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The Nonschlicht Mapping of Multiply Connected Domains

Niejednolistne odwzorowanie obszarów wielospójnych

Неоднолистные отображения многосвязных областей

Introduction. Let D be a bounded $N+1$ $(1 \leqslant N < \infty)$-connected domain in the (z)-plane with the boundary $\Gamma = \sum_{j=1}^{N+1} \Gamma_j \in C_\alpha$ $(0 < \alpha < 1)$, where Γ_j $(j=2,\ldots,N+1)$ form the boundary of a bounded domain contained in the interior of Γ_1, and $z=0 \in D$. It is known that there exists a schlicht function which maps conformally the domain D onto a $N+1$-connected circular domain G bounded by $N+1$ circles L_j $(j=1,\ldots,N+1)$, and there also exists a nonschlicht analytic function which maps the domain D onto a $N+1$-sheeted disk H (cf. [1]). In this paper we shall prove that there exists an analytic function which maps the domain D onto a $N+1-k$ $(0 \leqslant k \leqslant N)$-sheets Riemann surface in the unit disk, and give a uniqueness condition of the nonschlicht mapping. In particular the cases when $k = N$ and $k = 0$ are the known results stated above, respectively. Besides, we shall generalize the result obtained to the uniformly elliptic complex equation of first order.

I. The nonschlicht mappings for analytic functions.

Without loss of generality, we assume that the boundary component Γ_1 of the domain is $|z| = 1$.

Theorem 1.1. *Let D be a $N+1$-connected domain stated as above. Then there exists a unique analytic function $w = F(z)$, satisfying the following conditions:*

1) $w = F(z)$ maps D onto a $N+1-k$-sheeted Riemann surface G in $|w| < 1$ $(0 \leq k \leq N)$ with $k+1$ boundary components.

2) $w = F(z)$ maps three points a_1, b_1, $c_1 \in \Gamma_1$ onto 1, -1, $-i \in L_1$: $|w| = 1$ respectively, and maps Γ_j $(j=1,\dots,N+1-k)$ onto L_1, and maps $a_j \in \Gamma_j$ $(j=2,\dots,N+1-k)$ onto $w = 1$, where a_1, b_1, c_1 on Γ_1 are arranged in accordance with the positive direction.

Proof. We first prove that there exists a unique analytic function $W = f(z)$ satisfying the conditions:

1) $W = f(z)$ maps D onto a $N+1-k$-sheeted Riemann surface in $0 < \operatorname{Re} W < 1$ with the boundary: $\operatorname{Re} W = 0$, $\operatorname{Re} W = 1$ and k vertical rectilinear slits in $0 < \operatorname{Re} W < 1$.

2) $W = f(z)$ maps three points a_1, b_1, $c_1 \in \Gamma_1$ onto the boundary points $i\infty$, $-i\infty$, 1 and maps $a_j \in \Gamma_j$ $(j=2,\dots,N+1-k)$ onto $i\infty$, where a_1, b_1, c_1 are arranged on Γ_1 in accordance with the positive direction, for convenience we assume $a_1 = 1$, $b_1 = -1$, $c_1 = -i$.

In order to prove the existence of the above function

$W = f(z)$, we find a bounded harmonic function $u(z)$ in D satisfying the boundary condition

$$(1.1) \qquad u(z) = \begin{cases} 0\,, & \text{for } z \in \Gamma' = \sum_{j=1}^{N+1-k} \Gamma_j' \,, \\ 1\,, & \text{for } z \in \Gamma'' = \sum_{j=1}^{N+1-k} \Gamma_j'' \,, \\ d_j\,, & \text{for } z \in \Gamma_j\,, \quad j=N+2-k\,, \ldots, N+1\,, \end{cases}$$

where $\Gamma_j' = \widehat{a_j b_j}$ is the curve from a_j to b_j on Γ_j , $\Gamma_j'' = \widehat{b_j a_j}$ is another one on Γ_j , $j=1,\ldots,N+1-k$, and d_j $(j=N+2-k\,,\ldots, N+1)$ are unknown real constants and $b_j \in \Gamma_j$ $(j=1,\ldots,N+1-k)$ are points to be determined appropriately, so that the conjugate harmonic function $v(z)$ is single-valued in D . In fact, let $U(z)$ be a bounded harmonic function satisfying the boundary condition:

$$(1.2) \qquad U(z) = \begin{cases} 0\,, & \text{for } z \in \Gamma' \,, \\ 1\,, & \text{for } z \in \Gamma'' \,, \\ \alpha_j\,, & \text{for } z \in \Gamma_j\,, \quad j=N+2-k\,, \ldots, N+1\,, \end{cases}$$

