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Generalized Powers in the Theory of (ν, μ)-Solutions

Uogólnione potęgi w teorii (ν, μ)-rozwiązań

Обобщенные степенные функции в теории (ν, μ)-решений

Introduction. In view of the representation theorem, (ν, μ)-solutions, i.e. solutions of a system $f_{\bar z} = \nu f_z + \mu \overline{f_z}$, have a lot of properties in common with analytic functions. Especially, the notions of zeroes and poles and their orders are well defined. Namely, a (ν, μ)-solution f has a zero of order n at the point z_0 if and only if any representation of f, according to the representation theorem, reads $f(z) = F \circ \chi(z)$ with a quasiconformal mapping χ of a neighbourhood U of z_0 and a function $F(\chi)$ analytic in $\chi(U)$ which has a zero of n-th order at $\chi(z_0)$. Of course, the property of z_0 to be a zero of order n of a (ν, μ)-solution is independent of the choice of χ and F. In an analogous manner, poles and their orders are defined.

If there are no additional conditions on ν, μ, as

0.1 $\nu, \mu \in L_\infty(C)$, $|\nu| + |\mu| \leq k = \text{const.} < 1$ a.e. in C (C the finite complex plane), the order of a zero or pole has nothing to do with any asymptotic behaviour. This can be shown by rather simple examples (cf. [4], p. 72). But there is another question, where the answer is open in the most general case of ν, μ. Namely, let f, g be two (ν, μ)-solutions in a neighbourhood U of z_0 with zeroes of order n resp k at z_0. Of course, $f + g$ has again a zero at z_0 of a certain order m, but is always

0.2 $$m \geq \min(n, k) \quad ?$$

The corresponding question exists if z_0 is a pole of order n resp k of f resp g.

Both questions have an answer corresponding to the classical case if ν, μ satisfy some additional conditions, for example, if we have

0.3 $$\iint_U \left\{ \left| \frac{\nu(z)-\nu(z_0)}{z-z_0} \right|^p + \left| \frac{\mu(z)-\mu(z_0)}{z-z_0} \right|^p \right\} d\sigma_z ,$$

U any neighbourhood of z_0.

This condition assures the "natural" correspondence between the order of the zero or the pole z_0 and a certain asymptotic behavior at z_0. The asymptotic behavior of (ν, μ)-solutions at poles or zeroes is the topic of the next chapter, and these results are basic for the concept of generalized powers, introduced and treated in the second chapter. As one application we obtain an integral formula for the (first) derivatives of (ν, μ)-solutions which is the counterpart of the classical formula

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^2}\, dz \quad .$$

1. Asymptotic Expansions. The following function space plays an important role for the asymptotic behaviour of - solutions.

1.1 Definition. Let $G \subset C$ be a measurable set, D an arbitrary (not necessarily measurable) subset of G, and p any real number ≥ 1. By $HL_p(D,G)$ we denote the set of all functions f defined and measurable in G which satisfy

(I) $\frac{f(z)-f(z_0)}{z - z_0} \in L_p(G)$ (as a function of z) for each $z_0 \in D$

and

(II) $\|f\|_{HL_p(D,G)} = \|f\|_{L_\infty(G)} + \sup\left\{\left\|\frac{f(z)-f(z_0)}{z - z_0}\right\|_{L_p(G)} : z_0 \in D\right\} < \infty$.

Instead of $HL_p(D,C)$ we write $HL_p(D)$.

The space $HL_p(D,G)$ equipped with the norm (II) is a Banach space, but we will not use this fact in the following.

In [4], p. 65, it has been shown the

1.2 Theorem. Let G be a domain $\subset C$, D' an arbitrary set satisfying $D' \subset D$, $D' \subset\subset G$, and be $p > 2$. Then every $f \in HL_p(D,G)$ is continuous and bounded in D', and each bounded set in $HL_p(D,G)$ is compact in $C(D')$ ($C(D')$ the usual space of functions continuous in D' with the supremum norm).

