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# An Extension of a Sufficient Condition for $|p|$ – valence of Analytic Functions

O rozszerzeniu warunku dostatecznego $|p|$–listności funkcji analitycznych

**Abstract.** In this paper a sufficient condition for $|p|$–valence ($p$ being an integer) of functions $f$ analytic in the punctured unit disk $E \setminus \{0\}$ and satisfying $\lim_{z\to 0} z^{-p} f(z) = 1$ has been established (Theorem 1).

Using this condition we show that for a positive integer $p$ the function $F(z) = p\int_0^z t^{p-1}(f(t))^\alpha\, dt$ is $p$–valent in $E$, if $|\alpha| \le p/(6p-2)$.

**1. Introduction.** Let $E = \{z : |z| < 1\}$ and let $\Omega$ denote the class of analytic functions $\omega$ such that : $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in E$.

We denote by $S$ the class of functions $f$ of the form : $f(z) = z + a_2 z^2 + \cdots$ analytic and univalent in $E$ and by $P$ the class of analytic functions $h$ in $E$ such that : $\operatorname{Re} h(z) > 0$ for $z \in E$, $h(0) = 1$.

Let $f, F$ be analytic functions in $E$ and $f(0) = F(0)$. We say that the function $f$ is subordinate to the function $F$ in $E$ if there exists a function $\omega$ in $\Omega$ such that $f(z) = F(\omega(z))$ in $E$. This relation shall be denoted $f \prec F$.

**Definition** [3]. Let $I = [0, \infty)$. A family of functions $f(z,t)$, $z \in E$, $t \in I$ is called a $p$–subordination chain on $I$ if

(1.1) $f(z,t)$ is analytic in $E$ for each fixed $t \in I$

(1.2) $f^{(k)}(0,t) = 0$ , $k = 1, \ldots, p-1$ , and $f^{(p)}(0,t) \neq 0$,

(1.3) $f(z,s) \prec f(z,t)$ , $s \le t$ , $s,t \in I$ , $z \in E$.

The $p$–subordinaton chain is said to be normalized if $f(0,t) = 0$ and $f^{(p)}(0,t) = p! e^{pt}$ for each $t \in I$.

Folowing Pommerenke [5], Hallenbeck and Livingston [3] proved :

**Lemma 1.** *Let $f(z,t) = e^{pt} z^p + \cdots$ be analytic in $E$ for each $t \in I$. Then $f(z,t)$ is a normalized $p$–subordination chain on $I$ if and only if $f(z,t)$ is locally absolutely continuous in $I$, locally uniformly in $E$ and there exists a function $h(z,t)$ measurable in $t$ and analytic in $z$ with $\operatorname{Re} h(z,t) > 0$, $h(0,t) = 1$ such that for each $z \in E$ and* *almost all $t \in I$*

$$\dot{f}(z,t) = zf'(z,t)\,h(z,t)\,, \qquad (\dot{f} = \tfrac{\partial f}{\partial t}\,,\ f' = \tfrac{\partial f}{\partial z}) \tag{1.4}$$

**Lemma 2.** *Let $0 < r < 1$ and $f(z,t) = a_p(t)z^p + \sum_{n=p+1}^{\infty} a_n(t)z^p$, $a_p(t) \neq 0$ for each $t \in I$ be analytic in $E_r = \{z : |z| < r\}$ and locally absolutely cotinuous in $I$, locally uniformly in $E_r$. For almost all $t \in I$ suppose*

$$\dot{f}(z,t) = zf'(z,t)\,h(z,t)\,, \qquad z \in E_r\,,$$

*where $h(z,t)$ is analytic in $E$ and satisfies $\operatorname{Re} h(z,t) > 0$ for $z \in E$. If*

$$\lim_{n\to\infty} |a_n(t)| \left|\frac{a_p(t)}{a_p(0)}\right|^{-n/p} = 0 \tag{1.5}$$

*then for each $t \in I$, $f(z,t)$ is the $p$-th power of a univalent function.*

**2. $|p|$ – valence criterion.**

**Theorem 1.** *Let $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $p$ is an integer, be an analytic function different from zero in $E \setminus \{0\}$, and $\lim_{z\to 0} z^{-p}f(z) = 1$. Moreover, let $s$ be any complex number such that $\operatorname{Re} s = \alpha > 0$. If there exists an analytic function $g$, $g(z) \neq 0$ in $E$ satisfying the inequality*

$$\left|\frac{g(0)}{g(z)} \cdot \frac{zf'(z)}{f(z)} - p\,\frac{s+1}{2\alpha}\right| \leq |p|\,\frac{|s+1|}{2\alpha} \tag{2.1}$$

*and such that*

$$\left|\,|z|^a \frac{g(0)}{g(z)} \cdot \frac{zf'(z)}{f(z)} + (1 - |z|^a)\left(\frac{zf'(z)}{f(z)} + s\,\frac{zg'(z)}{g(z)}\right) - p\,\frac{s+1}{2\alpha}\right| \leq \tag{2.2}$$
$$\leq |p|\,\frac{|s+1|}{2\alpha}\,, \quad z \in E\,, \quad a = \frac{s+1}{\alpha s}\cdot p$$

