rho = (\delta+1)/(\delta+2)$ in Corollary 2 it follows that for $f \in H$ and $\delta \ge 0$ we have the sharp result $\mathrm{Re}\,\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} > \frac{\delta+1}{\delta+2}$ implies $\mathrm{Re}\,\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} > \frac{1}{F(1,2;\delta+2;\frac{1}{2})} > \frac{\delta}{\delta+1}$, $z \in U$. This improves Singh and Singh's result [24] obtained for $\delta = n \in N_0$.

Taking $A = 1 - 2\rho$ and $B = -1$ in Theorem 1, the following Corollary not only gives the correct form of the containment relation (7) but also shows that it is not possible to improve it further.

Corollary 3. *Let $\delta > -1$, $0 < \alpha < \delta + 2$ and $\frac{\alpha}{\delta+2} \le \rho < 1$. Then*

$$T_{\delta,\alpha}(1-2\rho,-1) \subset T_{\delta,0}\left(1 - 2\frac{(\delta+2)\rho-\alpha}{\delta+2-\alpha}, -1\right) \tag{30}$$

Further if $f(z) \in T_{\delta,\alpha}(1-2\rho,-1)$ then

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \frac{\alpha}{\delta+2-\alpha}\left[\int_0^1 \left(\frac{1-tz}{1-z}\right)^{\frac{-2(\delta+2)}{\alpha}(1-\rho)} t^{\frac{\delta+2}{\alpha}-2}\,dt\right]^{-1} \tag{31}$$

Further more if $\max\{\frac{\alpha}{\delta+2}, \frac{1}{2}\} \le \rho < 1$ then

$$T_{\delta,\alpha}(1-2\rho,-1) \subset T_{\delta,0}(1-2\rho''',-1) \tag{32}$$

where $\rho''' = \left[F(1, 2(\frac{\delta+2}{\alpha})(1-\rho), \frac{\delta+2}{\alpha}; \frac{1}{2}) \right]^{-1}$.
The result is sharp.

Taking $\delta = 0$, $\alpha = 2\mu/(\mu+1)$ and $\rho = \mu/(\mu+1)$ in Corollary 3 we, by (15), find that if f is μ–convex ($\mu \geq 1$), then $f \in S^*\big(\Gamma((2+\mu)/2\mu)/(\pi^{1/2}\Gamma(\mu+1)/\mu)\big)$ which in turn implies that $f \in K\big(((\mu-1)/\mu)\Gamma((\mu+2)/2\mu)/\pi^{1/2}\Gamma((\mu+1)/\mu)\big)$. This is due to Miller et. al. [17].

Similarly if we take $\delta = 0$, $\alpha = 1$ and $\rho = (1+\lambda)/2$ we obtain from Corollary 3 that for $0 \leq \lambda < 1$, $f \in K(\lambda)$ implies $f \in S^*(\beta(\lambda))$
where

$$\beta(\lambda) = \begin{cases} \dfrac{1-2\lambda}{2^{2-2\lambda}[1-2^{2\lambda-1}]} & \text{if } \lambda \neq 1/2 \\[2ex] \dfrac{1}{2\log 2} & \text{if } \lambda = 1/2. \end{cases}$$

The above expression for $\beta(\lambda)$ can be obtained by expanding $F(1, 2(1-\lambda); 2; 1/2)$. This is due to Goel [7] and MacGregor [15].

The classes $T_{\delta,\alpha}(A, B)$ have been defined for $\delta \geq -1$. However, in Theorem 1, δ has been taken to satisfy $\delta > -1$. The following theorem shows that it is possible to obtain an extension of Theorem 1 for the case $\delta = -1$ and α – real with $\alpha > 0$.

Theorem 2. *Let $f \in H$, $\delta \geq -1$, h be a convex univalent function in U with $h(0) = 1$. Then for a complex number α satisfying* $\operatorname{Re}\alpha > 0$,

$$(33) \qquad (1-\alpha)\frac{D^{\delta+1}f(z)}{z} + \alpha\frac{D^{\delta+2}f(z)}{z} \prec h(z)\,, \quad z \in U$$

implies

$$(34) \qquad \frac{D^{\delta+1}f(z)}{z} \prec \left(\frac{\delta+2}{\alpha}\right) z^{-\left(\frac{\delta+2}{\alpha}\right)} \int_0^z h(t)t^{\frac{\delta+2}{\alpha}-1}\,dt \prec h(z)\,, \quad z \in U$$

The above theorem can be proved on the same lines as those of Theorem 1 using, in place of Lemma 1 the following well known result due to Hallenbeck and Ruscheweyh [9].

