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On Solutions of a Stochastic Integral Equation of the Volterra-Fredholm Type

O rozwiązaniach stochastycznego równania całkowego typu Voltery-Fredholma

Abstract. The aim of this paper is the study of the mixed random Volterra-Frdholm equation of the form

$$x(t;\omega) = h(t, x(t,\omega)) + \int_0^t k_1(t,\tau;\omega) f_1(\tau, x(\tau;\omega))\, d\tau + \int_0^\infty k_2(t,\tau;\omega) f_2(\tau, x(\tau;\omega))\, d\tau\,,$$

under less restrictive conditions than those of [9] and [10]. Namely, we only assume that f_1 and f_2 are sublinear functions.

1. Introduction. The aim of this paper is to investigate the existence and the stability of stochastic integral equation of Volterra-Fredholm type. Problems concerning stochastic differential and integral equations have been treated in many papers and monographs (cf. [3], [4], [7], [8], [9], [10], [11], [12], [13]). The aim of this paper is to give a new existence theorem for a stochastic integral equation of the Volterra-Fredholm type of [9] and [10] (cf. also [13]) and to investigate the asymptotic behaviour and the stability of solutions of that equation.

The most important problem examined up to now is that one concerning the existence of solutions of considered equations. It is solved mostly by the Banach fixed point principle, the Schauder fixed point theorem and successive approximations (cf. [3], [4], [7], [9], [10], [11], [12], [13]). This paper uses the notion of measure of noncompactness in a Banach space and the fixed-point theorem of Darbo type, cf. [2], [6]. This approach allows us to weaken conditions of (cf. [9], [10], [13]). Namely, we replace the Lipschitz type conditions by those with sublinear functions. The asymptotic stability in mean square is also investigated here.

We shall deal with a stochastic integral equation of the Volterra-Fredholm type of the form

$$x(t;\omega) = h(t, x(t;\omega)) + \int_0^t k_1(t,\tau;\omega) f_1(\tau, x(\tau;\omega))\, d\tau + \int_0^\infty k_2(t,\tau;\omega) f_2(\tau, x(\tau;\omega))\, d\tau \,, \tag{1.1}$$

where $t \geq 0$ and

(i) $\omega \in \Omega$, where Ω is the supporting set of the complete probability measure space $(\Omega, \mathcal{A}, P)$;

(ii) $x(t;\omega)$ is the unknown random function for $t \in \mathbf{R}_+$ (= the set of nonnegative real numbers);

(iii) h is a scalar function $h : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$;

(iv) $f_1(t,x)$ is a scalar function of $t \in \mathbf{R}_+$ and $x \in \mathbf{R}$;

(v) $f_2(t,x)$ is a scalar function defined for $t \in \mathbf{R}_+$ and $x \in \mathbf{R}$, the real line;

(vi) $k_1(t,\tau;\omega)$ is a stochastic kernel defined for t and τ satisfying $0 \leq \tau \leq t < \infty$;

(vii) $k_2(t,\tau;\omega)$ is a stochastic kernel defined for t and τ in $\mathbf{R}_+$.

2. Mathematical preliminaries. We shall give here some mathematical concepts that are essential in understanding the details of this paper.

We now give the following definitions.

Definition 2.1. We shall call $x(t;\omega)$ a random solution of the stochastic integral equation (1.1) if for every fixed $t \in \mathbf{R}_+$, $x(t;\omega) \in L^2(\Omega, \mathcal{A}, P)$ and satisfies (1.1) P-a.e.

Definition 2.2. A random solution $x(t;\omega)$ is said to be asymptotically stable in mean square if

$$\lim_{t\to\infty} E|x(t;\omega)|^2 = 0 \,.$$

Throughout this paper $\mathcal{X}$ will denote an infinitely dimensional real Banach space with norm $\|\ \|$ and the zero element 0. $V(x,r)$ stands for the closed ball centered at x of radius r. Denote by $\mathcal{M}_{\mathcal{X}}$ the family of all nonempty bounded subsets of $\mathcal{X}$, and by $\mathcal{N}_{\mathcal{X}}$ the family of all relatively compact and nonempty subsets of $\mathcal{X}$.

The following axioms defining a measure of noncompactness are taken from Banaś and Goebel [2].

Definition 2.3. A nonempty family $\mathcal{B} \subset \mathcal{N}_{\mathcal{X}}$ is said to be the kernel (of measure of compactness), provided it satisfies the following conditions:

1° $U \in \mathcal{B} \Longrightarrow \overline{U} \in \mathcal{B}$;

2° $U \in \mathcal{B}, V \subset U, V \neq \emptyset \Longrightarrow V \in \mathcal{B}$;

3° $U, V \in \mathcal{B} \Longrightarrow \lambda U + (1-\lambda)V \in \mathcal{B}$, $\lambda \in [0,1]$;

4° $U \in \mathcal{B} \Longrightarrow \mathrm{Conv}\, U \in \mathcal{B}$;

5° $\mathcal{B}^c$ (the subfamily of $\mathcal{B}$ consisting of all closed sets) is closed in $\mathcal{N}^c$ with respect to the topology generated by Hausdorff metric.

Definition 2.4. The function $\mu : \mathcal{M}_{\mathcal{X}} \to [0, +\infty)$ is said to be a measure of noncompactness with the kernel ($\ker \mu = \mathcal{B}$) if it satisfies the following conditions :

1° $\mu(U) = 0 \Longleftrightarrow U \in \mathcal{B}$;

2° $\mu(U) = \mu(\overline{U})$;

3° $\mu(\text{Conv}\, U) = \mu(U)$;

4° $U \subset V \Longrightarrow \mu(U) \leq \mu(V)$;

5° $\mu(\lambda U + (1-\lambda)V) \leq \lambda\, \mu(U) + (1-\lambda)\mu(V), \lambda \in [0,1]$;

6° if $U_n \in \mathcal{M}_{\mathcal{X}}, U_n = \overline{U}_n$, and $U_{n+1} \subset U_n$, $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} \mu(U_n) = 0$, then $U = \bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

If a measure of noncompactness μ satisfies in addition the following two conditions :

7° $\mu(U + V) \leq \mu(U) + \mu(V)$;

8° $\mu(\lambda U) = |\lambda| \mu(U)$, $\lambda \in \mathbf{R}$;

it will be sublinear.