Perhaps it is possible to say much more about $HL_p(D,G)$. In any case, the kind of continuity of the functions from $HL_p(D,G)$ with $p > 2$ is anything between the usual continuity and Hölder continuity (perhaps equal to Hölder continuity under certain additional conditions on D, G, and p). Of course, theorem 1.2 makes no sense if D consists only of isolated points.

1.3 Definition. Let G be a domain, $z_0 \in G$, and $w(z)$ be a (ν, μ)-solution in $G \setminus \{z_0\}$. The point z_0 is called a point of order n of $w(z)$, if either n is a negative integer and $w(z)$ has a pole of order $-n$ at z_0 or n is a nonnegative integer and $w(z)$ has at z_0 a zero of order n (a zero of order 0 at z_0 means that $w(z_0) \neq 0$).

From now on we will assume additionally to 0.1 that ν, μ are defined not only a.e. but everywhere in $\overline{C} = C \cup \{\infty\}$. Obviously this is no loss of generality.

In [4], p. 72 (cf. also [3], p. 130) it has been shown the

1.4 Theorem. Let $z_0 \neq \infty$ be a point of order n of the (ν, μ)-solution $w(z)$, and be $\nu, \mu \in HL_p(\{z_0\})$, $p > 2$.

Then $w(z)$ has the asymptotic development

(I) $w(z) = c\left[z-z_0 + \mathfrak{b}(\overline{z-z_0})\right]^n - b\bar{c}\left[\overline{z-z_0} + \bar{\mathfrak{b}}(z-z_0)\right]^n + O(|z-z_0|^{n+\alpha})$

with certain constants $c \neq 0$, $\alpha > 0$ ($O(.)$ denotes the usual Bachmann-Landau symbol), and

(II) $b = \omega(\chi)$, $\mathfrak{b} = \omega(\sigma)$,

$$\chi = -\mu(z_0) / (1+|\mu(z_0)|^2 - |\nu(z_0)|^2) ,$$

$$\sigma = \nu(z_0) / (1+|\nu(z_0)|^2 - |\mu(z_0)|^2)$$

where $\omega(.)$ is the function $\omega(x) = 2x / (1+\sqrt{1-4x^2})$. These b , $\mathfrak{b}$ satisfy the estimate $|b| \leqslant k$, $|\mathfrak{b}| \leqslant k$ for every $z_0 \in \mathbb{C}$.

Under additional assumptions on ν, μ more can be said on α , cf. [4], p. 73.

We want now to extend this theorem to the case $z_0 = \infty$. This requires an assumption on ν, μ with respect to $z_0 = \infty$, which is analogous to ν , $\mu \in HL_p(\{z_0\})$ for finite z_0 .

1.5 Definition. (I) We will say that $f \in HL_p(\{\infty\})$ if and only if f is defined everywhere in $\bar{\mathbb{C}}$, and $f(1/z) - f(\infty)$ belongs to $HL_p(\{0\})$.

(II) Let D be an arbitrary subset of $\bar{\mathbb{C}}$, $\infty \in D$, $D \neq \emptyset$. We will say that $f \in HL_p(D)$ if and only if $f \in HL_p(D \setminus \{\infty\}) \cap HL_p(\{\infty\})$.

Then we have the following completion of theorem 1.4.

1.6 Theorem. Let ∞ be a point of order n of the (ν,μ)-solution $w(z)$, and be $\nu,\mu \in HL_p(\{\infty\})$, $p>2$. Then, with respect to the point ∞, the asymptotic expansion

$$w(z) = c\left[z+\beta\bar z\right]^{-n} - b\bar c\left[\bar z+\bar\beta z\right]^{-n} + O(|z|^{-n-\alpha})$$

holds with certain constants $c \neq 0$, $\alpha > 0$, and with b, β as in 1.4(II), taking there $z_0 = \infty$.

