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Zakład Zastosowań Matematyki

Instytut Teorii Rozwoju Społeczno Ekonomicznego UMCS

F. BOGOWSKI , CZ. BURNIAK

On the Relationship between the Majorization of Functions and the Majorization of Derivatives in Certain Classes of Holomorphic Functions

O zależności między majoryzacją funkcji a majoryzacją pochodnych w pewnych klasach funkcji holomorficznych

Abstract. In this paper we investigate the relationship between the majorization of functions and the majorization of derivatives in the class H of functions in the unit disc K, satisfying the condition

$$\mathrm{Re}\Big\{(1-z^2)\frac{F(z)}{z}\Big\} > 0 ,$$

as well as in the class H^* of close-to-star functions.

The results obtained are sharp.

1.Introduction. Let $\mathbf{C}$ denote the complex plane, $K_R = \{z \in \mathbf{C} : |z| < R\}$, $K_1 = K$ and f , F be two holomorphic functions in the disc K_R such that $f(0) = F(0)$. We say that f is majorized by F in the disc K_R and write $f \ll F$ in K_R if $|f(z)| \le |F(z)|$ for every $z \in K_R$. It means that there exists in K_R a holomorphic function Φ such that $|\Phi(z)| \le 1$ for $z \in K_R$ and

$$f(z) = \Phi(z) \cdot F(z) \tag{1}$$

for every $z \in K_R$.

Denote by $\mathcal{H}$, $\mathcal{F}$ two fixed and compact classes of holomorphic functions in the disc K. Suppose that $f \in \mathcal{H}$, $F \in \mathcal{F}$ and $f \ll F$ in K. Let $r_0 \in (0;1)$ denote the greatest number for which the following implication

$$f \ll F \quad \text{in} \quad K \Longrightarrow f' \ll F' \quad \text{in} \quad K_{r_0}$$

holds for every pair of functions $f \in \mathcal{H}$, $F \in \mathcal{F}$. The number $r_0 = r_0(\mathcal{H}, \mathcal{F})$ is called the radius of majorization of derivatives for the class $\mathcal{F}$. The determination of r_0 in various subclasses of univalent functions in the disc K was investigated by many authors. Among others Z. Lewandowski [4] proved that $r_0(\mathcal{H}, S) = 2-\sqrt{3}$, where S denote the class of functions F holomorphic and univalent in the disc K such that $F(0)=0$, $F'(0)=1$. This problem can be generalized in the following way. We will determine the smallest possible number $T(r) = T(r,\mathcal{H},\mathcal{F})$ such that for every pair of functions f, F $(f\in\mathcal{H}, F\in\mathcal{F})$ the implication

$$f \ll F \quad \text{in} \quad K \Longrightarrow |f'(z)| \leq T(r)\cdot|F'(z)|$$

for $|z| = r < 1$ holds.

$T(r)$ can be defined as follows

$$T(r) = \sup_{\substack{f\in\mathcal{H}\\ F\in\mathcal{F}\\ f\ll F}} \left(\max_{|z|=r}\left|\frac{f'(z)}{F'(z)}\right|\right). \tag{2}$$

Let us note that the radius of majorization of derivatives

$$r_0 = \sup_{r\in(0;1)} \{T(r)\leq 1\}.$$

It follows from the definition of the radius of majorization of derivatives and $T(r)$ that it must be $|z| < r_{\mathcal{F}}$, where $r_{\mathcal{F}}$ is the radius of local univalence in the class $\mathcal{F}$ i.e.

$$r_{\mathcal{F}} = \sup_r\{|z| < r\in\langle 0;1) : F'(z)\neq 0 \text{ for all } F\in\mathcal{F}\}.$$

Among others the following result was obtained (J. Janowski , J. Stankiewicz [3])

$$T(r,\mathcal{H},S) = \begin{cases} 1 & \text{for } r\in\langle 0;2-\sqrt{3}\rangle \\ \dfrac{4r^2+(1-r)^4}{4r(1-r)^2} & \text{for } r\in(2-\sqrt{3};1). \end{cases}$$

2. The majorization of derivatives in class H. Denote by H the class of functions F, holomorphic in the disc K, satisfying the condition

$$\operatorname{Re}\left\{(1-z^2)\frac{F(z)}{z}\right\} > 0 \quad , \quad z\in K \tag{3}$$

and such that $F(0)=0$, $F'(0)=1$.