where α_j $(j=N+2-k\,,\ldots, N+1)$ are all constants. We denote by $\omega_j(z)$ the harmonic measure in D with the boundary condition

$$(1.3) \qquad \omega_j(z) = \begin{cases} 0\,, & \text{for } z \in \Gamma - \Gamma_j \,, \\ 1\,, & \text{for } z \in \Gamma_j \,, \end{cases} \qquad j=N+2-k,\ldots,N+1\,,$$

and we can prove that

$$\begin{vmatrix} I_{N+2-k,N+2-k} & ,\dots, & I_{N+2-k,N+1} \\ \vdots & \vdots & \vdots \\ I_{N+1,N+2-k} & ,\dots, & I_{N+1,N+1} \end{vmatrix} \neq 0 \ ,$$

where $I_{ij} = \int_{\Gamma_i} \frac{\partial \omega_j}{\partial n} ds$, $i=N+2-k ,\dots, N+1$ and $\vec{n}$ is the outward normal on Γ_i . Hence there exist constants γ_j ($j=N+2-k ,\dots, N+1$) such that

$$(1.4) \qquad \sum_{j=N+2-k}^{N+1} \gamma_j I_{ij} = - \int_{\Gamma_i} \frac{\partial U}{\partial n} ds \ , \quad i=N+2-k ,\dots, N+1 \ ,$$

and the function

$$(1.5) \qquad u(z) = U(z) + \sum_{j=N+2-k}^{N+1} \gamma_j \omega_j(z)$$

is harmonic in D and satisfies the boundary condition

$$(1.6) \qquad u(z) = \begin{cases} 0 \ , & \text{for } z \in \Gamma' \ , \\ 1 \ , & \text{for } z \in \Gamma'' \ , \\ \alpha_j + \gamma_j \ , & \text{for } z \in \Gamma_j \ , \ j=N+2-k,\dots,N+1 \ . \end{cases}$$

Besides, according to the method described in [1] and [2], if we determin properly the points b_j on Γ_j, $j=2,\dots,N+1-k$, then the conjugate harmonic function $v(z)$ of $u(z)$ is single--valued in D.

Putting $f(z) = u(z) + iv(z)$, we are free to choose a complex number W_0, so that $\operatorname{Re} W_0 < 0$ or $\operatorname{Re} W_0 > 1$, it is clear that $\Delta_\Gamma \arg(f(z) - W_0) = 0$. Moreover, if the complex number W_0 satisfies the condition $0 < \operatorname{Re} W_0 < 1$ and $W_0 \bar{\in} f(\Gamma)$, then

$$\Delta_\Gamma \arg(f(z) - W_0) = 2(N+1-k)\pi \quad . \tag{1.7}$$

Therefore, $W = f(z)$ maps $N+1-k$ points ($N+1-k$ zero-points of the function $W_0 - f(z)$) in D to the point W_0. It is shown that $W = f(z)$ is an analytic function as required.

If we add the condition $v(c_1) = 0$, then the function $W = f(z)$ is unique.

It is easy to see that $w = g(W) = \dfrac{e^{i\pi W} - i}{e^{i\pi W} + i}$ maps the band domain onto the unit disk $|w| < 1$, and then the function $w = F(z) = g(f(z))$ is an analytic function desired.

<u>Theorem 1.2. Let</u> D_n $(n=0,1,\dots)$ <u>be a sequence of</u> $N+1$--<u>connected domains, which are of the same type as the domain</u> D <u>and</u> $\{D_n\}$ <u>converge to its kernel</u> D_0. <u>Then</u> $\{f_n(z)\}$ <u>uniformly converges to</u> $f_0(z)$ <u>on any closed point-set in</u> D, <u>where</u> $f_n(z)$

maps D_n onto the Riemann surface stated in Theorem 1.1 , $n=0,1,\ldots$.

