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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

J. G. KRZYŻ

Boundary Correspondence under Quasiconformal Mappings Revisited

Inne podejście do odpowiedniości brzegowej przy odwzorowaniach quasikonforemnych

Другой подход к соответствии границ для квазиконформных отображений

1. Introduction. Any quasiconformal (abbreviated : qc) automorphism H of a Jordan domain G admits a unique homeomorphic extension to its closure $\bar{G}$. In this way H generates a sense-preserving automorphism $h = H|\partial G$ of the boundary ∂G. The problem of boundary correspondence under qc mappings, i.e. the problem of characterizing generated automorphisms of ∂G can be reduced after a suitable conformal mapping to a corresponding problem for a standard Jordan domain, e.g. the unit disk D, or the upper half-plane U. In the latter case a very simple and elegant solution has been found by Beurling and Ahlfors [2].

Let us consider automorphisms of $R = \partial U$ determined by a continuous, strictly increasing function h which satisfies $h(-\infty) = -\infty$, $h(+\infty) = +\infty$. Then h can be extended to an automorphism of $\bar{U}$ which is K-qc in U, if and only if there exists a constant $M = M(K)$ such that

$$(1.1) \qquad M^{-1} \leqslant \frac{h(t+d) - h(t)}{h(t) - h(t-d)} \leqslant M \; ; \quad t,d \in R \; , \quad d \neq 0 \; .$$

Such automorphisms of R will be called functions M-quasisymmetric (abbreviated : M-qs) on R and the condition (1.1) will be referred to as the M-condition on R. The class of all

M-qs functions and its subclass consisting of functions h with the normalization $h(0) = 0$, $h(1) = 1$, will be denoted by $\mathcal{H}(M)$ and $\mathcal{H}_0(M)$, resp.

The necessity of the M-condition (see e.g. [1], [6]) was obtained by considering the modules of quadrilaterals arising from U by distinguishing four boundary points. The sufficiency has been established by an ingeneous construction, see ibid. There are, however, two one-parameter configurations involving points on the boundary, the quadrilateral and a punctured Jordan domain with two distinguished points on the boundary. Therefore we may expect that the boundary correspondence generated by qc automorphisms with one fixed interior point can be described in terms of harmonic measure - the characteristic conformal invariant of the latter configuration. In fact, we have obtained in [5] a counterpart of the M-condition for the unit disk punctured at its centre which is equally simple as (1.1).

Let $T = \partial \mathbb{D}$ and let $\tilde{h}$ be a sense-preserving automorphism of T. If β is an open subarc of T then $|\beta|$ will stand for the harmonic measure $\omega(0, \beta; \mathbb{D})$. With this notation we have

<u>Definition 1. A sense-preserving automorphism</u> $\tilde{h}$ <u>of</u> T <u>is said to be an</u> M-qs <u>automorphism of</u> T, <u>if and only if there exists a constant</u> M <u>such that for any pair</u> β_1, β_2 <u>of disjoint adjacent open subarcs of</u> T <u>with</u> $|\beta_1| = |\beta_2|$ <u>the following M-condition on</u> T

(1.2) $$|\tilde{h}(\beta_1)| / |\tilde{h}(\beta_2)| \leq M$$

<u>is satisfied</u>.

Let $\tilde{\mathcal{H}}(M)$ denote the class of all M-qs automorphisms of T. After suitable rotations any h in this class can be transformed into $\tilde{h}$ such that

(1.3) $\tilde{h}(1) = 1$, $\tilde{h}(-1) = -1$.

Let $\tilde{\mathcal{H}}_0(M)$ denote the subclass of $\tilde{\mathcal{H}}(M)$ consisting of all $\tilde{h} \in \tilde{\mathcal{H}}(M)$ normalized by (1.3). We have

Theorem A [5]. A sense-preserving automorphism h of T has a homeomorphic extension H on $\overline{\mathbb{D}}$ which is K-qc in $\mathbb{D}$ and satisfies $H(0) = 0$ if and only if there exists $M = M(K)$ such that $h \in \tilde{\mathcal{H}}(M)$.

Corollary 1. The condition $H(0) = 0$ is not essential for the existence of a qc extension, although the order K of qc-ity of H may possibly be lowered when this condition is omitted. E.g. the boundary correspondence generated by the Moebius transformation $z \mapsto (z - a)/(1 - \bar{a} z)$, $z \in T$, $0 < |a| < 1$, can be extended K-qcly on $\mathbb{D}$ with $H(0) = 0$, however, $K \geqslant K(a) > 1$. The same boundary correspondence obviously admits 1-qc extension which does not vanish at 0 .