Let $M \subset \mathcal{X}$ be a nonempty set and let μ be a measure of noncompactness on $\mathcal{X}$.

Definition 2.5. We say that a continuous mapping $T : M \to \mathcal{X}$ is a contraction with respect to μ (μ-contraction) if for any set $U \in \mathcal{M}_{\mathcal{X}}$ its image $TU \in \mathcal{M}_{\mathcal{X}}$, and there exists a constant $k \in [0, 1)$ such that

$$\mu(TU) \leq k \cdot \mu(U) .$$

We shall use the following modified version of the fixed-point theorem of Darbo type.

Theorem 2.1. *Let C be a nonempty, closed, convex and bounded set of $\mathcal{X}$ and let $T : C \to C$ be an arbitrary μ-contraction. Then T has at least one fixed point in C and the set* $\text{Fix}\, T = \{x \in C : Tx = x\}$ *of all fixed points of T belongs to* $\ker \mu$.

Let $C_p(\mathbf{R}_+, L^2(\Omega, \mathcal{A}, P), p)$ (or shortly C_p) denote a space of all continuous maps $x(t;\cdot)$ from $\mathbf{R}_+$ into $L^2(\Omega, \mathcal{A}, P)$ with the topology defined by the norm

$$\|x\|_p = \sup\{p(t)\|x(t)\|_{L^2} : t \geq 0\} < \infty .$$

The space C_p with norm $\| \; \|_p$ is a real Banach space (see Banaś [1], Zima [14]).

Now for $x \in C_p$, $U \in \mathcal{M}_C$, $T > 0$, and $\varepsilon > 0$, we put

$$\beta^T(x, \varepsilon) = \sup\{\|p(t)x(t) - p(s)x(s)\|_{L^2} : t, s \in [0, T]\,,\ |t - s| \leq \varepsilon\};$$

$\beta_0^T(U,\varepsilon) = \sup\{\beta^T(x,\varepsilon) : x \in U\}$;

$\beta_0^T(U) = \lim_{\varepsilon\to 0} \beta^T(U,\varepsilon)$;

$\beta_0(U) = \lim_{T\to\infty} \beta_0^T(U)$;

$a(U) = \lim_{T\to\infty} \sup_{x\in U} \sup_{t\geq T} \|x(t)\|_{L^2} p(t)$;

$b(U) = \lim_{T\to\infty} \sup_{s,t\geq T} \{\|p(t)x(t) - p(s)x(s)\|_{L^2}\}$;

$\mu_0(U) = \beta_0(U) + a(U) + \sup\{p(t)m(U(t)) : t \geq 0\}$;

$\mu_1(U) = \beta_0(U) + b(U) + \sup\{p(t)m(U(t)) : t \geq 0\}$,

where m is a sublinear measure of noncompactness on $\mathcal{M}_{L^2(\Omega,\mathcal{A},P)}$ and

$$U(t) = \{x(t) \in L^2(\Omega,\mathcal{A},P) : x \in U\}$$

The functions μ_0 and μ_1 define sublinear measure of noncompactness on $\mathcal{M}_{C_p}$ (see [1], [2]). It is also known (see [1], [2]) that ker μ_0 is the set of all sets $U \in \mathcal{M}_{C_p}$ such that the functions belonging to U are equicontinuous on any compact of $\mathbf{R}_+$ and

$$\lim_{t\to\infty} p(t)\|x(t)\|_{L^2} = 0$$

uniformly with respect to $x \in U$. Further properties of μ_0 and μ_1 can be found in [1] and [2].

3. Main results. We make the following assumptions concerning the equation (1.1).

For each t and τ such that $0 \leq \tau \leq t < \infty$ the stochastic kernel $k_1(t,\tau;\omega)$ has values in $L_\infty(\Omega,\mathcal{A},P)$ and the stochastic kernel $k_2(t,\tau;\omega)$ for each t and τ in $\mathbf{R}_+$ has values in $L_\infty(\Omega,\mathcal{A},P)$.

The mappings

$$(t,\tau) \to k_1(t,\tau;\omega) \quad \text{and} \quad (t,\tau) \to k_2(t,\tau;\omega)$$

from the sets

$$\Delta_1 = \{(t,\tau) : 0 \leq \tau \leq t < \infty\} \quad \text{and} \quad \Delta_2 = \{(t,\tau) : 0 \leq \tau < \infty,\ 0 \leq t < \infty\},$$

respectively, into $L_\infty(\Omega,\mathcal{A},P)$ are continuous.