Proof. Let $w(z)$ be any (ν,μ)-solution in a domain G. By the affine transformation

1.7 $\zeta = z + \beta\bar z + d$, β, d arbitrary but fixed constants $\in \mathbb{C}$, $|\beta|<1$,

the $w(z)$ is carried into a (ν_1,μ_1)-solution $w_1(\zeta)$ in the image domain G_1 of G under the mapping 1.7. At this we have

1.8 $\nu_1(\zeta) = (\bar\nu\beta^2 - (1+|\nu|^2-|\mu|^2)\beta + \nu)/N_1$,

$\mu_1(\zeta) = \mu(1-|\beta|^2)/N_1$,

$N_1 = |1-\bar\nu\beta|^2 - |\mu\beta|^2$, $\nu = \nu(z(\zeta))$, $\mu = \mu(z(\zeta))$.

Now we apply the affine mapping

1.9 $g = w_1 + b\bar w_1$, b an arbitrary but fixed constant with $|b|<1$,

to the (ν_1,μ_1)-solution $w_1(\zeta)$. By this transformation we carry $w_1(\zeta)$ into a (ν_2,μ_2)-solution $g(\zeta)$ in the domain

G_1 with

1.10
$$\nu_2(\zeta) = \left[\bar{\nu}\mathfrak{b}^2 - (1+|\nu|^2-|\mu|^2)\mathfrak{b} + \nu\right]\cdot(1-|b|^2)/N ,$$
$$\mu_2(\zeta) = \left[\bar{\mu}b^2 + (1+|\mu|^2-|\nu|^2)b + \mu\right]\cdot(1-|\mathfrak{b}|^2)/N ,$$
$$N = (1-|b|^2)\cdot\left[\tfrac{1}{2}(1+|\nu|^2-|\mu|^2)(1+|\mathfrak{b}|^2) - 2\,\mathrm{Re}\,\mathfrak{b}\bar{\nu}\right] +$$
$$+ (1-|\mathfrak{b}|^2)\left[\tfrac{1}{2}(1+|\mu|^2-|\nu|^2)(1+|b|^2) - 2\,\mathrm{Re}\,b\bar{\mu}\right] ,$$
$$\nu = \nu(z(\zeta)) , \quad \mu = \mu(z(\zeta)) .$$

As to ν_2, μ_2 we have the estimate

1.11
$$|\nu_2(\zeta)| + |\mu_2(\zeta)| \leqslant 1 - (1-k)\frac{(1-|\mathfrak{b}|)(1-|b|)}{(1+|\mathfrak{b}|)(1+|b|)}$$
for each $\zeta \in \bar{C}$,

which may be shown by means of the geometrical interpretation (i.e. arithmetically, by representing ν, μ by new parameters), cf. [4], p. 49.

According to 1.10 it is

1.12
$$\nu_2(\zeta_0) = \mu_2(\zeta_0) = 0 \quad \text{at the point} \quad \zeta_0 = z_0 + \mathfrak{b}\bar{z}_0 + a ,$$

if we take for $\mathfrak{b}$, b the expressions from 1.4(II), and this holds true especially if $z_0 = \zeta_0 = \infty$. Because the absolute values of that $\mathfrak{b}$, b are restricted by k , the estimate 1.11 yields in case of such $\mathfrak{b}$, b

1.13
$$|\nu_2(\zeta)| + |\mu_2(\zeta)| \leqslant 1 - \frac{(1-k)^3}{1+k^2} \equiv k' .$$

Now, let ∞ be a poin of order n of the (ϑ,μ)-solution $w(z)$. With the $\mathfrak{b}, b$ corresponding to $z_0 = \infty$ according to 1.4(II), we apply the transformations 1.7, 1.9 to $w(z)$. By a subsequent inversion, applied to the corresponding (ϑ_2, μ_2)--solution $g(\zeta)$, we obtain a (ϑ^*, μ^*)-solution $h(t) = g(1/t)$ in a certain set $0 < |t| < r$ with

1.14 $$\vartheta^*(t) = \vartheta_2(\tfrac{1}{t})\,\frac{t^2}{\bar{t}^2}\ , \qquad \mu^*(t) = \mu_2(\tfrac{1}{t})\ .$$

Because of $\vartheta, \mu \in HL_p(\{\infty\})$ we have $\vartheta_2, \mu_2 \in HL_p(\{\infty\})$ with $\vartheta_2(\infty) = \mu_2(\infty) = 0$. The ϑ^*, μ^* therefore belong to $HL_p(\{0\})$. Of course, $h(t)$ has a point of order n at $t=0$. Theorem 1.4 then gives

$$h(t) = c^* \cdot t^n + O(|t|^{n+\alpha})$$

in a neighbourhood of $t=0$, which is equivalent to the assertion of theorem 1.6.