*holds, then $f$ is the $p$-th power of a univalent function in $E$.*

**Proof.** We consider two cases
$1^\circ$. $p$ is a positive integer.

We put

$$f(z,t) = f(ze^{-st})\left[1 + (e^{\frac{s+1}{s}pt} - 1)\frac{g(ze^{-st})}{g(0)}\right]^s = e^{pt}z^p + \cdots \tag{2.3}$$
$$z \in E\,, \quad t \in I\,, \quad 1^s = 1\,,$$

where $g$ is the function which satisfies in $E$ the assumptions of theorem for some $t \in I$.

For each $t$ in $I$ $f(z,t)$ is an analytic function for $|z| < e^{\alpha t}$, $\alpha = \mathrm{Re}\, s$. It follows from Lemma 1 that $f(z,t)$ is a normalized $p$-subordination chain on $I$ if and only if

$$\frac{\dot{f}(z,t)}{zf'(z,t)} = \frac{1+\omega(z,t)}{1-\omega(z,t)}\,, \quad z \in E\,, \quad t \in I \tag{2.4}$$

where

$$\frac{1+\omega(z,t)}{1-\omega(z,t)} = h(z,t)\,, \quad z \in E\,, \text{ for each fixed } t \in I\,,$$

is a well-known relationship between the functions $\omega$ and $h$ of the class $\Omega$ and $P$, respectively.

From (2.4) and (2.3) after long but simple calculations we obtain :

$$\omega(z,t) = \frac{\dot{f}(z,t) - zf'(z,t)}{\dot{f}(z,t) + zf'(z,t)} = \frac{s+1}{s-1}\,\frac{w}{w - \frac{2p}{s-1}} \tag{2.5}$$

where

$$w = g(0)\,\frac{ze^{-st}f'(ze^{-st})}{f(ze^{-st})g(ze^{-st})}\,e^{-bt} + (1-e^{-bt})\Big(\frac{ze^{-st}f'(ze^{-st})}{f(ze^{-st})} + s\,\frac{ze^{-st}g'(ze^{-st})}{g(ze^{-st})}\Big) - p\,, \quad b = \frac{s+1}{s}\,p\,. \tag{2.6}$$

It follows from Lemma 1 and Lemma 2 that $f(z) = f(z,0)$ is the $p$-th power of univalent function if

$$\mathrm{Re}\left\{\frac{\dot{f}(z,t)}{zf'(z,t)}\right\} > 0\,, \quad z \in E\,, \quad t \in I\,,$$

thus for each $t \in I$ the function $\omega$ given by (2.5) will satisfy in $\overline{E}$ ($\overline{E}$ is the closure of $E$) the inequality : $|\omega(z,t)| \leq 1$.

The inequality (2.2) implies that for $z \in \overline{E}$ $\omega(z,t)$ given by (2.5) satisfies the inequality $|\omega(z,t)| \leq 1$ because it is enough to put $ze^{-st} = \varsigma$ in (2.6), $e^{-bt} = |\varsigma|^{b/\alpha} = |\varsigma|^a$ for $|z| = 1$, $\alpha = \mathrm{Re}\, s$ and apply the inequality (2.2) exchanging $z$ for $\varsigma$ there.

2°. The proof of theorem for $p$ being a negative integer is analogous. It is enough to consider the chain

$$f(z,t) = [f(ze^{-st})]^{-1}\Big[1 + (e^{\frac{s+1}{s}pt} - 1)\frac{g(ze^{-st})}{g(0)}\Big]^{-s} = e^{-pt}z^{-p} + \cdots$$

The proof is complete.

**Corollary 1.** *If we put $s = 1$ in the Theorem 1, and $g(z) = k(z)\dfrac{zf'(z)}{f(z)}$, $\mathrm{Re}\, k(z) \geq \frac{1}{2}$, $z \in E$, $k(0) = 1$ and*

$$\Big|\,|z|^{2p}\big(\frac{p}{k(z)} - 1\big) + (1-|z|^{2p})\Big(\frac{z'(z)}{k(z)} + \frac{zf''(z)}{f(z)}\Big) - p + 1\Big| \leq |p| \tag{2.7}$$

*then the function $f$ is the $|p|$-th power of a univalent function in $E$.*

This result for $p = 1$ and a suitably chosen $k$ was obtained by the present authors in [4]. The assumption $k(0) = 1$ can be dropped (see [5] Corollary 3).