We define for $0 \leq \tau < t < \infty$,

$$k_1(t,\tau) = P\text{–ess}\sup_{\omega\in\Omega} |k_1(t,\tau;\omega)|,$$

and for each t, and τ in $\mathbf{R}_+$

$$k_2(t,\tau) = P\text{–ess}\sup_{\omega\in\Omega} |k_2(t,\tau;\omega)|.$$

The above assumptions imply that if $x \in C_p$ then for each $t \in \mathbf{R}_+$

$$\|k_i(t,\tau)x(\tau)\|_{L^2} \leq k_i(t,\tau)\|x(\tau)\|_{L^2}, \quad i = 1,2.$$

Theorem 3.1. *Suppose that the functions f_i, $i = 1,2$, and h in the stochastic integral equation of the Volterra–Fredholm type (1.1) satisfy the following conditions :*

(i) *functions $f_i : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$, $i = 1,2$, are sublinear, that are $|f_i(t, x(t;\omega))| \leq u_i(t)|x(t;\omega)| + v_i(t)$ P-a.s., $i = 1,2$, for some nonnegative functions u_i and v_i, $i = 1,2$, are continuous and defined for $t \in \mathbf{R}_+$, and let us denote*

$$A = \sup\Big\{p(t)\Big(\int_0^t k_1(t,\tau)(u_1(\tau)/p(\tau))\,d\tau + \int_0^\infty k_2(t,\tau)(u_2(\tau)/p(\tau))\,d\tau\Big) : t \in \mathbf{R}_+\Big\},$$

$$B(t) = p(t)\Big(|h(t,0)| + \int_0^t k_1(t,\tau)v_1(\tau)\,d\tau + \int_0^\infty k_2(t,\tau)v_2(\tau)\,d\tau\Big) \quad \text{for } t \in \mathbf{R}_+,$$

$$B = \sup\{B(t) : t \in \mathbf{R}_+\} < \infty .$$

Suppose that

(ii) $\lim_{t\to\infty} B(t) = 0$;

(iii) $|ht, x(t;\omega)) - h(t, y(t;\omega))| \leq k|x(t;\omega) - y(t;\omega)|$ *P-a.s. for $k \in [0,1)$;*

(iv) $M := k + A < 1$;

(v) *for any given but fixed $T > 0$*

$$\lim_{\varepsilon\to 0} \sup\{\|h(t,x(s)) - h(s,x(s))\|_{L^2} : s,t \in [0,T], |t-s| \leq \varepsilon\} = 0$$

uniformly with respect to $x \in U \subset V(0,r)$, where $r = B/(1-M)$;

(vi) *the mappings $x(t;\omega) \to f_i(t, x(t;\omega))$, $i = 1,2$, $C_p(\mathbf{R}_+, L^2(\Omega,\mathcal{A},P),p)$ into $C_p(\mathbf{R}_+, L^2(\Omega,\mathcal{A},P),p)$ are continuous in the topology generated by the norm $\|\cdot\|_p$;*

(vii) $\lim_{t\to\infty} p(t)\|f_i(t,x(t)) - f_i(t,y(t))\|_{L^2} = 0$, *$i = 1,2$, uniformly with respect to x and y belonging to $V(0,r)$, $r = B/(1-M)$;*

(viii) *there exist L_i, $i = 1,2,3$, satisfying $0 \leq L_1 + L_2 + L_3 < 1$ such that*

$m\Big(\int_0^t k_1(t,\tau;\omega)f_1(\tau,U(\tau))\,d\tau\Big) \leq L_1 m(U(t))$,

$m\Big(\int_0^\infty k_2(t,\tau;\omega)f_2(\tau,U(\tau))\,d\tau\Big) \leq L_2 m(U(t))$,

$m(h(t,U(t))) \leq L_3 m(U(t))$, $L_i \in [0,1)$, $i = 1,2,3$,

$U(t) = \{x(s) \in L^2(\Omega,\mathcal{A},P),\ s \geq 0,\ x \in U \subset V(0,r)$:

$$p(t)\|x(t)\|_{L^2} \le \|U\|_p\}, \; t \ge 0, \text{ where } r = B/(1-M).$$

Then there exists at least one solution $x \in C_p$ of equation (1.1) such that

$$\lim_{t\to\infty} p(t)\|x(t)\|_{L^2} = 0 .$$

Proof. Define the H on C_p by

$$(Hx)(t;\omega) = h(t, x(t;\omega)) + \int_0^t k_1(t,\tau;\omega) f_1(\tau, x(\tau;\omega))\, d\tau + \int_0^\infty k_2(t,\tau;\omega) f_2(\tau, x(\tau;\omega))\, d\tau .$$

Using assumption (i), (ii) and (3.1) we get for $x \in C_p$

$$\begin{aligned} p(t)\|(Hx)(t)\|_{L^2} &\le p(t)\Big(k\|x(t)\|_{L^2} + |h(t,0)| \\ &+ \int_0^t k_2(t,\tau)\|f_1(\tau,x(\tau))\|_{L^2}\, d\tau + \int_0^\infty k_2(t,\tau)\|f_2(\tau,x(\tau))\|_{L^2}\, d\tau\Big) \\ &\le \|x\|_p\Big(k + p(t)\big(\int_0^t k_1(t,\tau)(u_1(\tau)/p(\tau))\, d\tau + \int_0^\infty k_2(t,\tau)(u_2(\tau)/p(\tau))d\tau\big)\Big) \\ &+ p(t)\Big(|h(t,0)| + \int_0^t k_1(t,\tau)v_1(\tau)\, d\tau + \int_0^\infty k_2(t,\tau)v_2(\tau)\, d\tau\Big) . \end{aligned}$$

Hence, we get

$$\|Hx\| \le M\|x\|_p + B ,$$

which implies that H maps C_p into C_p. Moreover, we note that

$$H : V(0,r) \to V(0,r) \quad \text{for } r = B/(1-M) .$$

We now prove that the map H is continuous in the ball $V(0,r)$. Let $x, y \in V(0,r)$. By assumption (vii) for any given $\varepsilon_i > 0$, $i = 1,2$, we can choose $T > 0$ such that

$$p(\tau)\|f_i(\tau,x(\tau)) - f_i(\tau,y(\tau))\|_{L^2} < \varepsilon_i , \quad \text{whenever } \tau > T . \tag{3.3}$$