Proof. We may assume that D_n $(n=0,1,\ldots)$ are the circular domains with a boundary component $\Gamma_1 : |z| = 1$, $G_n = f_n(D_n)$ is a Riemann surface in $0 < \operatorname{Re} W < 1$ stated as in the proof of Theorem 1.1 , and $f_n(1) = i\infty$, $f_n(-1) = -i\infty$, $f_n(-i) = 1$. In this case, we can seek an analytic function $f_n(z)$ in D_n , which satisfies the boundary condition

$$
(1.8) \qquad \operatorname{Re} f_n(z) = r_n(z) = \begin{cases} 0 \,, & \text{for } z \in \Gamma_n' \,, \\ 1 \,, & \text{for } z \in \Gamma_n'' \,, \\ d_j^n \,, & \text{for } z \in \Gamma_j^n \,,\ j=N+2-k,\ldots,N+1 \end{cases}
$$

where d_j^n $(j=N+2-k\,,\ldots,\,N+1)$ are all constants, and Γ_n' , Γ_n'' are circular arcs similar to Γ' , Γ'' in Theorem 1.1 . It is not difficult to see that $0 < d_j^n < 1$, $j=N+2-k\,,\ldots,\,N+1$. Having used Schwarz formula in the multiply connected domain (cf. [2]), the function $f_n(z)$ can be represented in the form

$$
(1.9) \qquad f_n(z) = \frac{1}{2\pi} \int_{\Gamma_n} T_n(z,t) r_n(t) d\theta + \Phi_n(z) = F_n(z) + \Phi_n(z) \,,
$$

where $T_n(z,t)$ is the Schwarz kernel of the type

$$
(1.10) \qquad T_n(z,t) = \sum_{j=1}^{N+1} P_j(z,t) + P_*(z,t) \,, \quad t \in \Gamma^n \,,
$$

in which

$$(1.11) \qquad P_j(z,t) = \frac{t+z-2z_j^n}{t-z} \;, \quad t \in \Gamma_j^n \;, \quad j=1,\dots,N+1 \;,$$

z_j^n is the centre of Γ_j^n, and $P_*(z,t)$ is an analytic function in D_n with the boundary condition:

$$(1.12) \qquad \begin{cases} \operatorname{Re} P_*(z,t) = -\operatorname{Re} Q(z,t) + h(z,t) \;, & t \in \Gamma^n \;, \\ Q(z,t) = \sum\limits_{\substack{m=1 \\ m \neq j}}^{N+1} P_m(z,t) \;, & z \in \Gamma_j^n \;, \quad j=1,\dots,N+1 \;, \\ \operatorname{Im} P_*(t_j^n,t) = -\operatorname{Im} Q(t_j^n,t) \;, & t_j^n \in \Gamma^n \;, \; j=2,\dots,N+1, \end{cases}$$

where t_j^n $(\neq a_j^n, b_j^n)$ is a fixed point on Γ_j^n, $j=2,\dots,N+1$. Applying Lemma 5.1, Chapter 5 in [2], we can prove the estimate:

$$(1.13) \qquad C_\alpha\left[F_n(z) \;, D_n^m\right] \leq M_1 = M_1(a,b,c,d,\alpha,m) \;,$$

in which α $(0 < \alpha < 1)$ is a constant, a,b,c represent a_j^n, b_j^n $(j=1,\dots,N+1-k)$, c_1^n respectively, d (>0) is the greatest lower bound of the distances between Γ_j $(j=1,\dots,N+1)$, D_n^m is the point-set in D_n whose distance from a_j^n, b_j^n $(j=1,\dots,N+1-k)$, c_1^n is not less than $\frac{1}{m}$ (m is a integer). According to the method of the proof of Theorem 3.3, Chap. 5 in [2], we can obtain the following estimate of the analytic function $\Phi_n(z)$:

$$(1.14) \qquad C_\alpha\left[\Phi_n(z),\ \bar{D}_n\right] \leqslant M_2 = M_2(M_1) \ .$$

Consequently we may select a subsequence of $\{f_n(z)\}$ which converges uniformly to an analytic function $f_0(z)$ on any closed set D_* in D_0, it is clear that $f_0(z)$ is not a constant, and such function is unique, and $f_n(z)$ uniformly converges to $f_0(z)$ on D_*.

II. The nonschlicht mappings for elliptic complex equations. In this section, we discuss the nonlinear uniformly elliptic complex equation of first order

$$(2.1) \qquad w_{\bar{z}} = F(z,w,w_z) \ , \quad F = Q(z,w,w_z)w_z \ ,$$

in a $N+1$ -connected domain D. We suppose that the equation (2.1) satisfies the condition C in D, i.e. $Q(z,w,V(z))$ is measurable in z for all functions $w(z)\in W^1_{p_0}(D)$ $(2 < p_0 < \infty)$ and $V(z)\in L_{p_0}(\bar{D})$, and is continuous in $w\in E$ (the whole plane) for almost every point $z\in D$ and $V\in E$, and the equation (2.1) satisfies the uniformly elliptic condition

$$(2.2) \qquad |F(z,w,V_1) - F(z,w,V_2)| \leqslant q_0|V_1 - V_2| \ ,$$

for almost every point $z\in D$ and $w\in E$, where q_0 $(0\leqslant q_0 < 1)$ is a real constant.