Corollary 2. Given any qc automorphism H of $\mathbb{D}$, $H(0) \neq 0$, we can always find another qc automorphism H_0 of $\mathbb{D}$ with the same boundary values as H and such that $H_0(0) = 0$.

In fact, we may replace 0 , $\mathbb{D}$ by i and U , resp., using a suitable homography. Note that the sense-preserving affine mapping $l(w) = b w + c \bar{w}$, $b + c = 1$, $|b| > |c|$, satisfies $l(U) = U$ and keeps the points of ∂U unchanged. Given arbitary $w_1, w_2 \in U$, $w_1 \neq w_2$, we can choose $l(w)$ so that $l(w_1) = w_2$. Take now $w_2 = i$, $w_1 = H(i)$. Then $H_0 = l \circ H$ satisfies $H_0(i) = i$ and $H_0 | \mathbb{R} = H | \mathbb{R}$.

Definition 2. An automorphism h of the real line $\mathbb{R}$ which satisfies the conditions (1.1) and (1.4) :

(1.4) $h(x + a) = a + h(x)$, $a > 0$ is fixed ,

will be called a periodic M-qs function with period a .

This definition is justified by the fact that the function

(1.5) $\sigma(x) = h(x) - x$

is periodic with period a.

Let $\mathcal{H}(M, a)$ denote the class of all periodic M-qs functions with period a, normalized by the conditions

(1.6) $h(0) = 0 \quad , \quad h(a/2) = a/2$.

Since any $h \in \mathcal{H}(M,a)$ satisfies (1.1), it has a qc extension to $\overline{U}$, i.e. $h \in \mathcal{H}(M)$. Suppose now that $h_1 \in \mathcal{H}(M)$ and $h_1(0) = 0$. Then by Corollary 2 there exists a qc automorphism of $\overline{U}$ with boundary correspondence h_1 and the fixed point i. The mapping $w \mapsto (w - i)/(w + i) = z$ generates an automorphism of $\overline{D}$ with 0 as a fixed point and boundary values normalized as in (1.3). Now, the mapping $z \mapsto -i \log z$ with $\log 1 = 0$ induces a qc automorphism of $\overline{U}$ whose boundary values are in the class $\mathcal{H}_1(M, 2\pi)$. This mapping establishes obviously an 1 : 1 correspondence between the classes $\tilde{\mathcal{H}}_0(M)$ and $\mathcal{H}(M, 2\pi)$. As shown in [5] the order M of quasisymmetry remains unchanged. Using a similarity transformation we may obtain a periodic M-qs function with an arbitrary period a and normalization (1.6).

These remarks imply

Theorem B. Any qs function $h_1 \in \mathcal{H}(M)$, $h_1(0) = 0$, is associated with a periodic M-qs function $h \in \mathcal{H}(M, a)$, where $a > 0$ can be arbitrarily chosen.

Theorem B has some important aspects. The possibility of representing the boundary correspondence by periodic qs functions makes many problems much easier. E.g. we don't need consider separately the behaviour of functions at finite points and near infinity because of a more homogeneous structure due to the fact that no boundary point is distinguished. As pointed out in [5], any function h

with continuous and periodic $h(x) - x$ and the period a satisfies the M-condition on the whole $\mathbb{R}$, as soon as the double inequality in (1.1) holds for all $t \in [0; a]$ and $0 < |d| \leqslant a/2$. This makes verifying the M-symmetry much easier.

Let $\Sigma(M,a)$ stand for the class of all continuous, periodic functions σ with period a and normalization

(1.7) $\sigma(0) = 0 = \sigma(a/2)$

which satisfy the condition

$$(1.8) \qquad M^{-1} \leqslant \frac{1 + d^{-1}[\sigma(t+d) - \sigma(t)]}{1 + d^{-1}[\sigma(t) - \sigma(t-d)]} \leqslant M$$

for all $0 \leqslant t \leqslant a$, $0 < |d| \leqslant a/2$.

Then obviously $h(x) = x + \sigma(x)$ belongs to the class $\mathcal{H}(M,a)$ so that actually σ represents any boundary correspondence under qc mappings up to a translation. Taking $a = 2\pi$, we can make use of the Fourier series expansion for σ. This leads to interesting connections between the boundary correspondence under qc mappings and the harmonic analysis.

Since this paper is considered to be a preliminary report on research being done, we state in the next section without proof some results already established which have been presented during the Ninth Conference on Analytic Functions in Lublin, June 1 - 8, 1986.