If the coefficients of the function F satisfying condition (3) are real, then the function is typically real. The function F holomorphic in K and such that $F(0)=0$, $F'(0)=1$, is said to be typically real if it takes the real values on the segment $(-1;1)$ of the real axis and satisfies the condition $\operatorname{Im} z\cdot\operatorname{Im} F(z) > 0$ for $z\in K\setminus(-1;1)$. The class of typically real functions will be denoted by TR.

The class H contains then the class TR of typically real functions.

The condition (3) can be written in the equivalent form

$$F(z) = \frac{z}{1-z^2}\cdot p(z) \quad , \quad z\in K, \tag{4}$$

where $p \in \mathcal{P}$, $\mathcal{P}$ being the class of functions p holomorphic in K and such that $\operatorname{Re} p(z) > 0$ for $z \in K$ and $p(0) = 1$.

Theorem 1 [1]. *The radius of local univalence in the class H is equal*

$$r_H = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} \approx 0,346 .$$

Remark . The radius of local univalence r_H is simultaneously the radius of univalence (i.e. in disc K_{r_H} every function $F \in H$ is univalent) as well as the radius of starlikeness in the class H (i.e. in disc K_{r_H} every function $F \in H$ satisfies the condition $\operatorname{Re} \dfrac{zF'(z)}{F(z)} > 0$).

Theorem 2. *If $F \in H$ then for $z \in K_{r_H}$*

$$\left|\frac{F(z)}{F'(z)}\right| \le \frac{r(1-r^4)}{(1-r^2) - 2r(1+r^2)}, \quad r = |z|. \tag{5}$$

The estimation is sharp and the equality holds for the function F of the form

$$F(z) = \frac{z}{1-z^2} \cdot \frac{1+iz}{1-iz} \tag{6}$$

at the point $z = ir$.

Proof. It follows from the condition (4) that

$$\frac{zF'(z)}{F(z)} - \frac{1+z^2}{1-z^2} = \frac{zp'(z)}{p(z)}$$

where $p \in \mathcal{P}$. Making use of the well-known sharp estimation

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2|z|}{1-|z|^2} \quad \text{for} \quad p \in \mathcal{P}$$

we obtain

$$\left|\frac{zF'(z)}{F(z)} - \frac{1+z^2}{1-z^2}\right| \le \frac{2|z|}{1-|z|^2} \tag{7}$$

From (7) the inequality (5) follows.

A simple calculation shows that the function of the form (6) for $z = ir$ gives the equality in (5).

Theorem 3. *Let $f \in \mathcal{H}$, $F \in H$. If $f \ll F$ in the disc K and*

$$|z| = r < r_H = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} .$$

then

$$(8)\qquad |f'(z)| \le T(r)\cdot |F'(z)|\,,$$

where

$$(9)\qquad T(r) = \begin{cases} 1 & \text{for } r \in \langle 0, r_0\rangle \\ \dfrac{[(1-r^2)^2 - 2r(1+r^2)]^2 + 4r^2(1+r^2)^2}{4r(1+r^2)[(1-r^2)^2 - 2r(1+r^2)]} & \text{for } r \in (r_0, r_H)\,. \end{cases}$$

The number $r_0 = 1+\sqrt{2}-\sqrt{2+2\sqrt{2}} \approx 0,217$ is the unique positive root of the equation

$$(10)\qquad \frac{(1-r^2)^2 - 2r(1+r^2)}{2r(1+r^2)} = 1$$

in the interval $(0, r_H)$.

The result is sharp.

For $r \in \langle 0, r_0\rangle$ and for every pair of functions f, F such that $f(z) \equiv F(z)$, $F \in H$, we have $T(r) = 1$.

For $r_1 \in (r_0, r_H)$ the equality in (8) holds at the point $z = ir_1$ for the pair of functions f, F such that

$$f(z) = \Phi(z)\cdot F(z)\,,$$

where F is given by (6) and

$$(11)\qquad \Phi(z) = \frac{\dfrac{z - ir_1}{1 + ir_1 z} + M(r_1)}{1 + M(r_1)\dfrac{z - ir_1}{1 + ir_1 z}}\,,$$

$$(12)\qquad M(r) = \frac{(1-r^2)^2 - 2r(1+r^2)}{2r(1+r^2)}\,.$$

Proof. If $f \ll F$ in the disc K, then there exists a function Φ, $|\Phi(z)| \le 1$ for $z \in K$ such that

$$f(z) = \Phi(z)\cdot F(z) \quad \text{for } z \in K\,.$$

Hence

$$(13)\qquad \frac{f'(z)}{F'(z)} = \Phi'(z)\cdot \frac{F(z)}{F'(z)} + \Phi(z)\,.$$

It is known (cf. see [2] p.319) that

$$(14)\qquad |\Phi'(z)| \le \frac{1 - |\Phi(z)|^2}{1 - |z|^2} \quad \text{for } z \in K\,.$$

Taking account of the inequalities (5) and (14) in (13) we obtain

$$\left|\frac{f'(z)}{F'(z)}\right| \le \frac{-|\Phi(z)|^2}{2M(r)} + |\Phi(z)| + \frac{1}{2M(r)}\,, \tag{15}$$

where $M(r)$ is given by the formula (12).