Theorem 2.1. Let the equation (2.1) satisfy the condition C. Then there exists a solution $w(z)$ for the equation (2.1) which maps the domain D onto a $N+1-k$ -sheeted Riemann surface in $|w| < 1$, $0 \leqslant k \leqslant N$.

Proof. Let us introduce a bounded closed and convex set B in the Banach space $L_{p_0}(D)$ $(2 < p_0 < \infty)$, in which the elements are measurable functions $Q(z)$ satisfying the condition

$$(2.3) \qquad L_\infty[Q(z), D] \leqslant q_0 < 1 .$$

We choose arbitrarily $Q(z) \in B$. By the principle of contraction, the integral equation

$$(2.4) \qquad h(z) - Q(z)\Pi h = Q(z) \quad , \quad h = -\frac{1}{\pi}\iint_D \frac{h(\zeta)}{(\zeta - z)^2}\, d\sigma_\zeta$$

has a unique solution $h(z) \in L_{p_0}(\bar{D})$, $2 < p_0 < p$. We can verify that

$$\chi(z) = z + Th = z - \frac{1}{\pi}\iint_D \frac{h(\zeta)}{\zeta - z}\, d\sigma_\zeta$$

is a homeomorphism on D (cf. [3]). Next we find a univalent analytic function $\zeta(\chi)$ which maps the domain $\chi(D)$ topologically onto a circular domain H in $|\zeta| < 1$, and maps $\chi(a_1)$, $\chi(b_1)$, $\chi(c_1)$ onto the three points a' , b' , c' on $|\zeta| = 1$. Afterwards, applying Theorem 1.1, we can seek a unique analytic function $w(\chi)$ in H , which maps H onto a $N+1-k$ -sheeted Riemann surface in $|w| < 1$, so that $w(\zeta(\chi(a_j))) = 1$, $w(\zeta(\chi(b_1))) = -1$, $w(\zeta(\chi(c_1))) = -i$ and $w(\zeta(\chi(\Gamma_j))) = L_1$, $|w| = 1$, $j=1,\ldots,N+1-k$.

Putting $w(z) = w(\zeta(\chi(z)))$, $w(\chi) = w(\zeta(\chi))$ and using the principle of contraction, we can find a unique solution $h^*(z) \in L_{p_0}(\bar{D})$ for the integral equation

$$(2.5) \qquad h^*(z) = Q(z,w(z),w'(\chi)(1+\Pi h^*))\cdot(1+\Pi h^*) \quad .$$

Let $Q^*(z) = h^*(z)/(1+\Pi h^*)$. It is obvious that $Q^*(z)$ satisfies $L_\infty[Q(z), \bar{D}] \leqslant q_0 < 1$. We denote by $h = S_1(Q)$ a mapping from $Q(z)$ to $h(z)$, by $h^* = S_2(w)$ a mapping from $w(z)$ to $h^*(z)$, and by $Q^* = S(Q)$ a mapping from $Q(z)$ to $Q^*(z)$.

To prove that $Q^* = S(Q)$ is a continuous mapping, we select $Q_n(z) \in L_\infty(D)$, $n=0,1,2,\ldots$, where $\lim\limits_{n\to\infty} L_{p_0}[Q_n(z)-Q_0(z), \bar{D}] = 0$. According to the method of §3 in [4], we know that $\lim\limits_{n\to\infty} L_{p_0}[h_n(z)-h_0(z), \bar{D}] = 0$, where $h_n = S_1(Q_n)$, $n=0,1,2,\ldots$ and the corresponding sequences of functions $\{\zeta_n(\chi_n(z))\}$ uniformly converges to $\zeta_0(\chi_0(z))$ on $\bar{D}$. On the basis of Theorem 1.2, we can see that $w_n(z) = w_n(\zeta_n(\chi_n(z)))$ uniformly converges to $w_0(z) = w_0(\zeta_0(\chi_0(z)))$ on $\bar{D}$. In the following, we shall derive

$$(2.6) \qquad \lim_{n\to\infty} L_{p_0}[h_n^*(z)-h_0^*(z), \bar{D}] = 0 \quad ,$$

where $h_n^* = S_2(w_n)$, $n=0,1,2,\ldots$. For two arbitrary positive constants ε_1 and ε_2 , there exists a subset D_* in D , so that $\operatorname{mes} D_* < \varepsilon_1$ and $|c_n| < \varepsilon_2$, $z \in D-D_*$, for $n > N$, where

