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Duality Applied to Meromorphic Functions with a Simple Pole at the Origin

Zasada dualności dla funkcji meromorficznych z biegunem pierwszego rzędu w początku układu

Принцип дуальности для мероморфных функций из простым полюсом в точке 0

1. Introduction. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be analytic in the unit disk $D = \{z : |z| < 1\}$ and normalized by $f(0) = g(0) = 1$. We denote the class of functions with this property by A_0. The convolution (Hadamard product) of f and g is defined by

$$(f*g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k .$$

For $U \subset A_0$ the dual set U^* is defined in the following way

$$U^* = \{ g \in A_0 : \text{for each } f \in U \; (f*g)(z) \neq 0 , \; z \in D \} .$$

We denote $(U^*)^*$ by U^{**} and call it the second dual of U.

This concept was introduced by Ruscheweyh [5] in connection with the work leading to the proof of the Pólya-Schoenberg conjecture [6]. The central reference on convolutions and properties of duality in A_0 is the book of Ruscheweyh: Convolutions in Geometric Function Theory [4].

We introduce the class B of functions analytic in $0 < |z| < 1$ with a simple pole at the origin and the subclass B_0 consisting of functions with the series expansion

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \quad . \tag{1.1}$$

I.e. $f \in B_0$ if and only if $zf \in A_0$.

The purpose of the present paper is to show some results from the transfer of the theory of convolutions and duality from A_0 to B_0.

For $f, g \in B_0$ the convolution is defined in the obvious way

$$(f*g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k \quad . \tag{1.2}$$

The concept of duality can also be transformed in a natural way, because for f and $g \in B_0$ we have

$$f*g \neq 0 \; , \quad 0 < |z| < 1$$

if and only if

$$zf * zg \neq 0 \quad , \qquad z \in D \quad .$$

The basic theorem for the duality theory in A_0 is the Duality Principle which is stated and proved in [4]. Before stating the Duality Principle in B_0 we shall give two definitions which will be needed later.

Definition.

1. $U \subset B_0$ is called complete if for all $f \in U$, $0 < |x| \leq 1$ we have $f_x \in U$, where $f_x(z) = xf(xz)$.
2. Let $U \subset B_0$. $T \subset B_0$ is called a test set for U if

$$T \subset U \subset T^{**}$$

and write $T \rightsquigarrow U$.

The definition of a test set in B_0 is exactly the same as in A_0. In the definition of completeness the function $f_x(z)$ is defined slightly different in B_0 because we want to keep the normalization on f_x.
(The corresponding definitions in A_0 are in [4].)

2. The duality principle in B_0. In the topology of uniform convergence on compact subsets of the punctured disk $0 < |z| < 1$, B is a locally convex topological vector space. Let Λ be the space of continuous linear functionals on B.

Theorem 1. (Duality Principle). Let $U \subset B_0$ be compact and complete. Then

(i) for each $\lambda \in \Lambda$: $\lambda(U) = \lambda(U^{**})$;

(ii) $\overline{co}\,(U) = \overline{co}\,(U^{**})$.

The Duality Principle in A_0 is stated in exactly the same way [4], and the proof runs the same way for both A_0 and B_0 . We will therefore not go into details of the proof, but only point out that the proof rests on the representation theorem for continuous linear functionals by Caccioppoli [1] which in our case is a slight modification of the theorem of Toeplitz [7]. This theorem will be formulated in the following way for the class B .

Theorem. $\lambda \in \Lambda$ if and only if there is a function $g \in B$ such that for $f \in B$

$$\lambda(f) = (f*g)(1) \quad .$$

3. Applications to univalent functions. We now turn to the class of univalent functions in B_0 , here denoted by Σ . By Σ_0 we denote the subclass of Σ which consists of the functions with constant term zero. The following theorem shows how Σ can be described as the dual set of a two parameter family of functions.