Let z be fixed, $|z| = r$, then the right-hand side of inequality (15) is a function of the variable $u = |\Phi(z)|$, $u \in \langle 0, 1)$,

$$\Psi(u) = \frac{-u^2}{2M(r)} + u + \frac{1}{2M(r)}\,. \tag{16}$$

It should be noted that $M(r)$ is a decreasing function in the interval $(0, 1)$. If $r \in \langle 0, r_0)$, where r_0 is the root of the equation (10), then $M(r) \ge 1$. Then Ψ is increasing in the interval $\langle 0, 1\rangle$ and at the point $u = 1$ it attains the greatest value $\Psi(1) = 1$. If $r \in (r_0, r_H)$, then

$$0 = M(r_H) < M(r) < 1$$

and at the point $u_0 = M(r)$ the function Ψ attains the greatest value equal

$$\Psi(u_0) = \frac{(1-r^2)^2 - 2r(1+r^2)^2 + 4r^2(1+r^2)^2}{4r(1+r^2)[(1-r^2)^2 - 2r(1+r^2)]}\,. \tag{17}$$

It is easily seen that for $|z| = r \in \langle 0, r_0\rangle$ and for every pair of functions f, F, $F \in H$ such that $f(z) \equiv F(z)$

$$\left|\frac{f'(z)}{F'(z)}\right| = 1\,.$$

An elementary calculation shows that for $r_1 \in (r_0, r_H)$ and for the pair of functions f, F, where F is given by formula (6),

$$f(z) = \Phi(z) \cdot F(z)$$

and Φ defined by (11) for $z = ir_1$ we have

$$\left|\frac{f'(z)}{F'(z)}\right| = \Psi(u_0)\,,$$

where $\Psi(u_0)$ is defined by (17).

3. The majorization of derivatives in the class of close–to–star functions. Let

$$S^* = \left\{G \in \mathcal{H} : \operatorname{Re} \frac{zG'(z)}{G(z)} > 0 \text{ for } z \in K,\ G(0) = 0,\ G'(0) = 1\right\}.$$

It is the well-known class of starlike functions.

The function F holomorphic in disc K and such that $F(0) = 0$, $F'(0) = 1$, is said to be close to star if there exists a function $G \in S^*$ such that

$$\operatorname{Re} \frac{F(z)}{G(z)} > 0 \quad \text{for } z \in K . \tag{18}$$

The class of close-to-star functions is denoted by H^*.

It is easy to observe that if $G(z) = \dfrac{z}{1-z^2}$, $G \in S^*$, then the condition (18) takes the form

$$\operatorname{Re}\left\{(1-z^2)\frac{F(z)}{z}\right\} > 0 .$$

It means that $H \subset H^*$.

Theorem 4. *The radius of starlikeness in the class H^**

$$r^* = \sup_r \left\{ |z| < r \in (0,1) : \operatorname{Re} \frac{zF'(z)}{F(z)} > 0 \text{ for each function } F \in H^* \right\} = 2 - \sqrt{3} .$$

Proof. From the condition (18) it follows that

$$\frac{F(z)}{G(z)} = p(z) \quad , \quad p \in \mathcal{P} .$$

Hence

$$\frac{zF'(z)}{F(z)} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)} .$$

Using the well-known and sharp estimations

$$\operatorname{Re} \frac{zG'(z)}{G(z)} \geq \frac{1-r}{1+r} , \quad r = |z| , \quad G \in S^*$$

and

$$\operatorname{Re} \frac{zp'(z)}{p(z)} \geq \frac{-2r}{1-r^2} , \quad r = |z| , \quad p \in \mathcal{P}$$

we get

$$\operatorname{Re} \frac{zF'(z)}{F(z)} \geq \frac{(1-r)^2 - 2r}{1-r^2} . \tag{19}$$

This estimation is sharp. The equality holds for the function F

$$F(z) = \frac{z}{(1+z)^2} \cdot \frac{1-z}{1+z} \tag{20}$$

at the point $z = r$.