2. Generalized Powers. We now ask for a certain converse of theorem 1.4. Let be given a constant $c \neq 0$ and the integer n. Are there (ϑ,μ)-solutions with an expansion 1.4(I), and what additional conditions may be prescribed for such (ϑ,μ)-solutions if they do exist ? Of course, the case $n = 0$ is uninteresting and may be omitted because, together with $w(z)$, $w(z)$+const. is again a (ϑ,μ)-solution.

The following considerations are based on the

2.1 Condition. Let $z_0 \neq \infty$ be an arbitrary but fixed point, and ϑ, μ are to satisfy $|\vartheta(z)| + |\mu(z)| \leqslant k = \text{const.} < 1$

for each $z \in \bar{C}$ as well as $\vartheta, \mu \in HL_p(\{z_0, \infty\})$ with $p > 2$.

The following theorem holds.

2.2 Theorem. Let $n \neq 0$ be an integer and $c \neq 0$ an arbitrary complex number. Under condition 2.1 there is exactly one (ϑ,μ) -solution $w(z)$ in $C \setminus \{z_0\}$ with the properties

(I) $w(z)$ has at z_0 the asymptotic expansion

$$w(z) = c\left[z-z_0 + b(\overline{z-z_0})\right]^n - b\bar{c}\left[\overline{z-z_0} + \bar{b}(z-z_0)\right]^n + O(|z-z_0|^{n+\alpha}),$$

with b, b, α as in theorem 1.4, and

(II) the point ∞ is a point of order $-n$ of $w(z)$.

This unique $w(z)$ will be called generalized n-th power and will be denoted by $\left[c(z-z_0)^n\right]_{(\vartheta,\mu)}$ or simply by $\left[c(z-z_0)^n\right]$, if no misunderstanding is possible.

As the following proof will show, the existence of a $w(z)$ with the properties (I) and (II) in 2.2 is assured under 2.1 without $\vartheta, \mu \in HL_p(\{\infty\})$ but we cannot prove uniqueness.

At first, let us notice a certain topological property of $\left[c(z-z_0)^n\right]$.

2.3 Corollary. There are exactly $|n|$ quasiconformal mappings $\chi_1, \ldots, \chi_{|n|}$ of $\bar{C}$ onto itself with $\chi_j(z_0) = 0$, $\chi_j(\infty) = \infty$, and

$$\left[c(z-z_0)^n\right]_{(\vartheta,\mu)} = (\chi_j(z))^n \quad , \quad j=1,\ldots,|n| \quad ,$$

and these χ_j may be arranged in such an order that we have

$$\chi_j(z) = e^{2\pi ij/n}\cdot\chi_1(z) \quad , \quad j=1,\dots,|n| \ .$$

Proof of the corollary. Without loss of generality we may assume $z_0 = 0$. By the representation theorem we have

$$[cz^n] = f\circ\chi(z)$$

with a quasiconformal mapping χ of $\overline{C}$ onto itself with $\chi(0) = 0$, $\chi(\infty) = \infty$, and f an analytic function in $C\setminus\{0\}$. Because of 2.2(I), (II), f has at $z = 0$ a point of order n and at $z = \infty$ a point of order $-n$. Hence,

$$f(\chi) = a\cdot\chi^n \text{ with a certain constant } a \neq 0 \ .$$

Putting $\sqrt[n]{a}\cdot\chi = \chi_1$ we obtain $[cz^n] = (\chi_1(z))^n$. Of course, this is also valid for $\chi_j = e^{2\pi ij/n}\cdot\chi_1$, $j=1,\dots,|n|$. It remains to show that there are no further such $\chi = \chi^*$. For continuity, for each $z^* \in C\setminus\{0\}$ there is a whole neighbourhood $U(z^*)$ and a j with $\chi^*(z) \equiv \chi_j(z)$ for each $z \in U(z^*)$. Because there is no continuous change from a χ_j to a χ_m with $j \neq m$ in $C\setminus\{0\}$, the assertion 2.3 follows.