Lemma 4. *If $p(z) = 1 + p_1 z + \cdots$ is analytic in U and h is a convex univalent function in U with $h(0) = 1$ and γ is a complex number such that* $\operatorname{Re}\gamma > 0$, *then*

$$(35) \qquad p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

implies

$$p(z) \prec q(z) = \gamma z^{-\gamma} \int_0^z h(t)t^{\gamma-1}\,dt \prec h(z)$$

and $q(z)$ is the best dominant.

By giving different values to the parameters δ, α and choosing suitable convex function h in the above theorem we get the improved form of the results obtained by the authors [11], Chichra [5], Owa and Obradovic [21], Singh and Singh [26] and others.

4. Integral transforms. For a function $f \in H$, Libera [14] defined the integral transform $F_1(z)$ by

$$F_1(z) = \frac{2}{z}\int_0^z f(t)\,dt \tag{36}$$

and showed that

$$f \in S^* \text{ or } K \quad \text{implies that } F_1 \in S^* \text{ or } K \tag{37}$$

respectively. Bernardi [4] showed that the above result (37) continues to hold for the more general integral transform

$$F_c(z) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,dt \tag{38}$$

where $c \in N$. Bajpai and Srivastava [3] extended the result of Bernardi to $S^*(\beta)$ and $K(\beta)$ $(0 \le \beta < 1)$. From Lewandowski et. al. [13] it follows that (37) continues to hold for $F_c(z)$ if c in (38) is taken to be a complex number satisfying $\operatorname{Re} c \geq 0$.

Ruscheweyh [22] considered the Bernardi transform of functions in K_n defined by (38) and showed that $f \in K_n$ implies $F_c(z) \in K_n$ provided $\operatorname{Re} c > \frac{n-1}{2}$. Goel and Sohi [8] attempted to extend this result for $T_n(1-2\beta,-1)$ for $\operatorname{Re} c \geq (1-\beta)n - \beta$ $(n \in N_0,\ 0 \le \beta \le 1/2)$. Al-Amiri [2] showed that $f \in T_\delta(0,-1)$ implies $F_c(z) \in T_\delta(0,-1)$ provided $\operatorname{Re} c > \frac{\delta-1}{2}$ whereas Singh and Singh [24] showed that $f \in T_n(\frac{2-n}{1+n},-1)$ implies $F_1(z) \in T_n(\frac{1-n}{1+n},-1)$.

In our next theorem by showing that the class $T_{\delta,0}(A,B)$ is preserved under the Bernardi Transform (38) we not only get refinements of aforesaid results but also show that it is not possible to improve them further.

Theorem 3. *Let $\delta > -1$, $-1 \le B < A \le 1$ and c be a complex number satisfying*

$$\operatorname{Re} c \geq \frac{\delta(A-B)+A-1}{1-B}. \tag{39}$$

(a) *If $f \in T_{\delta,0}(A,B)$ then the function $F_c(z)$ defined by (38) satisfies $F_c \in T_{\delta,0}(A,B)$. Furthermore we have*

$$\frac{D^{\delta+1}F_c(z)}{D^\delta F_c(z)} \prec \frac{1}{\delta+1}\left[\frac{1}{Q(z)} - (c-\delta)\right] \equiv \tilde{q}(z)\,, \quad z \in U \tag{40}$$

where

(41) $$Q(z) = \begin{cases} \int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{(\delta+1)\frac{(A-B)}{B}} t^c\,dt & \text{if } B \neq 0 \\ \int_0^1 \exp\{(1+\delta)A(t-1)z\} t^c\,dt & \text{if } B = 0. \end{cases}$$

(b) *If in addition to* (39), c *is real and* $\frac{A}{B} > \frac{\delta-(c+1)}{1+\delta}$ *with* $B < 0$, *then for* $f \in T_{\delta,0}(A,B)$ *we have*

$$\operatorname{Re} \frac{D^{\delta+1}F_c(z)}{D^\delta F_c(z)} > \frac{1}{\delta+1}\left[\frac{c+1}{F(1,(1+\delta)((B-A)/B;c+2;-B/(1-B))} - (c-\delta)\right],$$

the bound is sharp.