Theorem 2. Let

$$(3.1) \qquad V = \left\{ f \in B_0 : f(z) = \frac{1}{z} - \frac{xyz}{(1-xz)(1-yz)} \; ; \; x,y \in \overline{D} \; , \; x \neq y \right\}.$$

Then $V^* = \Sigma$ and $\Sigma_0 \subset V^{**}$.

Proof. Let $g \in B_0$ and $f \in V$. Then we have

$$(f*g)(z) = \left(\frac{1}{z} - \frac{xyz}{(1-xz)(1-yz)}\right) * g(z) =$$

$$= \frac{xy}{x-y}\left(\frac{1}{yz} + \frac{1}{1-yz} - \frac{1}{xz} - \frac{1}{1-xz}\right) * g(z) =$$

$$= xy\,\frac{g(yz)-g(xz)}{x-y} .$$

From this computation we see that $f*g \neq 0$ if and only if g is univalent. Thus we have proved that $V^* = \Sigma$.

To prove that $\Sigma_0 \subset V^{**}$ we use the following well known fact:

If $f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \in \Sigma$ and $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \in \Sigma_0$,

then $\sum_{k=1}^{\infty} k|a_k b_k| \leq 1$, and this implies that $f*g$ is starlike.

(For reference, see e.g. Goodman [2], p. 134-135.)

This means in particular that $(f*g)(z) \neq 0$, $0 < |z| < 1$, so $\Sigma_0 \subset \Sigma^* = V^{**}$.

A relevant type of problem in this context is to find a suitable test set for a given set. It would in particular be an interesting problem to try to find a test set for Σ. Because of the Duality Principle we then could get information about Σ by investigating the functions in the test set. Theorem 2 is a small step in this direction because of the inclusion $\Sigma_0 \subset V^{**}$.

But we obviously do not have $V \subset \Sigma_0$ since by appropriate choices of x and y the coefficients of the functions in V will be so large that the area theorem is violated.

Our next idea is to introduce a set of functions related to V, but with smaller coefficients. A function $f \in V$ can be written $f(z) = \frac{1}{z} - xyz - xy(x+y)z^2 - xy(x^2+xy+y^2)z^3 - \dots$. Let $\ell(z) = \frac{1}{z} + \log(1-z) = \frac{1}{z} - z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \dots$. Define a function $h(z) = (f(z) * \ell(z)) * \ell(z)$, $f \in V$, and let W be the set consisting of functions of this form. That is

$$(3.2) \qquad W = \left\{ h \in B_0 ;\ h(z) = \frac{1}{z} - \frac{xy}{x-y} \sum_{k=1}^{\infty} \frac{x^k - y^k}{k^2} z^k \ ;\ x,y \in \overline{D},\ x \neq y \right\}.$$

For functions in W it is clear that the k^{th} coefficient is bounded by $\frac{1}{k}$ in absolute value, so these functions are "closer" to the univalent functions as far as the size of the coefficients is concerned.

For $h \in W$, $f \in B_0$ we get

$$(3.3) \qquad z(z(f*h)')'(z) = xy\,\frac{f(yz) - f(xz)}{x-y}$$

which means that for $f \in \Sigma$, $h \in W$

$$(3.4) \qquad z(z(f*h)')'(z) \neq 0 \quad , \quad 0 < |z| < 1 \ .$$

A natural question is to ask whether for arbitrary f and h in Σ, (3.4) will be true. As previously mentioned the convolution of two functions f and g in Σ_0 is starlike. In particular we will then have $(f*g)'(z) \neq 0$, $0 < |z| < 1$. If f and g are in Σ_0 (or in Σ), it is therefore clear that

$$(3.5) \qquad z(z(f\ g)')'(z) \neq 0 \ , \quad 0 < |z| < 1$$

is equivalent to

$$(3.6) \qquad 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \neq 0 \ , \quad 0 < |z| < 1 \ .$$

We notice the similarity between (3.6) and the condition for convexity of $f*g$ which is

$$(3.7) \qquad \operatorname{Re}\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)}\right\} < 0 \ , \quad 0 < |z| < 1 \ .$$

This is a stronger condition than (3.6), and it is tempting to ask whether the convolution of two meromorphic univalent functions (in Σ) is a convex function. If this were true, it would indeed be a surprising result, but we will soon give an example showing that neither (3.7) nor (3.5) is true in general.

