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# Distortion Problem for Bounded Convex Functions Normalized by Montel's Conditions

Problem zniekształcenia dla funkcji wypukłych ograniczonych unormowanych przez warunki Montela

Проблема искажения для выпуклых ограниченных функций нормированных по Монтелю

1. Let $C(M,z_0)$ denote the class of functions $f(z)$ regular and univalent in the unit disk $K = \{z : |z| < 1\}$ normalized by conditions $f(0) = 0$, $f(z_0) = z_0$, where $z_0 \neq 0$ is a fixed point of the disk $K$ (we can assume that $z_0 > 0$) mapping the disk $K$ onto a convex domain $f(K) = G(f)$ contained in $K(M) = \{w : |w| < M\}$, $M > 1$. The boundary of $G(f)$ is a simple closed and convex Jordan curve $\Gamma(f)$ having one-sided tangents everywhere. In addition the set of points with different one-sided tangents is at most enumerable.

In this paper is considered the problem of determining estimations of $|f(z)|$, $(1-|z|^2)|f'(z)|$, $|a_1| = |f'(0)|$ for functions of the class $C(M,z_0)$. The form of the domain onto which the extremal function maps the disk $K$ is defined by using variational methods and the general form of such functions is given. The idea of the proof is based on a method given by J. Krzyż in [1], where the solution is given for the analogous problem in the class $C(M)$ of bounded convex functions with $f(0) = 0$, $|f'(0)| = 1$.

2. Let $\eta \in K(M)$, $\eta \neq z_0$ be fixed and let $U$ denote the class of closed convex domains containing the points $0$, $z_0$ and $\eta$ included in the disk $K(M)$ and such that $g(0,z_0,G) = -\log z_0$, where $g(0,z_0,G)$ is the classical Green's function of the domain $G$ with the pole $0$. We may confine our consideration to the classical Green's function, since the boundary of the domain $G$ is a Jordan curve. It is known that for $G \in U$ there exists a function $f \in C(M,z_0)$ such that $f(re^{i\theta}) = \eta$ and $f(K) = G$ and conversely if $f \in C(M,z_0)$, $f(re^{i\theta}) = \eta$ and $f(K) = G$ then $G \in U$. This implies that if $\eta \in G$ and $\eta$ is fixed then $\eta = f(re^{i\theta})$, $f \in C(M,z_0)$ and we have:

(2.1) $$g(0,\eta,G) = -\log r \quad .$$

It is noticeable that the function $f_0$ satisfying $\sup_{f,\theta} |f(r_0e^{i\theta})| = |\eta|$ where $0 < r_0 < 1$ fixed, $0 \leq \theta \leq 2\pi$, $f \in C(M,z_0)$ attains the value in such a point $z \in K$ that the modulus $|z|$ is the least possible. It is easy to see that the problem of finding $f_0$ is equivalent to that of finding the domain $G_0 \in U$ for which:

(2.2) $$\sup g(0,\eta,G) = g(0,\eta G_0) \quad , \quad G \in U \ .$$

Similarly, the problem of determining the function $f_1 \in G(M,z_0)$, attaining a fixed value $\eta$, $|\eta| < \delta(M,z_0)$ ( $\delta(M,z_0)$ denotes the Koebe constant for the class $C(M,z_0)$ ) at $z$ with the smallest possible modulus is equivalent to that of finding the domain $G_1$, $G_1 \in U$, which satisfies the following condition:

$$(2.3) \qquad \inf g(0,\eta,G) = g(0,\eta,G_1) \ , \qquad G \in U \ .$$

The assumption $|\eta| < \delta(M,z_0)$ is essential, since when $|\eta| \geqslant \delta(M,z_0)$ then the infimum (2.3) is equal to zero.

The extremal domains $G_0$ and $G_1$ will be obtained by using Hadamards formula [2] for the variations of Green's function.

Suppose that the function $z = h(w)$ maps conformally the domain $G$ $U$, with the boundary $\Gamma$ consisting of a finite number of analytic arcs onto the unit disk $K$ so that $h(0) = 0$. Then Green's function is:

$$(2.4) \qquad g(w,\eta,G) = \log \left| \frac{1 - h(w)\overline{h(\eta)}}{h(w) - h(\eta)} \right| \ ,$$

and Hadamard's formula can be written in the following form [1]:

$$(2.5) \qquad \delta g(0,\eta,G) = \frac{1}{2\pi} \int_{\Gamma} |h'(w)|^2 \frac{1-|h(\eta)|^2}{|h(w)-h(\eta)|^2} \delta n(s)\, ds \ ,$$

where $\delta n(s) = \varepsilon p(s)$ is the normal displacement, which is to be taken positive, if the displacement vector coincides with the outward pointing normal, and negative, if it has the opposite direction. Furthermore, $p(s)$ is a piecewise continuous function of the arc length $s$ on $\Gamma$.