On the other hand, by (vi), for any given $\varepsilon^{(i)} > 0$, $i = 1,2$, there exist $\delta_i > 0$, $i = 1,2$, such that for all $\tau \in [0,T]$

$$p(\tau)\|f_i(\tau,x(\tau)) - f_i(\tau,y(\tau))\|_{L^2} < \varepsilon^{(i)} , \quad i = 1,2, \quad \text{whenever } \|x-y\|_p < \delta_i , \quad i = 1,2. \tag{3.4}$$

Moreover, by (iii), for any given $\varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$p(t)\|h(t,x(t)) - h(t,y(t))\|_{L^2} < \varepsilon_3 , \quad \text{whenever } \|x-y\|_p < \delta . \tag{3.5}$$

Furthermore, we can assume without loss of generality that there exists $T > 0$ such that $u_i(t) \geq 1$, $i = 1,2$, whenever $t \geq T$, and $u_T^i = \min\{u_i(\tau) : 0 \leq \tau \leq T\} > 0$, $i = 1,2$.

Hence, using (3.5), and putting $p_T = \max\{p(\tau) : 0 \leq \tau \leq T\}$ we have for $t \geq T$

$$
\begin{aligned}
(3.6)\qquad & p(t)\|(Hx)(t) - (Hy)(t)\|_{L^2} \leq p(t)\Big(\|h(t,x(t)) - h(t,y(t))\|_{L^2} \\
&+ \int_0^t k_1(t,\tau)\|f_1(\tau,x(\tau)) - f_1(\tau,y(\tau))\|_{L^2}\, d\tau \\
&+ \int_0^\infty k_2(t,\tau)\|f_2(\tau,x(\tau)) - f_2(\tau,y(\tau))\|_{L^2}\, d\tau < \varepsilon_3 \\
&+ p(t)((p_T/u_1^T)\cdot \int_0^T k_1(t,\tau)(u_1(\tau)/p(\tau))\big\|f_1(\tau,x(\tau)) \\
&- f_1(\tau,y(\tau))\big\|_{L^2}\, d\tau + \int_T^t k_1(t,\tau)(u_1(\tau)/p(\tau))p(\tau)\cdot \\
&\cdot\|f_1(\tau,x(\tau)) - f_1(\tau,y(\tau))\|_{L^2}\, d\tau\Big) \\
&+ p(t)\Big((p_T/u_2^T)\cdot \int_0^T k_2(t,\tau)(u_2(\tau)/p(\tau))p(\tau)\|f_2(\tau,x(\tau)) \\
&- f_2(\tau,y(\tau))\|_{L^2}\, d\tau + \int_T^\infty k_2(t,\tau)(u_2(\tau)/p(\tau))p(\tau)\cdot \\
&\cdot\|f_2(\tau,y(\tau)) - f_2(\tau,y(\tau))\|_{L^2}\, d\tau\Big)\,.
\end{aligned}
$$

Therefore, by (iv), (3.3), (3.4) and (3.6) we obtain

$$
\begin{aligned}
(3.7)\qquad & \sup_{t\geq T} p(t)\|(Hx)(t) - (Hy)(t)\|_{L^2} \leq \varepsilon_3 + M((p_T/u_1^T)\varepsilon^{(1)} \\
&+ \varepsilon_1 + (p_T/u_2^T)\varepsilon^{(2)} + \varepsilon_2)\,.
\end{aligned}
$$

Moreover, it can be seen that for any given $\varepsilon_4 > 0$ one has

$$
\sup_{0\leq t\leq T} p(t)\|(Hx)(t) - (Hy)(t)\|_{L^2} < \varepsilon_4\,, \quad \text{whenever } \|x - y\|_p < \delta\,.
$$

Thus by (3.7) and (3.8), for any given $\varepsilon > 0$ $\|Hx - Hy\|_p < \varepsilon$, whenever $\|x-y\|_p < \delta$, $x, y \in V(0,r)$.

Let now be given $\varepsilon > 0, T > 0$ and $t, s \in [0,T]$, $|t-s| < \varepsilon$. By (3.2) for $0 \leq s \leq t$

and $x \in U \subset V(0,r)$, we have

$$(3.9) \qquad \|p(t)(Hx)(t) - p(s)(Hx)(s)\|_{L^2} \le |p(t)-p(s)| \cdot \\ \cdot \|h(t,x(t))\|_{L^2} + p(s)\|h(t,x(t)) - h(s,x(s))\|_{L^2} \\ + |p(t)-p(s)| \left\| \int_0^t k_1(t,\tau) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} \\ + p(s) \left\| \int_0^t (k_1(t,\tau) - k_1(s,\tau)) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} \\ + p(s) \left\| \int_s^t k_1(t,\tau) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} + |p(t)-p(s)| \cdot \\ \cdot \left\| \int_0^\infty k_2(t,\tau) f_2(\tau,x(\tau))\, d\tau \right\|_{L^2} + p(s) \cdot \\ \cdot \left\| \int_0^\infty (k_2(t,\tau) - k_2(s,\tau)) f_2(\tau,x(\tau))\, d\tau \right\|_{L^2}.$$

But using (3.1) with $i = 1$ and x replaced by $f_1(\tau, x(\tau))$, we obtain

$$(3.10) \qquad |p(t)-p(s)| \left\| \int_0^t k_1(t,\tau) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} \\ \le |p(t)-p(s)| \cdot \int_0^t k_1(t,\tau) \|u_1(\tau)|x(\tau)| + v_1(\tau)\|_{L^2}\, d\tau \\ \le T|p(t)-p(s)| \cdot (r \cdot \max\{k_1(t,\tau)(u_1(\tau)/p(\tau)) : 0 \le \tau \le T\} \\ + \max\{k_1(t,\tau)v_1(\tau) : 0 \le \tau \le T\}).$$