Proof of theorem 2.2. Without loss of generality let be $z_0 = 0$, and let us at first assume $\nu(0) = \mu(0) = 0$, $\nu, \mu \in HL_p(\{0\})$. For each (ν,μ)-solution f in $C\setminus\{0\}$, $H(z) = f(z)\, z^{-n}$ is a solution of

$$2.4 \qquad H_{\bar z} = \nu H_z + \mu\cdot\left(\frac{z}{\bar z}\right)^{-n}\overline{H_z} + \nu\frac{n}{z}H + \mu\cdot\left(\frac{z}{\bar z}\right)^{-n}\frac{n}{\bar z}\overline{H} \quad \text{in} \quad C\setminus\{0\},$$

and conversly, if H is any solution of 2.4, then Hz^n is a (ν,μ)-solution in $C\setminus\{0\}$. Of course, the same is true if we replace ν, μ by ν_m, μ_m with

2.5 $$\nu_m = \nu \quad \text{and} \quad \mu_m = \mu \quad \text{for} \quad |z| \leq m ,$$

$$\nu_m = \mu_m = 0 \quad \text{for} \quad |z| > m ,$$

m a positive integer.

<u>2.6 Lemma. For each number $c \in C$ there exists exactly one solution h of the system</u>

2.6.1 $$h_{\bar z} = \nu_m h_z + \mu_m \left(\frac{z}{\bar z}\right)^{-n} \overline{h_z} + \nu_m \left(\frac{n}{z}\right) h + \mu_m \left(\frac{z}{\bar z}\right)^{-n} \overline{\left(\frac{n}{z}\right) h} \quad \text{in} \quad C$$

<u>with $h(0) = c$ and h bounded in C. If $c \neq 0$, then $h(z) \neq 0$ for each $z \in C$.</u>

<u>Proof.</u> We try determine a $g \in L_p = L_p(C)$ in such a way that

$$h(z) = c - \frac{1}{\pi} \iint_C g(t) \left[\frac{1}{t-z} - \frac{1}{t}\right] d\sigma_t = c + P_0 g(z)$$

will become a solution of 2.6.1. This leds to the equation for g

2.7 $$g = A + Sg + Rg$$

with $$A = n \left[\frac{\nu_m}{z} c + \frac{\mu_m}{\bar z} \cdot \left(\frac{z}{\bar z}\right)^{-n} \cdot \bar c\right] , \quad Rg = n \left[\frac{\nu_m}{z} P_0 g + \frac{\mu_m}{\bar z} \left(\frac{z}{\bar z}\right)^{-n} \overline{P_0 g}\right]$$

$$Sg = \nu_m Tg + \mu_m \left(\frac{z}{\bar z}\right)^{-n} \cdot \overline{Tg} , \quad Tg(z) = -\frac{1}{\pi} \iint_C \frac{g(t)}{(t-z)^2} d\sigma_t$$

the two-dimensional Hilbert transformation.

Because 2.1 remains true if we let p decrease we may assume p sufficiently near to 2 such that the norm $C(p)$ of Γ in L_p satisfies

2.8 $kC(p) < 1$ (cf., e.g., [1], chapter V).