Still we fill that it would be of interest to characterize the subset of the univalent functions for which (3.5) holds.

We denote this subset by C and define it in the following way.

$$(3.8) \qquad C = \left\{ g \in \Sigma_0 : \text{for each } f \in \Sigma ,\ z(z(f*g)')'(z) \neq 0 ,\ 0 < |z| < 1 \right\} .$$

From (3.3) it is immediately clear that $C \supset W \cap \Sigma$. We notice that it is no restriction to define C as a subset of Σ_0 because if g is a function that satisfies (3.5), then any function $g + c_0$, c_0 a complex constant, will also satisfy (3.5).

It is well known that the convolution of a convex univalent function with an arbitrary univalent function (both in B_0) is convex [3], so we know that C contains all functions in B_0 that are convex.

We will return to the class C later. First we give the example showing that C does not contain all of Σ .

4. The counterexample. Let

$$F(z) = \frac{1}{z}(1+z^4)^{1/2} = \frac{1}{z} + \sum_{k=1}^{\infty} a_{4k-1} z^{4k-1}$$

and

$$k_r(z) = \frac{1}{z(1-(rz)^4)} = \frac{1}{z} + \sum_{k=1}^{\infty} r^{4k} z^{4k-1}$$

Then

$$F(z) * k_r(z) = \frac{1}{z}(1 + (rz)^4)^{1/2} .$$

F is univalent, so if we can find some r for which k_r is univalent and the same time $z(z(F\,k_r)')'(z) = 0$ for some z, $0 < |z| < 1$, then we have constructed a counterexample to (3.5).

In order to decide when k_r is univalent we choose $z_1 \neq z_2$ and compute

$$k_r(z_1) - k_r(z_2) = \frac{1}{z_1(1-(rz_1)^4)} - \frac{1}{z_2(1-(rz_2)^4)} =$$

$$\text{(4.1)} \qquad = \frac{z_2 - z_1 - (r^4 z_2^5 - r^4 z_1^5)}{z_1 z_2 (1-(rz_1)^4)(1-(rz_2)^4)} =$$

$$= \frac{(z_2 - z_1)\left[1 - r^4(z_2^4 + z_1 z_2^3 + z_1^2 z_2^2 + z_1^3 z_2 + z_1^4)\right]}{z_1 z_2 (1-(rz_1)^4)(1-(rz_2)^4)}$$

If $r^4 \leq \frac{1}{5}$, (4.1) will never be zero for $0 < |z_1|, |z_2| < 1$, and if $r^4 > \frac{1}{5}$, it will be possible to find z_1, z_2 such that $k_r(z_1) - k_r(z_2) = 0$. Thus we have found that $k_r(z)$ is univalent if and only if $r < 5^{-1/4} \approx 0.6687$.

In order to decide when $z(z(F*k_r)')'(z) = 0$ we get the equation

$$\text{(4.2)} \qquad \frac{1}{z}(1 + (rz)^4)^{1/2} + 4r^4 z^3 (1 + (rz)^4)^{-1/2} -$$

$$- 4r^8 z^7 (1 + (rz)^4)^{-3/2} = 0 .$$

This equation has a solution in $0 < |z| < 1$ if and only if $r > (3 - 2\sqrt{2})^{1/4} \approx 0.6436$.

By geometric considerations one can see that this r value also will be the radius of convexity for the functions $\frac{1}{z}(1 + (rz)^4)^{1/2}$. So in this case condition (3.5) and condition (3.7) will be equivalent.

The conclusion is that if we choose r in the interval $0.6436 < r < 0.6687$, then $F(z) * k_r(z)$ will be the convolution of two functions from Σ , and there is a z , $0 < |z| < 1$, such that $z(z(F * k_r)')'(z) = 0$.