Similarly we get

$$(3.11) \qquad p(s) \left\| \int_0^t (k_1(t,\tau) - k_1(s,\tau)) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} \\ \le Trp(s) \max\{|k_1(t,\tau) - k_1(s,\tau|(u_1(\tau)/p(\tau)) : 0 \le \tau \le T\} \\ + Tp(s) \max\{|k_1(t,\tau) - k_1(s,\tau)|v_1(\tau) : 0 \le \tau \le T\},$$

and

$$(3.12) \qquad p(s) \left\| \int_s^t k_1(t,\tau) f_1(\tau,x(\tau))\, d\tau \right\|_{L^2} \le |t-s| p(s) \cdot \\ \cdot (r \cdot \max\{k_1(t,\tau)(u_1(\tau)/p(\tau)) : 0 \le \tau \le T\} \\ + \max\{k_1(t,\tau)v_1(\tau) : 0 \le \tau \le T\}).$$

Now using (i), (iv) and (3.1) for $i = 2$, we have the following estimates

$$(3.13) \ |p(t)-p(s)| \left\| \int_0^\infty k_2(t,\tau) f_2(\tau,x(\tau))\, d\tau \right\|_{L^2} \\ \le |p(t)-p(s)| \cdot \left(\int_0^\infty k_2(t,\tau)(u_2(\tau)/p(\tau))\, d\tau \right. \\ \left. + \int_0^\infty k_2(t,\tau) v_2(\tau) d\tau \right) \le \max\left\{ \frac{1}{p(t)} : 0 \le t \le T \right\} |p(t)-p(s)|(Mr + B).$$

Now we note that

$$\int_0^\infty k_2(t,\tau)-k_2(s,\tau)\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau<\infty\,.$$

Indeed, we have

$$\begin{aligned}
&p(s)\int_0^\infty |k_2(t,\tau)-k_2(s,\tau)|\,\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau\\
&\le p(s)\Big(r\int_0^\infty k_2(t,\tau)(u_2(\tau)/p(\tau))\,d\tau+\int_0^\infty k_2(t,\tau)v_2(\tau)\,d\tau\\
&+r\int_0^\infty k_2(s,\tau)(u_2(\tau)/p(\tau))\,d\tau+\int_0^\infty k_2(s,\tau)v_2(\tau)\,d\tau\Big)\\
&\le p(s)\,2\cdot\max\{1/p(t):0\le t\le T\}(Mr+B)\,.
\end{aligned}$$

By the estimate given above, for any given $\delta>0$ and sufficiently large T we have

$$\begin{aligned}
(3.14)\qquad &p(s)\int_0^\infty |k_2(t,\tau)-k_2(s,\tau)|\,\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau\\
&\le p(s)\Big(r\int_0^T |k_2(t,\tau)-k_2(s,\tau)|(u_2(\tau)/p(\tau))\,d\tau\\
&+\int_0^T |k_2(t,\tau)-k_2(s,\tau)|v_2(\tau)\,d\tau\Big)+p(s)\cdot\\
&\cdot\int_T^\infty |k_2(t,\tau)-k_2(s,\tau)|\,\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau\\
&\le \max\{k_2(t,\tau)-k_2(s,\tau):0\le\tau\le T\}\cdot\max\{k_2(t,\tau)^{-1}:\\
&0\le\tau\le T\}\Big(rp(s)\int_0^T k_2(t,\tau)(u_2(\tau)/p(\tau))\,d\tau\\
&+p(s)\int_0^T k_2(t,\tau)v_2(\tau)\,d\tau\Big)+\sup\{p(s):0\le s\le T\}\cdot\\
&\cdot\int_T^\infty |k_2(t,\tau)-k_2(s,\tau)|\,\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau\\
&\le \max\{k_2(t,\tau)-k_2(s,\tau):0\le\tau\le T\}\cdot\max\{k_2(t,\tau)^{-1}:\\
&0\le\tau\le T\}\ \sup\{p(s)/p(t):s,t\in[0,T],\ s\le t\}(Mr+B)\\
&+\sup\{p(s):0\le s\le T\}\cdot\int_T^\infty |k_2(t,\tau)-k_2(s,\tau)|\cdot\\
&\cdot\|f_2(\tau,x(\tau))\|_{L^2}\,d\tau\le C(T)(Mr+B)+\delta\,,
\end{aligned}$$

where $C(T)$ is a positive constant.

We need to recall the definition of the modulus of continuity which is defined for all real functions w as:

$$(3.15)\qquad \nu_T(w;\varepsilon)=\sup\{|w(t)-w(s)|:t,s\in[0,T],\ |t-s|\le\varepsilon\}\,,\quad \varepsilon>0\,.$$

Using now (3.15) , the properties of the functions k_1, k_2 and p we have

(3.16)
$$\lim_{\varepsilon\to 0} \nu_T(k_1;\varepsilon) = \lim_{\varepsilon\to 0} \nu_T(k_2;\varepsilon) = \lim_{\varepsilon\to 0} \nu_T(p;\varepsilon) = 0 .$$

Moreover, by the assumption (iii) and (v), we see that

(3.17)
$$\lim_{\varepsilon\to 0} \nu_T(\|h(t,x(t)) - h(s,x(s))\|_{L^2};\varepsilon) = 0 .$$