**Corollary 2.** *If in Corollary 1 we put additionally $\frac{p}{k(z)} - 1 = c =$ const., $|c - p + 1| \le |p|$ and*

$$\left| c|z|^{2p} + (1 - |z|^{2p}) \frac{zf''(z)}{f'(z)} - p + 1 \right| \le |p| \tag{2.8}$$

*then the function $f$ is the $p$-th power of a univalent function in $E$.*

Putting $c = p - 1$ in the inequality (2.8) we obtain Theorem 21, 4, 1° given by Avkhadiev [2].

For $p = 1$ the condition (2.8) is a well-known sufficient condition of univalence given by Ahlfors [1].

A simple corollary of Theorem 1 is :

**Theorem 2.** *Let $f(z) = z + a_2 z^2 + \cdots$ be an analytic function in $E$ and $s$ be any complex number such that* Re $s = \alpha > 0$. *If there exists an analytic function $g$ and $g(z) \ne 0$ in $E$ satisfying the inequality*

$$\left| \frac{g(0)}{g(z)} \frac{zf'(z)}{f(z)} - \frac{s+1}{2\alpha} \right| \le \frac{|s+1|}{2\alpha} \tag{2.9}$$

*and such that for a fixed, positive integer $p$*

$$\begin{aligned} \left| p|z|^{a} \frac{g(0)}{g(z)} \cdot \frac{zf'(z)}{f(z)} + (1 - |z|^{a})\left(p \frac{zf'(z)}{f(z)} + s \frac{zg'(z)}{g(z)}\right) - p \frac{s+1}{2\alpha} \right| &\le \\ \le p \frac{|s+1|}{2\alpha}, \quad z \in E, \quad a = \frac{s+1}{2\alpha}, \quad \alpha = \text{Re}\, s, \end{aligned} \tag{2.10}$$

*holds, then $f$ is a univalent function in $E$.*

**Proof.** Let $h(z) = [f(z)]^p$, $z \in E$. It is easy to see that $h(z)$ satisfies the assumption of Theorem 1 if $f$ satisfies the assumption of this theorem. Thus $f$ is univalent function in $E$ because $h$ in view of Theorem 1 is the $p$-th power of a univalent function.

**3. An application.**

**Theorem 3.** *If $f \in S$ and $\alpha$ is any complex number such that $|\alpha| \le \frac{p}{6p-3}$, where $p$ is a positive integer, then the function*

$$F(z) = p \int_0^z t^{p-1} (f'(t))^{\alpha}\, dt, \qquad z \in E \tag{3.1}$$

*is p-valent in E.*

**Proof.** In the proof we use Corollary 2 and a well-known Bieberbach transformation preserving the class of univalent functions $S$

$$g(z)=\frac{f(\frac{z+z_0}{1+z\bar{z}_0})-f(z_0)}{f'(z_0)(1-|z_0|^2)}\,,\qquad z\in E\,,\quad f,g\in S\,, \tag{3.2}$$

$z_0$ is a fixed point of the disk $E$.

From (3.2) the value of the functional $\frac{-z_0F''(-z_0)}{F'(-z_0)}$ obtained from (3.1) at the point $-z_0=z$ is equal to :

$$\frac{zF''(z)}{F'(z)}=p-1+\alpha\,\frac{zf''(z)}{f'(z)}=p-1+\alpha\,\frac{2|z|^2+2b_2z}{1-|z|^2}$$

where $b_2$ is the second coefficient of the function $g\in S$ in Maclaurin's expansion.

Putting $c=p-1-2\alpha$ in (2.8) and using the above equality we have

$$\left|(p-1-2\alpha)|z|^{2p}+(1-|z|^{2p})\frac{zF''(z)}{F'(z)}-p+1\right|=$$
$$=\left|(p-1-2\alpha)|z|^{2p}+(1-|z|^{2p})\alpha\,\frac{2|z|^2+2b_2z}{1-|z|^2}-(p-1)|z|^{2p}\right|=$$
$$=2|\alpha|\,\left|b_2z(1+|z|^2+\cdots+|z|^{2(p-1)})+|z|^2(1+|z|^2+\cdots+|z|^{2(p-2)})\right|\le$$
$$\le 2|\alpha|(3p-1)\,.$$

In view of the assumption $|\alpha|\le\frac{p}{6p-2}$ and the Corollary 2 we obtain the assertion of the theorem.

**Remark.** Using the criterion of $p$-valence stated in [2] for the function $F(z)$ given by (3.1) we conclude that this function is $p$-valent if $|\alpha|\le\frac{1}{6}$.

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## STRESZCZENIE

W pracy podano warunek dostateczny $|p|$-listności ($p$ liczba całkowita) funkcji analitycznych $f(z)$ w $E \setminus \{0\}$ , $\lim\limits_{z \to 0} |z^{-p} f(z)| = 1$ (Twierdzenie 1).

Stosując ten warunek pokazano, że dla dodatniego i całkowitego $p$ funkcje

$$F(z) = p \int_0^z t^{p-1} (f(t))^{\alpha} \, dt$$

są $p$-listne w kole jednostkowym jeśli $|\alpha| \leq \frac{p}{6p-3}$.