5. More about the class C . From the preceding example we have seen that C , as defined in (3.8), does not contain all univalent functions. But we notice that the interval of permissable r values was rather small, and that could be a hint towards guessing that C is a fairly big subset of Σ_0 . It would therefore be interesting to find a good characterization of C .

The following result, although not very informative, is an immediate consequence of the definitions we have made.

Theorem 3. As before let

$$W = \left\{ h \in B_0 : h(z) = \frac{1}{z} - \frac{xy}{x-y} \sum_{k=1}^{\infty} \frac{x^k - y^k}{k^2} z^k \quad ; \quad x, y \in \bar{D} , \quad x \neq y \right\}$$

and

$$C = \left\{ g \in \Sigma_0 : \text{for each } f \in \Sigma , \quad z(z(f * g)')'(z) \neq 0 , \ 0 < |z| < 1 \right\}$$

Then $C = \Sigma_0 \cap W^{**}$.

Proof. Let $f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$ and define

$$F(z) = (f(z) * \ell(z)) * \ell(z) , \quad \text{where} \quad \ell(z) = \frac{1}{z} + \log(1-z) .$$

Keeping in mind that a function from W can be expressed as a function from the set V, as defined in (3.1), convolved twice with $\ell(z)$, it is clear that with $h \in W$

$$(5.1) \qquad f(z) * h(z) = \frac{xy}{y-x}\left[F(xz) - F(yz)\right] .$$

From (5.1) we deduce that $f \in W^*$ if and only if F is univalent ($\in \Sigma_0$). With F as above we can write

$$(5.2) \qquad (f * g)(z) = z(z(F * g)')'(z)$$

for arbitrary $g \in B_0$ with constant term zero.

Now assume that F is univalent ($f \in W^*$) and $g \in C$. Then $z(z(F*g)')'(z) \neq 0$, $0 < |z| < 1$, and from (5.2) we get $(f*g)(z) \neq 0$, $0 < |z| < 1$. Thus we have proved that $C \subset W^{**}$. In fact we have $z(z(F*g)')'(z) \neq 0$ if and only if $g \in W^{**}$, so W^{**} consists exactly of those functions g having the property that $z(z(F*g)')'(z) \neq 0$ for any $F \in \Sigma_0$. Since we have defined C to be a subset of Σ_0, we get that $C = \Sigma_0 \cap W^{**}$.

Remark. The ultimate goal of the investigations of the present type is to find a suitable test set for Σ. From what we now have seen, it is clear that W^{**} will not contain all of Σ_0, so W is not a suitable candidate for a test set. Nevertheless the class C seems to be an interesting and fairly large subset of Σ_0, and hence it would be interesting to make further investigations of W^{**} in order to get a better characterisation of the class C.

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STRESZCZENIE

W pracy tej wprowadzone przez Ruscheweyh'a pojęcie zbiorów dualnych (ze względu na splot Hadamarda) funkcji holomorficznych f w kole jednostkowym, f (0) = 1, przeniesiono na klasę B_0 funkcji holomorficznych w obszarze $\{z : 0 < |z| < 1\}$, mających w zerze biegun pierwszego rzędu z residuum 1. Sformułowana została zasada dualności dla B_0. Badana była podklasa $\Sigma \subset B_0$ składająca się z funkcji jednolistnych. Wyznaczono zbiór dualny do Σ .

РЕЗЮМЕ

В этой работе введено Рушевайом понятие дуальных множеств /по отношении к свертке Адамара/ аналитических функций f в единичном круге f(o)=1, переносится на класс B_0 функций аналитических в области $\{z: 0 < |z| < 1\}$ имеющих в z=0 простой полюс с вычетом 1. Формулирован принцип дуальности для B_0. Исследован подкласс $\Sigma \subset B_0$ однолистных функций и определено его дуальное множество.