Therefore, by (3.9)-(3.14) and (3.16), (3.17), we get for $U \subset V(0,r)$

(3.18)
$$\beta_0(HU) = 0 .$$

Fix now $U \subset V(0,r)$, $r = B/(1-M)$. We prove that

(3.19)
$$a(HU) \le Ma(U) .$$

It is clear, by the definition of the integral, that for any given $\eta_1 > 0$ there exists a positive integer $n_1 = n_1(\eta_1)$ such that for $n \ge n_1$

$$\Big| \int_0^t k_1(t,\tau)(u_1(\tau)\|x(\tau)\|_{L^2}/p(\tau))p(\tau)d\tau - \sum_{k=0}^{n-1} \frac{t}{n} k_1\Big(t,\frac{kt}{n}\Big)\Big(u_1\Big(\frac{kt}{n}\Big)\Big\|\Big(\frac{kt}{n}\Big)\Big\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big)\Big| < \eta_1 .$$

Let now $T < t$. Put $k_1^* = \max\{k : 0 \le k \le n,\ \frac{kt}{n} < T\}$, then we have

$$\begin{aligned}
&\int_0^t k_1(t,\tau)(u_1(\tau)\|x(\tau)\|_{L^2}/p(\tau))p(\tau)\,d\tau \\
&\le \eta_1 + \sum_{k=0}^{k_1^*} \frac{t}{n} k_1\Big(t,\frac{kt}{n}\Big)\Big(u_1\Big(\frac{kt}{n}\Big)\Big\|\Big(\frac{kt}{n}\Big)\Big\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big) \\
&+ \sum_{k=k_1^*+1}^{n-1} \frac{t}{n} k_1\Big(t,\frac{kt}{n}\Big)\Big(u_1\Big(\frac{kt}{n}\Big)\Big\|\Big(\frac{kt}{n}\Big)\Big\|_{L^2}/p\Big(\frac{kt}{n}\Big)\Big)p\Big(\frac{kt}{n}\Big) \\
&= \eta_1 + I_1 + I_2 .
\end{aligned}$$

Now for any given $\eta_2 > 0$ there exists $n_2 = n(\eta_2)$ such that for $n \ge n_2$

$$\begin{aligned}
I_1 \le& k_1 t \cdot \max\Big\{p\Big(\frac{kt}{n}\Big)\Big\|\Big(\frac{kt}{n}\Big)\Big\|_{L^2} : \frac{kt}{n} < T\Big\} \max\{k_1(t,\tau) \\
&(u_1(\tau)/p(\tau)) : 0 \le \tau \le T\}\, n^{-1} < \eta_2 .
\end{aligned}$$

Similarly, for any given $\eta_3 > 0$

$$I_2 \le \sup\{p(t)\|x(t)\|_{L^2} : t \ge T\}\Big(\int_0^t k_1(t,\tau)(u_1(\tau)/p(\tau))\,d\tau + \eta_3\Big)$$

for sufficiently large n.

We put

$$g_t(\tau) = k_1(t,\tau)(u_2(\tau)/p(\tau)) ,$$
$$g_t^*(\tau) = g(\tau,t)\|x(\tau)\|_{L^2} p(\tau) , \text{ where } t \in \mathbf{R}_+ , x \in V(0,r) .$$

By the assumption of Theorem 3.1 we get

$$\int_0^\infty g_t(\tau)\, d\tau < \infty , \quad \int_0^\infty g_t^*(\tau)\, d\tau < \infty \quad \text{for } t \in \mathbf{R}_+ .$$

This fact allows us to find functions $\widetilde{g}_t$, $\widetilde{g}_t^*$ which are nonnegative decreasing and

$$\lim_{\tau\to\infty} \widetilde{g}_t(\tau) = \lim_{\tau\to\infty} \widetilde{g}_t^*(\tau) = 0 .$$

These functions satisfy additionally the following conditions:

$$g_t(\tau) \leq \widetilde{g}_t(\tau) , \quad g_t^*(\tau) \leq \widetilde{g}_t^*(\tau)$$

and

$$\int_0^\infty \widetilde{g}_t(\tau)\, d\tau < \infty , \quad \int_0^\infty \widetilde{g}_t^*(\tau)\, d\tau < \infty .$$

Hence, we can write

$$\int_0^\infty \widetilde{g}(\tau)\, d\tau = \lim_{h\to 0_+} h \sum_{n=1}^\infty \widetilde{g}_t(nh)$$

and

$$\int_0^\infty \widetilde{g}_t^*(\tau)\, d\tau = \lim_{h\to 0_+} h \sum_{n=1}^\infty \widetilde{g}_t^*(nh) \quad \text{(see [15])} .$$

Moreover, $\widetilde{g}_t(\tau)$ can be choosen such that

$$\lim_{0<h\to 0} h \left| \sum_{n=1}^\infty \widetilde{g}_t(nh) - \sum_{n=1}^\infty g_t(nh) \right| = 0$$

and

$$\lim_{0<h\to 0} h \left| \sum_{n=1}^\infty \widetilde{g}_t^*(nh) - \sum_{n=1}^\infty g_t^*(nh) \right| = 0 .$$

Let $T > 0$ be fixed. Choose m such large that $m + 1 > T$. Then, by the assumptions, we have

$$p(t)\Big\|\int_0^\infty k_2(t,\tau)f_2(t,x(\tau))\,d\tau\Big\|_{L^2} \le p(t)\int_0^\infty g_t^*(\tau)\,d\tau$$

$$+\,p(t)\int_0^\infty k_2(t,\tau)\cdot v_2(\tau)\,d\tau \le p(t)\Big|\int_0^\infty \widetilde{g}_t^*(\tau)\,d\tau - h\sum_{n=1}^\infty \widetilde{g}_t^*(nh)\Big|$$

$$+\,p(t)h\Big|\sum_{n=1}^\infty \widetilde{g}_t^*(nh) - \sum_{n=1}^\infty g_t^*(nh)\Big| + p(t)h\sum_{n=1}^\infty g_t^*(nh)$$