Then S is a contraction operator in L_p. The operator R is compact in L_p (cf., e.g., [4], section 0.4, and note that $\frac{\nu_m}{z}$, $\frac{\mu_m}{z}$ have bounded support and belong to L_p. Consequently, 2.7 has exactly one solution for each $A \in L_p$ if the corresponding homogenous equation $g = Sg + Rg$ has only the trivial solution $g \equiv 0$ in L_p, cf. [5], p. 176. The latter follows as in [5], p. 176f. :

Suppose $g \not\equiv 0$ is a solution of $g = Sg + Rg$ in L_p. This means that $h = P_0 g$ is a solution of 2.6.1. Therefore, according to the Bers-Nirenberg representation theorem (cf. [4], p. 46), $h = P_0 g$ may be represented by

2.9 $$h(z) = e^{s(z)} F(\chi(z)) ,$$

where $s(z)$ is bounded in C, F is an analytic function in C, and χ is a quasiconformal mapping of C onto itself. Because $g = 0$ for $|z| > m$, $P_0 g$ as well as F must be bounded in C, hence $F(\chi) \equiv$ const. Then $F \equiv 0$ follows from $h(0) = 0$, and this gives $g \equiv 0$ because of $(P_0 g)_{\bar{z}} = h_{\bar{z}} = g$. Since the homogeneous equation has only the trivial solution, the inhomogeneous equation 2.7 has a unique solution g for each $A \in L_p$, and this g is $= 0$ for $|z| > m$. Hence, $h(z) = c + P_0 g$ is indeed bounded in C. Because this h is a solution of 2.6.1 and has therefore a representation 2.9, the corresponding F is always $\equiv$ $\equiv$ const. This means that h is identically zero if and only if

$h(z)$ vanishes for any $z \in C$. The lemma is proved.

With this $h(z) = h_m(z)$ we obtain a (ν_m, μ_m) -solution $f_m(z) = h_m(z)z^n$ in $C\setminus\{0\}$. Because of the Hölder continuity of $P_0 g(z)$, especially at $z=0$, this $f_m(z)$ has at $z=0$ the asymptotic expansion

2.10 $$f_m(z) = cz^n(1 + O(|z|^{\alpha})) \text{ with a positive } \alpha > 1-\frac{2}{p},$$

where p has to satisfy only 2.1 and 2.8.

The function f_m is analytic for $m < |z| < \infty$ and has there a expansion (note that $P_0 g$ has a zero at ∞)

2.11 $$f_m(z) = cz^n + c_{n-1}^{(m)} z^{n-1} + \dots = cz^n(1 + O(|z|^{-1})) \quad .$$

By the representation theorem it holds $f_m(z) = F_m(\chi_m(z))$ with a certain Beltrami homeomorphism χ_m of C onto itself with $\chi_m(0) = 0$ and a function F_m analytic in $C\setminus\{0\}$. Together with ν, μ also ν_m, μ_m belong to $HL_p(\{0\})$ for each $m=1,2,\dots$, and $\nu_m(0) = \mu_m(0) = 0$. Hence, χ_m has at $z=0$ an asymptotic expansion $\chi_m(z) = d_m z + O(|z|^{1+\alpha})$ with a positive α (cf. [4], theorem III.5.2), and $d_m \neq 0$. By simple change of F_m we may assume $d_m = 1$, i.e.,

2.12 $$\chi_m(z) = z(1 + O(|z|^{\alpha})) \text{ with } \alpha > 0 \quad .$$

This χ_m is conformal for $|z| > m$, hence

2.13 $$\chi_m(z) = a^{(m)} z + a_0^{(m)} + \frac{a_{-1}^{(m)}}{z} + \dots \text{ with } a^{(m)} \neq 0 \quad .$$

Together with f_m also F_m has a point of order n at $z = 0$ and a point of order $-n$ at ∞. Then 2.10–2.13 give $F_m(\chi) = c\chi^n + O(|\chi|^{n+1})$ in a neighbourhood of $\chi = 0$, $F_m(\chi) = A^{(m)}\chi^n + O(|\chi|^{n-1})$ in a neighbourhood of ∞, hence $F_m(\chi) = c\chi^n$ for each $m=1,2,\ldots$, and this means

2.14 $$f_m(z) = c\left[\chi_m(z)\right]^n .$$

We need now a lemma on convergence.