Proof. Since $F_c(z) = \left(\sum_{j=1}^{\infty} \frac{1+c}{c+j} z^j\right) * f(z) \equiv F(z)$, say, and

$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(\delta+1)} a_n z^n$, it can be easily seen from (38) that

(42) $$z\left(D^\delta F(z)\right)' = (c+1)D^\delta f(z) - CD^\delta F(z)\,.$$

We put

(43) $$g(z) = z\left(\frac{D^\delta F(z)}{z}\right)^{1/(1+\delta)}$$

and $r_1 = \sup\{r : g(z) \neq 0,\ 0 < |z| < r\}$.
Then $g(z)$ is single valued and analytic in $|z| < r_1$ and

(44) $$p(z) = \frac{z\,g'(z)}{g(z)} = \frac{D^{\delta+1}F(z)}{D^\delta F(z)}$$

is analytic in $|z| < r_1$, $p(0) = 1$. (22) and (42) easily lead to

(45) $$(1+\delta)\frac{D^{\delta+1}F(z)}{D^\delta F(z)} + c - \delta = (1+c)\frac{D^{\delta+1}f(z)}{D^\delta F(z)}\,.$$

If $f \in T_{\delta,0}(A,B)$ then it is clear that $D^\delta f(z) \neq 0$ in $0 < |z| < 1$. So (44) and (45) give

(46) $$\frac{D^\delta F(z)}{D^\delta f(z)} = \frac{1+c}{c-\delta+(1+\delta)p(z)}\,.$$

Differentiating (44) and using (42) and (46), we get

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} = p(z) + \frac{zp'(z)}{c-\delta+(1+\delta)p(z)}, \quad |z| < r_1 . \tag{47}$$

Since $f \in T_{\delta,0}(A,B)$, we have by (47) that

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz}, \quad \beta = \delta+1, \quad \gamma = c-\delta . \tag{48}$$

Using Lemma 1, we deduce that

$$p(z) \prec \tilde{q}(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1$$

where $Q(z)$ is given by (41) and $\tilde{q}(z)$ is the best dominant. Since for $-1 \le B < A \le 1$ it is easy to see that Re $\{(1+Az)/(1+Bz)\} > 0$ in U, we have Re $p(z) > 0$ in $|z| < r_1$. Now (44) gives that $g(z)$ is univalent in $|z| < r_1$. Thus it is not possible that $g(z)$ vanishes on $|z| = r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore $p(z)$ is analytic in U and hence by (44) and (48) we obtain the first part of the theorem.

Proceeding as in Theorem 1 the second part follows. Putting $A = 1-2\rho$, $B = -1$ in Theorem 3, we obtain

Corollary 4. *Let δ and ρ be real numbers satisfying $\delta > -1$ and $0 \le \rho < 1$.*

(a) *If $f \in T_{\delta,0}(1-2\rho,-1)$ and c is a complex number satisfying* Re $c \ge \delta-(1+\delta)\rho$ *then $F_c(z)$ defined by* (38) *satisfies $F_c \in T_{\delta,0}(1-2\rho,-1)$. Furthermore we have*

$$\frac{D^{\delta+1}F_c(z)}{D^{\delta}F_c(z)} \prec \frac{1}{\delta+1}\left[\frac{1}{Q(z)} - (c-\delta\right] = \tilde{q}(z), \quad z \in U$$

where $Q(z)$ is obtained from (41) *with $A = 1-2\rho$ and $B = -1$.*

(b) *If c is a real number satisfying* $c \ge \max\{\delta-(1+\delta)\rho, 2[\delta-(1+\delta)\rho]\}$ *and* $f \in T_{\delta,0}(1-2\rho,-1)$, *then* $F_c \in T_{\delta,0}(1-2\rho',-1)$ *with* $\rho' = \frac{1}{\delta+1}\left[\frac{c+1}{F(1,2(1+\delta)(1-\rho);c+2;1/2)} - (c-\rho)\right]$. *The result is sharp.*

Remarks : (i) Substituting $\delta = 0$, in part (b) of Corollary 4 we see that $f \in S^*(\rho)$ $(0 \le \rho < 1)$ implies $F_c \in S^*\left(\frac{c+1}{F(1,2(1-\rho);c+2,1/2)} - c\right)$ provided $c \ge -\rho$, which is the improvement of a result of Bajpai and Srivastava [3] and Bernardi [4] for $c = 1,2,..$