$$+\,p(t)\int_0^\infty k_2(t,\tau)\cdot v_2(\tau)\,d\tau \le p(t)\Big|\int_0^\infty \widetilde{g}_t^*(\tau)\,d\tau - h\sum_{n=1}^\infty \widetilde{g}_t^*(nh)\Big|$$

$$+\,p(t)h\Big|\sum_{n=1}^\infty \widetilde{g}_t^*(nh) - \sum_{n=1}^\infty g_t^*(nh)\Big| + p(t)hr\sum_{n=1}^m g_t(nh)$$

$$+\,p(t)\sup\{\|x(nh)\|_{L^2}p(nh) : n \ge m+1\}h\sum_{n=m+1}^\infty g_t(nh)$$

$$+\,p(t)\int_0^\infty k_2(t,\tau)v_2(\tau)\,d\tau\,.$$

Letting now $h \to 0$, we have

$$p(t)\Big\|\int_0^\infty k_2(t,\tau)f_2(\tau,x(\tau))\,d\tau\Big\|_{L^2} \le \sup\{p(t)\|x(t)\|_{L^2} : t \ge T\}$$

$$+\,p(t)\int_0^\infty g_t(\tau)\,d\tau) + p(t)\int_0^\infty k_2(t,\tau)v_2(\tau)\,d\tau\,.$$

By (iii), we have

$$p(t)\|h(t,x(t))\|_{L^2} \le k\cdot\sup\{p(t)\|x(t)\|_{L^2} : t \ge T\} + p(t)|h(t,0)|\,.$$

Therefore, by the above considerations, we get

$$p(t)\|(Hx)(t)\|_{L^2} \le p(t)|h(t,0)| + M\cdot\sup\{p(t)\|x(t)\|_{L^2} : t \ge T\}$$

$$+\,p(t)(\eta_1+\eta_2+r\eta_3) + p(t)\Big(\int_0^t k_1(t,\tau)v_1(\tau)\,d\tau + \int_0^\infty k_2(t,\tau)v_2(\tau)\,d\tau\Big).$$

Thus, by the assumptions of Theorem 3.1, we obtain

$$\lim_{T\to\infty}\sup_{x\in U}\{\sup\{p(t)\|(Hx)(t)\|_{L^2} : t \ge T\}\} \le (\eta_1+\eta_2+r\eta_3)C_1$$

$$+\,M\cdot\lim_{T\to\infty}\sup_{x\in U}\{\sup\{p(t)\|x(t)\|_{L^2} : t \ge T\}\}\,.$$

Let now $\eta_i \to 0$, $i = 1,2,3$. Then we get (3.19). Finally, by (3.18), (3.19) and the assumptions (viii) we obtain

$$\mu_0(HU) \le D\cdot\mu_0(U)\,,$$

where $D = \max\{L_1 + L_2 + L_3, M\}$, which proves that H is a μ_0-contraction. This fact by Theorem 2.1 ends the proof.

4. Remarks. In the monograph [13] the authors study the stochastic integral equation of the Fredholm type of the form

$$x(t;\omega) = h(t, x(t;\omega)) + \int_0^\infty k(t,\tau;\omega) f(\tau, x(\tau;\omega))\, d\tau\ . \tag{4.1}$$

(a) Theorem (3.1) extends the following result of Theorem 4.5.3 gven in [13].

Theorem 4.1. ([9], [13] p.125). *Consider the random integral equation* (4.1) *subject the following condtions:*

(i) *H_1 and H_2 are Hilbert spaces stronger than C_c and the pair (H_1, H_2) is admissible with respect to the completely continuous integral operator*

$$(Wx)(t;\omega) = \int_0^\infty k(t,\tau;\omega) x(\tau;\omega)\, d\tau\ , \quad t \in \mathbf{R}_+,$$

where $k(t,\tau;\omega)$ behaves as described previously and the integral

$$\int_0^\infty \int_0^\infty k(t,\tau)\, d\tau\, dt$$

exists and is finite;

(ii) *$x(;\omega) \to f(t, x(t;\omega))$ is a continuous operator on*

$$S = \{x(t;\omega) : x(t;\omega) \in H_1\ ,\ \|x(t;\omega)\|_{H_1} \le \rho\}$$

for some $\rho > 0$ with values in H_2 such that $\|f(t, x(t;\omega))\|_{H_2} \le \gamma$ for some $\gamma > 0$ a constant;

(iii) *$x(t;\omega) \to h(t, x(t;\omega))$ is the contraction on S.*

Then there exists at least one bounded (by ρ) random solution of equation (4.1) *provided*

$$\|h(t, x(t;\omega))\| H_1 + \gamma K \le \rho\ ,$$

where K is the norm of the operator W.

We see that the assumptions (i)–(iii) of Theorem imply the conditions (i)–(viii) of Theorem 3.1 if we put $p(t) \equiv 1$ for $t \in \mathbf{R}_+$, $u_i(t) \equiv 0$, $i = 1,2$ and $v_1(t) \equiv 0$, $v_2(t) \equiv \gamma$ for $t \in \mathbf{R}_+$.

Analogously, we prove that Theorem 3.1 generalizes the Theorems 4.5, 4.5.4, and 4.5.6 of [13].

(b) The proof of Theorem 3.1 can be extended to the case when $x(\cdot\ ;\omega) \in L_p(\Omega, \mathcal{A}, P)$.

(c) If $p(t) \equiv 1$ then a random solution $x(t;\omega)$ of (1.1) is asymptotically stable in the sense of Definition 2.2.

5. Example. First we give examples functions which are sublinear, but they do not satisfy the Lipschiz condition.

Lemma 1. *If a real function satisfies the Lipschitz condition and it is differentiable then the derivative is bounded.*

We omit the proof.