2. Statement of results

The functions $\sigma \in \Sigma(M,2\pi)$ being continuous and of bounded variation over $[0;2\pi]$ can be represented by an everywhere convergent Fourier series. With any $\sigma \in \Sigma(M,2\pi)$ we may associate $\sigma_0 \in \Sigma(M,2\pi)$ such that

$\sigma_0(x) = \sigma(x) - \frac{1}{2\pi}\int_0^{2\pi} \sigma(t)\,dt$. Obviously

$$(2.1) \qquad \int_0^{2\pi} \sigma_0(x)\,dx = 0 .$$

This normalization seems to be most natural and we shall denote by $\mathcal{E}(M)$ the subclass of $\mathcal{E}(M,2\pi)$ of functions σ with the normalization (2.1). The functions $\sigma \in \mathcal{E}(M)$ have some nice properties from the point of view of the classical harmonic analysis. They are of monotonic type and of bounded variation. Moreover, they belong to the familiar class Λ_α (cf. [3]), where

$$(2.2) \qquad \alpha = \log_2(1 + \tfrac{1}{M}) .$$

In fact, any $\sigma \in \mathcal{E}(M,1)$ satisfies the uniform Hölder condition

$$(2.3) \qquad |\sigma(x+t) - \sigma(x)| \leqslant (1 + \tfrac{1}{M})|t|^{\alpha} , \quad x,t \in \mathbb{R} , \quad |t| \leqslant \tfrac{1}{2} .$$

The Fourier series of σ may be written in the form

$$(2.4) \qquad \sigma(x) = \sum_{n=1}^{\infty} \rho_n \sin(nx + x_n) , \quad \text{where } \rho_n = \rho_n(\sigma) \geqslant 0 .$$

If we put for short $m = \pi(M-1)/(M+1)$, then the series $\sum \rho_n$ is convergent and we have the following estimate of its sum :

$$(2.5) \qquad \sum_{n=1}^{\infty} \rho_n \leqslant m + \sqrt{m}\left[M + \sqrt{M(M+1)}\right] := \rho(M) .$$

The Beurling - Ahlfors construction of a qc extension of $x+\sigma(x)$ onto the upper half-plane (cf. [1]) has the following form :

$$u(x,y) = x + \sum_{n=1}^{\infty} \rho_n \sin(nx + x_n) \frac{\sin ny}{ny} ,$$

$$v(x,y) = \frac{1}{2}y + \sum_{n=1}^{\infty} \rho_n \cos(nx + x_n) \frac{1 - \cos ny}{ny} .$$

The convergence- and sum-preserving factors $\frac{\sin ny}{ny}$ correspond to the Lebesgue method of summability.

We have the following estimate for the Fourier coefficients ρ_n :

$$(2.6) \qquad n\,\rho_n(\sigma) \leqslant \sup\left\{\rho_1(\sigma) : \sigma \in \mathcal{E}(M)\right\} := C(M) ,$$

the bound being sharp for any $n \in \mathbb{N}$.

The following estimates could be obtained for $C(M)$:

$\frac{m}{\pi} \leqslant C(M) \leqslant m$ and also a more precise one :

$$\frac{2m}{\pi}\left[1 + \frac{2\sqrt{M}}{1+M}\right]^{-1} \leqslant C(M) \leqslant 2\left[1 - \exp(-\rho(M))\right] .$$

It follows from (2.5) and (2.6) that the harmonic extension of $u(e^{ix}) = \sigma(x)$ has a finite Dirichlet integral $D[u] \leqslant \pi m\, \rho(M)$.

Various norm estimates were helpful in obtaining the above mentioned results.

(i) If $\sigma \in \mathcal{E}(M)$, then the total variation of σ over $[0;2\pi]$ satisfies : $V[\sigma] \leqslant 4m$;
hence $\sup\left\{|\sigma(x)| : x \in \mathbb{R}\right\} \leqslant m$;

(ii) $\int_0^{2\pi} |\sigma(x)|\,dx \leqslant \pi m$; $\int_0^{2\pi} |\sigma(x)|^2 dx \leqslant \pi m^2$.

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STRESZCZENIE

Z wykazanego przez autora we wcześniejszej pracy i cytowanego jako Twierdzenie A wyniku wypływa następujący wniosek: odpowiedniość brzegową dla dowolnego quasikonforemnego automorfizmu obszaru Jordanowskiego można scharakteryzować przez funkcję quasisymetryczną $x \mapsto x + \sigma(x)$, gdzie σ jest funkcją ciągłą i okresowe o okresie 2π . Podano szereg twierdzeń dotyczących funkcji σ .

РЕЗЮМЕ

Из полученного автором в раньшей работе результата, представленного здесь как Теорема А следует, что соответствие границ полученного из любого квазиконформного автоморфизма ердановой области может быть характеризовано квазисимметрической функцией $x \mapsto x + \sigma(x)$, где σ непрерывна и периодическа с периодом 2π. Получено несколько теорем относящихся к функции σ.