(ii) Corollary 4 includes an improvement of the recent result of the authors [12, Theorem 3] including the Singh and Singh's [24] result. That is for $f \in H$, $\rho = \frac{2n-1}{2(n+1)}$, $\delta = n \in N$ and $c = 1$ it follows that

$$\text{Re}\,\frac{D^{n+1}f(z)}{D^{n}f(z)} > \frac{2n-1}{2(n+1)} \text{ implies Re}\,\frac{D^{n+1}F_1(z)}{D^{n}F_1(z)} > \frac{n}{n+1}, \quad z \in U$$

where $F_1(z)$ is defined by (36). This for $n = 1$ extends the results of Libera [14].

(iii) It can be easily seen that by giving different values to the parameters δ, A, B and c, the results obtained in this direction by Al-Amiri [2], Ruscheweyh and Singh [23], Goel and Sohi [8], etc. get improved considerably in many cases.

In the case where c in Bernardi transform (38) is taken to be δ, Theorem 3 gets improved as follows and can be proved on the same line as that of Theorem 3 (see also [24]). So we omit its proof.

Theorem 4. *Let $\delta > -1$ and $-1 \le B < A \le 1$. If $f \in T_{\delta,0}(A,B)$ then the function $F_\delta(z)$ defined by*

$$F_\delta(z) = \frac{\delta+1}{z^\delta} \int_0^z t^{\delta-1} f(t)\, dt$$

belongs to $T_{\delta,1}(A'',B)$ where $A'' = \dfrac{B + A(\delta+1)}{\delta+2}$

It is to be noted that for $A = 1 - 2\rho$, $B = -1$ and $0 \le \rho < 1$ we obtain the earlier result of the authors in [12, Theorem 5]. For $A = 1 - 2\left(\dfrac{n}{n+1}\right)$, $B = -1$ and $\delta = n \in N_0$ Theorem 4 leads to the result of Singh and Singh [24].

The following theorem shows that it is possible to obtain an extension of Theorem 3, to the case $\delta = -1$ also. Since this can be easily proved using Lemma 4, so we omit its proof.

Theorem 5. *Let $\delta \ge -1$, $\mathrm{Re}\,(1+c) > 0$, h be a convex univalent function in U with $h(0) = 1$ and $f \in H$. Then we have*

$$\frac{D^{\delta+1} f(z)}{z} \prec h(z) \text{ implies } \frac{D^{\delta+1} F_c(z)}{z} \prec q(z) = (1+c) z^{-(1+c)} \int_0^z h(t) t^c\, dt$$

where F_c is defined by (38).

The above theorem generalizes and improves the earlier results obtained in [11]. For the case $\delta = 0$ and $c = 1$, Theorem 5 not only generalizes an earlier results of Libera [14] but also shows the result obtained is sharp. Further, Theorem 5 extends the result of Singh and Singh [26, Theorem 4] for suitably chosen $h(z)$.

4. Recently, Mocanu [20] showed that for $f \in H$

$$\left|\frac{f''(z)}{f'(z)}\right| < \tfrac{3}{2} \text{ implies } \left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U \tag{49}$$

and in [25] Singh and Singh proved that if for some $\gamma \ge 0$,

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\gamma} \left|\frac{zf''(z)}{f'(z)}\right| < \left(\tfrac{3}{2}\right)^\gamma, \quad z \in U \tag{50}$$

holds then $f \in S^*$. The results (49) and an improved form of (50) can be obtained from the following more general theorem.

Theorem 6. *If $\delta \geq -1$ and $f \in H$ satisfies the condition*

$$\left|\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-1\right|^{1-\gamma}\left|\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}-M\right|^{\gamma}<\beta(M,\delta,\gamma)\,,\quad z\in U \tag{51}$$

for some $\gamma \geq 0$ and $M \leq 1$, where $\beta(M,\gamma) \equiv \left[\dfrac{2M(\delta+2)-3)^2+8(\delta+1)}{4(\delta+2)^2}\right]^{\gamma/2}$, then

$$\left|\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-1\right|<1\,,\quad z\in U\,. \tag{52}$$

For the proof of the theorem we need the following lemma.