2.15 Lemma. Let M be a set of quasiconformal mappings χ of C onto itself satisfying the following conditions: there exists a $k(z) \in HL_p(\{0\})$ with $p>2$, $k(0)=0$, $0 \leqslant k(z) \leqslant k = \text{const.} < 1$ for each $z \in C$, such that holds

2.15.1 $$|\chi_{\bar z}/\chi_z| \leqslant k(z) \quad \text{a.e. in } C \text{ for each } \chi \in M,$$

and each $\chi \in M$ has at $z = 0$ an asymptotic expansion

2.15.2 $$\chi(z) = z + O(|z|^{1+\alpha'}) \quad \text{with an } \alpha' > 0$$

(possibly depending on χ).

Then M is compact in the set Q of $\frac{1+k}{1-k}$ - quasiconformal mappings of C onto itself, there are two positive constants K and α, such that in $\{|z| < 1\}$ the inequality

2.15.3 $$\left|\frac{\chi(z)}{z} - 1\right| \leqslant K|z|^\alpha \quad \text{for each } \chi \in M$$

holds, and each limit χ^* of any convergent sequence $\chi_m \in M$

has an asymptotic expansion 2.15.2 at $z = 0$.

Proof. By well-known compactness criteria for quasiconformal mappings, M is compact in $\mathcal{C}$ if (additionally to $\chi(0) = 0$ and $\chi(C) = C$ for each $\chi \in M$) the set $M(1) = \{\chi(1) : \chi \in M\}$ is bounded away from the points 0 and ∞ . Consequently, the set of mappings $\psi = \chi/\chi(1)$ with $\chi \in M$ is compact in $\mathcal{C}$. By theorems II.5.2. II and II.5.47 of [4] there are two constants m_1, m_2 with $m_1 \leq |\psi_z(0)| \leq m_2$. But we have also cf. [4], II.5.22 $\psi_z(0) = 1/\chi(1)$, and this means that $M(1)$ cannot have the limit points 0 or ∞ . Furthemore, by the compactness just proved, the $\chi \in M$ are uniformly bounded, say for $|z| \leqslant 1$. The assertion 2.15.3 is then a consequence of the result in [4], II.5.22 . Finally, 2.15.3 implies the last statement of the lemma. The lemma is proved.

Of course, the χ_m mentioned in 2.12-2.14 satisfy the conditions of the last lemma (especially, $|(\chi_m)_{\bar z}/(\chi_m)_z| \leqslant \leqslant |\nu(z)| + |\mu(z)|$ a.e. in C).
Therefore, if $m \to \infty$, there is a subsequence of the χ_m which is convergent to a mapping χ of C onto itself with an expansion as in 2.12, and which yields simultaneously a (ν, μ) -solution $f(z) = c(\chi(z))^n$ in $C \setminus \{0\}$ (because this $f(z)$ is the limit of a subsequence of (ν_m, μ_m) -solutions $f_m = c\chi_m^n$, cf. [4], II.4.1). Obviously this f has a point of order n at zero and a point of order $-n$ at ∞ . Because of the asymptotic expansion of χ at $z = 0$, this f has at $z = 0$ the expansion

$$f(z) = cz^n + O(|z|^{n+\alpha}) \quad .$$

Therefore the existence of a $w(z)$ with the properties 2.2(I), (II) is proved under the additional assumptions $z_0 = 0$, $\vartheta(0) = = \mu(0) = 0$. By means of the transformations 1.7, 1.9, this implies the validity of 2.2(I), (II) for arbitrary ϑ, μ under the conditions 2.1, but without $\vartheta, \mu \in HL_p(\{\infty\})$.

To prove uniqueness let ϑ, μ now satisfy all of 2.1, and let us assume that there are two $w(z)$, $\overset{*}{w}(z)$ mentioned in 2.2(I), (II). By theorem 1.4 it follows that the (ϑ, μ)-solution $H(z) = w(z) - w^*(z)$ has a point of at least $(n+1)$th order at $z = 0$. On the other hand, $H(z)$ has at ∞ a point of a certain order, say of order $-j$, and by theorem 1.6 we have

$$K_1 \leqslant |H(z)\, z^{-j}| \leqslant K_2 \quad \text{in a neighbourhood of } \infty ,$$

where K_1, K_2 are certain positive constants. But $w(z)$ as well as $w^*(z)$ have a point of order $-n$ at ∞, and this means in conjunction with 1.6 that $H(z)\, z^{-n}$ is bounded in a neighbourhood of ∞. This implies $n \geqslant j$. By the representation theorem and because every rational function $\not\equiv 0$ takes each value $\in \overline{C}$ equally often, we arrive at a contradiction if $H \not\equiv 0$. Theorem 2.2 is proved.