$$c_n(z) = Q(z,w_n(z),w_n'(\chi)(1+\Pi h_0^*))\cdot(1+\Pi h_0^*) - \\ - Q(z,w_0(z), w_0'(\chi)(1+\Pi h_0^*))\cdot(1+\Pi h_0^*)$$

and N is a sufficiently large positive number. By the Hölder inequality and the Minkowski inequality, we have

$$(2.7)\quad \begin{cases} L_{p_o}\left[c_n, \overline{D}\right] \leqslant L_{p_o}\left[c_n, D_*\right] + L_{p_o}\left[c_n, \overline{D}-D_*\right] \leqslant \\ \leqslant L_{p_1}\left[c_n, D_*\right]\cdot L_{p_2}\left[1, D_*\right] + \varepsilon_2 L_{p_o}\left[1, \overline{D}-D_*\right] \leqslant \\ \leqslant 2L_{p_1}\left[1+\pi h_o, D_*\right] \varepsilon_1^{1/p_2} + \varepsilon_2 \pi^{1/p_o} \leqslant \\ \leqslant 2(\varepsilon_1^{1/p_1} + \Lambda_{p_1} L_{p_1}\left[h_o, D_*\right])\cdot \varepsilon_1^{1/p_2} + \varepsilon_2 \pi^{1/p_o} = \\ = \varepsilon, \end{cases}$$

where $p_2 = p_o p_1/(p_1-p_o)$ $(2 < p_o < p_1 < p_2 < \infty)$, $n > N$ and Λ_{p_1} is a constant satisfying $\delta_o \Lambda_{p_1} < 1$. Next, from the integral equation

$$(2.8)\quad \begin{cases} h_n^*(z) - h_o^*(z) = Q(z,w_n,w_n'(\chi)(1+\pi h_n^*))\cdot(1+\pi h_n^*) - \\ \qquad - Q(z,w_n,w_n'(\chi)(1+\pi h_o^*))\cdot(1+\pi h_o^*) + c_n(z), \end{cases}$$

we can conclude

$$(2.9)\quad L_{p_o}\left[h_n^*-h_o^*, \overline{D}\right] \leqslant L_{p_o}\left[c_n, \overline{D}\right]/(1-\delta\Lambda_{p_o}).$$

In virtue of $\lim_{n\to\infty} L_{p_o}\left[c_n, \overline{D}\right] = 0$, it follows $\lim_{n\to\infty} L_{p_o}\left[h_n^*-h_o^*, \overline{D}\right] = 0$. Afterwards, by using Lemma 3.3 in [4], it is easy to see that $\lim_{n\to\infty} L_{p_o}\left[Q_n^*(z)-Q_o^*(z), \overline{D}\right] = 0$. Therefore, $Q^* = S(Q)$ is a continuous mapping on $L_{p_o}(\overline{D})$.

Similarly, we can verify that $Q^* = S(Q)$ maps B into a compact set in B. It follows from the Schauder fix point theorem that there exists a measurable function $Q(z) \in B$, so that $Q = S(Q)$. We denote $h(z) = S_1(Q)$ and $\chi(z) = z + Th$, and the corresponding function $w(z) = w(\zeta(\chi(z)))$ is exactly a solution of (2.1) stated as in Theorem 2.1.

Making use of the same method as that in the above theorem, we can prove the following result.

Theorem 2.2. If the nonlinear equation (2.1) satisfies the condition C, then it has a solution $w(z)$, which maps the domain D onto one of the following Riemann surfaces:

(1) $N+1-k$ $(0 \leqslant k \leqslant N)$-sheeted Riemann surface, the boundary of which consists of rectilinear slits.

(2) $N+1-k$ $(0 \leqslant k \leqslant N)$-sheeted Riemann surface whose boundary consists of some spiral slits.

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STRESZCZENIE

W pracy tej wykazano, że istnieje funkcja analityczna niejednolistna, która odwzorowuje obszar D o rzędzie spójności N + 1 na (N + 1 - k) listną powierzchnię Riemanna nad kołem jednostkowym, $0 \leqslant k \leqslant N$. Podano warunki jedyności takiego odwzorowania. Wynik daje się uogólnić na rozwiązanie układów jednostajnie eliptycznych pierwszego rzędu.

РЕЗЮМЕ

В данной работе доказано, что существует аналитическая неоднолистная функция, которая отображает (N+1) - связную область D на (N+1-k) - листную Римановую поверхность над одиничным кругом, $0 \leq k \leq N$. Полученро условия единства этого отображения. Полученные результаты обобщаются на решении равномерных эллиптических систем первого порядка.