Lemma 5 [20]. *Let Ω be a set in the complex plane $\mathbb{C}$. Suppose that the function $\Psi:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition*

$$\Psi(ir_2,s_1)\notin\Omega$$

for all real r_2 and $s_1 \leq -\dfrac{(1+r_2^2)}{2}$. If $p(z)=1+p_1z+\cdots$ is analytic in U and $\Psi\big(p(z),zp'(z)\big)\in\Omega$ for $z\in U$, then $\operatorname{Re}p(z)>0$ in U.

Proof of Theorem 6. Set $p(z)=2\dfrac{D^{\delta}f(z)}{D^{\delta+1}f(z)}-1$, then $p(z)$ is regular in U with $p(0)=1$. A simple calculation shows that

$$\begin{aligned}&\left[\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-1\right]^{1-\gamma}\left[\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}-M\right]^{\gamma}=\\&=\left[\frac{1-p(z)}{1+p(z)}\right]^{1-\gamma}\left[\frac{1}{\delta+2}\left\{(\delta+1)\left(\frac{1-p(z)}{1+p(z)}\right)-\frac{zp'(z)}{p(z)+1}\right\}+1-M\right]^{\gamma}\equiv\\&\equiv\Psi\big(p(z),zp'(z)\big)\,,\end{aligned} \tag{53}$$

where

$$\Psi(r,s)=\left[\frac{1-r}{1+r}\right]^{1-\gamma}\left[\frac{1}{\delta+2}\left\{\frac{\delta+1+(\delta+2)(1-M)-s-r\big(M(\delta+2)-1\big)}{1+r}\right\}\right]^{\gamma}$$

with $r=p(z)$ and $s=zp'(z)$.

By (53), we have to prove that $|\Psi\big(p(z),zp'(z)\big)|<\beta(M,\delta,\gamma)$, $z\in U$ implies that $\operatorname{Re}p(z)>0$ in U which is equivalent to showing (52). Now for all real r_2, and

$s_1 \le -(1+r_2^2)/2$, we have

$$
\begin{aligned}
|\Psi(ir_2, s_1)|^2 &= \Big[\frac{((\delta+1)+(\delta+2)(1-M)-s_1)^2+r_2^2(M(\delta+2)-1)^2}{1+r_2^2}\Big]^\gamma = \\
&= \frac{1}{(\delta+2)^{2\gamma}}\Big[\frac{s_1^2-2\{(\delta+1)+(\delta+2)(1-M)\}s_1}{1+r_2^2}+ \\
&+\frac{4(\delta+1)(\delta+2)(1-M)}{1+r_2^2}+(M(\delta+2)-1)^2\Big]^\gamma \ge \\
&\ge \frac{1}{(s+2)^{2\gamma}}\Big[\tfrac{1}{4}+\{(\delta+1)+(\delta+2)(1-M)\}+(M(\delta+2)-1)^2\Big]^\gamma = \\
&= \Big[\frac{(2M(\delta+2)-3)^2+8(\delta+1)}{4(\delta+2)^2}\Big]^\gamma \equiv \\
&\equiv \beta(M,\delta,\gamma)
\end{aligned}
$$

Taking Ω to be the set $\Omega = \{w \in \mathbb{C} : |w| < \beta(M,\delta,\gamma)\}$, we see by lemma 5 that $\operatorname{Re} p(z) > 0$ in U. Hence the theorem.

Remark. Taking $\delta = 0$ and $M = 1$ in Theorem 6, it folows that for $\gamma \ge 0$ and $f \in H$,

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\gamma}\left|\frac{zf''(z)}{f'(z)}\right| < \left(\tfrac{3}{2}\right)^\gamma \text{ implies } \left|\frac{zf'(z)}{f(z)}-1\right| < 1, \quad z \in U. \tag{54}$$

whereas for $\delta = -1$ and $M = 0$, Theorem 6 gives

$$|f'(z)-1|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}\right|^\gamma < \left(\tfrac{3}{2}\right)^\gamma \text{ implies } |f'(z)-1| < 1, \ z \in U. \tag{55}$$

(54) improves Theorem 3 in [25] while (55) gives Theorem 4 in [25]. The case $\gamma = 1$ in (54) is due to Mocanu [20].

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STRESZCZENIE

Otrzymano pewne zastosowania podporządkowania różniczkowego Briota–Bouqueta, które ulepszają i zaostrzają kilka znanych twierdzeń teorii funkcji jednolistnych. W szczególności otrzymano w ten sposób zaostrzenie pewnych wyników dotyczących transformat Libery i Bernardiego.