As an obvious consequence of theorem 2.2 it follows that every (ϑ, μ)-solution has a unique representation, analogously to the Laurent expansion of analytic functions, at each of its poles.

3. Some Further Results on Generalized Powers. Surprisingly, generalized powers have a certain important property in common with the usual powers. This is stated in the following

3.1 Theorem. Let $\nu, \mu \in HL_p(\overline{C})$, $p>2$, and $|\nu|+|\mu| \leq k = \text{const.} < 1$ for each $z \in \overline{C}$, n, j be arbitrary integers. Then we have

$$\operatorname{Re} \frac{1}{2\pi i} \oint_{|z-z_0|=r} \left[a(z-z_0)^n\right]_{(\nu,\mu)} d\left[c(z-z_0)^j\right]_{(\nu,\mu)} =$$

$$= (1-|b|^2) j \delta_{n,-j} \operatorname{Re} ac$$

for each $r \in (0,\infty)$ with b as in 1.4(II), $\delta_{n,m}$ the Kronecker symbol (=1 if n=m, and =0 otherwise).

This theorem is obviously a generalization of the classical relation

$$\frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{dz}{(z-z_0)^m} = \delta_{1,m} .$$

The proof of 3.1 is anything but a brief matter and may be omitted here. As a consequence of 3.1 one obtains a generalized Cauchy integral formula for the derivatives of (ν,μ)-solutions.

3.2 Theorem. Let ν, μ be as in 3.1, and f be a (ν,μ)-solution in the domain $G \subset C$. With the constants ℓ, b corresponding to z_0, ν, μ according to 1.4(II), we have

$$\operatorname{Re} \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\, d\left[c(z-z_0)^{-1}\right]_{(\nu,\overline{\mu})} =$$

$$= -\frac{1-|b|^2}{1-|b\ell|} \operatorname{Re}\left\{f_z(z_0)(c-\overline{c}b\ell)\right\}$$

for each $z_0 \in G$ and any $r > 0$ such that $\{|z-z_0| \leq r\}$ is contained in G .

Note that, in view of the generalized Cauchy integral theorem (cf. [4], p. 66), the special shape of the contour of integration in 3.1, 3.2 is unessential. The proofs of 3.1, 3.2 are to be published elsewhere.

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STRESZCZENIE

Wprowadzono pojęcie n-tej uogólnionej potęgi (ν, μ) - rozwiązania (tzn. rozwiązania układu $f_{\bar{z}} = \nu f_z + \mu \bar{f}_{\bar{z}}$) i wykazano istnienie i jedyność takich potęg dla dowolnego, całkowitego n. Z topologicznego punktu widzenia potęgi te są w istocie zwykłymi superpozycjami quasikonforemnych homeomorfizmów płaszczyzny. Superpozycje te mają pewne własności zwykłych potęg, jeśli chodzi o pewne całki po konturze. Pociąga to za sobą, m.in. uogólniony wzór całkowy Canchy'ego dla pochodnych (ν, μ) - rozwiązań.

РЕЗЮМЕ

Введено понятие n -той обобщенной степени (ν, μ) -решения (т.е. решения системы $f_{\bar{z}} = \nu f_z + \mu \bar{f}_{\bar{z}}$) и доказано существование и единство таких степенных функций для любого целого n . С топологической точки зрения эти степени это обыкновенные суперпозиции квазиконформных отображений плоскости. Эти суперпозиции имеют несколько свойств обычных степеней, например по отношении к некоторым контурным интегралам. Это влечет за собой, нр. обобщенную интегральную формулу Коши для производных (ν, μ)-решений.