Example 5.1. Let $f : \mathbf{R}_+ \to \mathbf{R}$ be defined as follows:

$$f(x) = x \cdot \exp(\sin x) .$$

We see that

$$|f(x)| \leq e \cdot |x| \quad \text{for} \quad x \in \mathbf{R}_+ ,$$

and

$$f'(x) = \exp(\sin x) + x \cdot \cos x \cdot \exp(\sin x) .$$

Hence, by Lemma 5.1, the function f does not satisfy the Lipschitz condition.

Example 5.2. Let $f : \mathbf{R}_+ \to \mathbf{R}$ be defined as follows:

$$f(x) = (x - n + 1)^n + n - 1 \quad \text{for} \quad x \in [n-1, n) ,$$

where $n \in \mathcal{N}$.

It is clear that

$$x - n + 1 \geq (x - n + 1)^n \quad \text{for} \quad x \in [n-1, n), \quad \text{where} \quad n \in \mathcal{N} .$$

By the above inequality we have

$$|f(x)| \leq |x| \quad \text{for} \quad x \in \mathbf{R}_+ .$$

Moreover,

$$f'(x) = n(x - n + 1)^{n-1} \quad \text{for} \quad x \in [n-1, n), \quad \text{where} \quad n \in \mathcal{N} .$$

By Lemma 5.1 the function f does not satisfy the Lipschitz condition.

Using the functions of Example 5.1, and 5.2 one can prove the following result.

Theorem 5.1. *Let in the equation* (1.1)

$$f_1(t, x(;\omega)) = x(t;\omega)\exp(\sin[x(t;\omega)]) ,$$
$$f_2(t, x(t;\omega)) = (x(t;\omega) - n + 1)^n + n - 1 , \quad x(t;\omega) \in [n-1, n) ,$$

where $n \in \mathcal{N}$, *and*

$$h(t, x(t;\omega)) \equiv h(t;\omega) .$$

Suppose that

(i) $k_1(t,\tau) \le \tilde{k}_1(t)\bar{k}_1(\tau)$, $k_2(t,\tau) \le \tilde{k}_1(t)\bar{k}_2(\tau)$,
where $\bar{k}_1$, k_1, k_2 *are positive functions and* $\tilde{k}_1$ *is differentiable function;*

(ii) $D_1(t) := p(t)\tilde{k}_1(t)\Big(\int_0 t(\bar{k}_1(\tau)/p(\tau))\,d\tau + \int_0^t(\bar{k}_2(\tau)/p(\tau))\,d\tau\Big)$ *positive, differentiable and* $D_2 = \sup\{D_1(t) : t \in \mathbf{R}_+\}$, $D_2 \in [0,1)$;

(iii) $D_1(0) = \tilde{k}_1(0)\int_0^\infty \bar{k}_2(\tau)\,d\tau$,
and

$$\int_0^\infty (\bar{k}(\tau)/D_2(\tau))\exp\Big(\int_0^\tau (e\bar{k}_1(\tau_1)\tilde{k}_1^2(\tau_1)+\tilde{k}_1'(\tau_1)D_1(\tau_1))\cdot(D_1(\tau_1)\tilde{k}_1(\tau_1))^{-1}d\tau_1\Big)d\tau<\infty;$$

(iv) $\lim_{t\to\infty} p(t)\|x(t)-y(t)\|_{L^2} = 0$
uniformly with respect to x *and* y *belonging to* $V(0,r)$, *where* $r = \|h\|_p/(1-D_2)$;

(v) $\lim_{t\to\infty} p(t)\|h(t)\|_{L^2} = 0$;

(vi) *there exists* L_i *for* $i = 1,2$ *satisfying* $0 \le L_1 + L_2 < 1$ *such that*

$$m\Big(\int_0^t k_1(,\tau;\omega)f_1(\tau,U(\tau))\,d\tau\Big) \le L_1 m(U(t)),$$

$$m\Big(\int_0^\infty k_2(t,\tau;\omega)gf_2(\tau,U(\tau))\,d\tau\Big) \le L_2 m(U(t)),\ \text{where } L_1, L_2 \in [0,1),$$

$$U(t) = \{x(s) \in L^2(\Omega,\mathcal{A},P),\ s \ge 0,\ x \in U \subset V(0,r) : p(t)\|x(t)\|_2 \le \|U\|_p\},$$
$$t \ge 0,\ r = \|h\|_p/(1-D_2).$$

Then there exists at least one solution $x \in C_p$ *of equation* (1.1) *such that*

$$\|x(t)\|_{L^2} = o((1/D_1(t))\exp\Big(\int_0^t ((e k_1(\tau)\tilde{k}_1^2(\tau) + \tilde{k}_1'(\tau)D_1(\tau))\cdot(D_1(\tau)\tilde{k}_1(\tau))^{-1}\,d\tau)\Big).$$

Proof. By differentiating $D_1(t)$ we obtain

$$p(t) = D_1(t)\exp-\Big(\int_0^t ((e\bar{k}_1(\tau)\tilde{k}_1 2(\tau) + \tilde{k}_1'(\tau)D_1(\tau))/D_1(\tau))\cdot\tilde{k}_1(\tau))\,d\tau\Big).$$

Hence, using Theorem 3.1 we get the statement of Theorem 5.1.

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STRESZCZENIE

W pracy bada się losowe równanie Volterry-Fredholma postaci:

$$x(t;\omega) = h(t, x(t,\omega)) + \int_0^t k_1(t,\tau;\omega) f_1(\tau, x(\tau;\omega))\, d\tau + \int_0^\infty k_2(t,\tau;\omega) f_2(\tau, x(\tau;\omega))\, d\tau ,$$

przy słabszych założeniach niż rozważane w pracach [9] i [10]. Zakładamy jedynie, że f_1 i f_2 są funkcjami subliniowymi.