In order that the function F is starlike it must be

$$\frac{(1-r)^2 - 2r}{1-r^2} \geq 0 .$$

Hence, it follows that $r^* = 2 - \sqrt{3} \approx 0,268$.

Lemma . *If $F \in H^*$, then for $|z| < r^* = 2 - \sqrt{3}$ we have the sharp estimate*

(21) $$\left|\frac{F(z)}{F'(z)}\right| \le \frac{r(1-r^2)}{(1-r)^2 - 2r}, \quad r = |z|.$$

The equality takes place for the function (20) at the point $z = r$.

The above estimate follows from inequality

$$\left|\frac{zF'(z)}{F(z)}\right| \ge \operatorname{Re}\frac{zF'(z)}{F(z)}$$

and from the estimate (19).

Theorem 5. *Let $f \in \mathcal{H}$, $F \in H^*$. If $f \ll F$ in the disc K and $|z| = r < r^* = 2 - \sqrt{3}$, then*

(22) $$|f'(z)| \le T(r) \cdot |F'(z)|,$$

where

$$T(r) = \begin{cases} 1 & \text{for } r \in \langle 0, r_0 \rangle \\ \dfrac{[(1-r)^2 - 2r]^2 + 4r^2}{4r[(1-r)^2 - 2r]} & \text{for } r \in (r_0, 2 - \sqrt{3}) \end{cases}.$$

The number $r_0 = 3 - \sqrt{8} \approx 0,172$ is the unique positive root of the equation

(23) $$\frac{(1-r)^2 - 2r}{2r} = 1$$

in the interval $(0, 2 - \sqrt{3})$.

The result is sharp.

For $r \in \langle 0, r_0 \rangle$ the equality in (22) is attained for every pair of functions f, F, $F \in H^*$ such that $f(z) \equiv F(z)$.

If $r_1 \in (r_0, 2 - \sqrt{3})$ then the equality in (22) is attained at the point $z = r_1$ for the pair of functions f, F and such that $f(z) = \Phi(z) \cdot F(z)$, where

(24) $$F(z) = \frac{z}{(1+z)^2} \cdot \frac{1-z}{1+z}, \quad z \in K$$

(25) $$\Phi(z) = \frac{\dfrac{z - r_1}{1 - r_1 z} + M(r_1)}{1 + M(r_1)\dfrac{z - r_1}{1 - r_1 z}}, \quad z \in K$$

(26) $$M(r) = \frac{(1-r)^2 - 2r}{2r}.$$

Proof. From the equality $f(z) = \Phi(z) \cdot F(z)$, $z \in K$, and from the estimations (14), (21) it follows that

$$\left|\frac{f'(z)}{F'(z)}\right| \leq \frac{-u^2}{2M(r)} + u + \frac{1}{2M(r)} = \Psi(u) , \tag{27}$$

where $u = |\Phi(z)|$, $u \in \langle 0, 1)$, z is fixed, $|z| = r$. Proceeding analogously as in the proof of Theorem 3 we obtain

$$\max_{u \in \langle 0;1)} \Psi(u) = \begin{cases} 1 & \text{for } r \in \langle 0, 3-\sqrt{8}) \\ \dfrac{[(1-r)^2 - 2r]^2 + 4r^2}{4r[(1-r)^2 - 2r]} & \text{for } r \in (3-\sqrt{8}, 2-\sqrt{3}) . \end{cases}$$

Hence our theorem follows.

It is easily seen that for $r \in \langle 0, 3-\sqrt{8})$ the equality in (22) is attained for each pair of functions f, F such that $f(z) \equiv F(z)$, $F \in H^*$.

An elementary calculation shows that for $r_1 \in (3-\sqrt{8}, 2-\sqrt{3})$ and for the pair of functions f, F, $f(z) = \Phi(z) \cdot F(z)$, where F is given by formula (24), Φ defined by (25) we obtain equality in (22) at the point $z = r_1$.

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STRESZCZENIE

W pracy badana jest zależność między majoryzacją funkcji a majoryzacją pochodnych w klasie H, funkcji spełniających w kole jednostkowym K warunek

$$\operatorname{Re}\left\{(1-z^2)\frac{F(z)}{z}\right\} > 0 ,$$

oraz w klasie H^* funkcji prawie gwiaździstych.

Otrzymane rezultaty są dokładne.