Notice that:

$$g(0,z_0,G) = \frac{1}{2\pi}\int_{\Gamma} |h'(w)|^2 \frac{1-|h(z_0)|^2}{|h(w)-h(z_0)|^2}\,\delta n(s)\,ds . \tag{2.6}$$

Function

$$H(w) = \frac{H_\eta(w)}{H_{z_0}(w)} , \tag{2.7}$$

where $H_\eta(w) = \dfrac{1-|h(\eta)|^2}{|h(w)-h(\eta)|^2}$ and $H_{z_0}(w) = \dfrac{1-|h(z_0)|^2}{|h(w)-h(z_0)|^2}$, varies in a certain monotonic manner for fixed $\eta$ and $z_0$ and for $w$ moving on $\Gamma$. It gives the following:

Lemma 1. If the boundary $\Gamma$ of the domain $G$ is a Jordan curve and the points $A$, $B$, $C$ divide $L$ into three arcs not reducing to points, then among these arcs there exist two arcs $L_1$ and $L_2$ such that for any arcs $l_1$ and $l_2$, $l_1 \subset L_1$ and $l_2 \subset L_2$ the following inequality holds:

$$\max_{w \in l_1} H(w) \leqslant \min_{w \in l_2} H(w) . \tag{2.8}$$

The proof is similar to the proof of the lemma 3.1 in [1]. If the domain $G_0$ satisfies (2.2) i. e. it gives the maximal value for Green's function, then there exists a constant $\lambda$ such that for the admissible variation $G_0$ giving the domains from the considered class the condition:

$$(2.9) \qquad g(0,\eta,G_0) + \lambda\,\delta g(0,z_0,G_0) = 0$$

must be fulfilled.

Hence, when the variation of the domain $G$ is admissible, $\delta g(0,z_0,G) = 0$ and $\delta g(0,\eta,G) > 0$ then the domain $G$ cannot give the maximum of Green's function.

3. Theorem I. The boundary of the domain $G_0$ satisfying condition (2.2) consists of the one arc of the circumference $|w| = M$ and one straight line segment connecting its endpoints.

Proof. We first note that the domain $G_0$ with respect to its normalization is not identical with the whole disk $K(M)$, $M > 1$. Let $U_n$ denote the class of the closed, convex polygons $G$ with at most $n$ vertices containing points $0$, $z_0$ and fixed $\eta$, included in the disk $K(M)$ and such that $g(0,z_0,G) = -\log z_0$. For each polygon we can find such a function $f \in C(M,z_0)$ that $f(K) = G$. Hence $U_n$ contains the extremal domain $G_n$ for which:

$$(3.1) \qquad g(0,\eta,G_n) = \sup g(0,\eta,G) \quad , \quad G \in U_n .$$

If $G_0$ is the extremal domain satisfying (2.2) then there exists subsequence $\{G_{n_k}\}$ which converges to $G_0$ in the sense of nucleus convergence. The domains $G_n$ and $G_0$ we determine by eliminating from the classes $U_n$ and $U$ those domains which cannot be extremal i. e. by admissible variation the condition (2.9) is not satisfied. We define the boundary variation at the same way as in [1]. In each individual case of the domain eliminating we take the angles of rotation to satisfy the following condition:

$$(3.2) \qquad \int_{l_1} |h'(w)|^2 \frac{1-|h(z_0)|^2}{|h(w)-h(z_0)|^2}\, p(s)\, ds = $$

$$= \int_{l_2} |h'(w)|^2 \frac{1-|h(z_0)|^2}{|h(w)-h(z_0)|^2}\, (-p(s))\, ds$$

where $l_1$ and $l_2$ are the arcs of the curve such that:

$$\max_{w \in l_2} H(w) < \min_{w \in l_1} H(w) .$$

The existence of such arcs follows from lemma 1.
The above inequality implies:

$$(3.3) \qquad \int_{l_2} |h'(w)|^2 H_\eta(w)\,(-\delta n(s))ds < \int_{l_1} |h'(w)|^2 H_\eta(w)\,\delta n(s)ds ,$$

and it means that $\delta g(0,z_0,G) = 0$ while $\delta g(0,\eta,G) > 0$ . Therefore condition (2.9) is not satisfied, so the domain $G$ does not maximize Green's function. In order to present the eay of the elimination the domains $G \in U$ which do not satisfy condition (2.9) we prove:

<u>Lemma 2. If the polygon $G_n$ satisfies the condition (3.1), then at most one vertex with the angle less than $\pi$ lies inside $K(M)$ and all the others lie on the circumference $\{w : |w| = M\}$ .</u>

<u>Proof.</u> Suppose that, contrary to this, $A$ and $B$ are vertices of $G_n$ with angles less than $\pi$ which lie inside $K(M)$ . Let $C$ be an arbitrary vertex of $G_n$ different from $A$ and $B$ .

The points A , B , C divide $\Gamma_n$ into three parts and in view of lemma 1 there exists segments $l_1$ and $l_2$ each having A or B as one of its endpoints such that (2.8) holds. We now turn $l_1$ outwards and $l_2$ inwards about their endpoints by moving A or B (see [1] p. 12) and the angles of rotations are chosen so that (3.2) holds. Such a treatment leads to domains within the class $U_n$, and does not change Green's function in the point $z_0$ while it increases in the point $\eta$. We see that Green's function cannot attain a maximum at $\eta$ for such a domain. In other cases the way of treatment is similar as in [1].

We can show likewise by putting $p_1(s) = -p(s)$ that $G_1$ satisfying (2.3) has the same shape as $G_0$ i. e. the boundary of the domain $G_1$ consists of the one arc of circumference $|w| = M$ and the segment connecting its endpoints.

The function $f_0(z)$ maps conformally the disk K onto the domain of this type such that $f_0(0) = 0$ has representation:

$$f_0(z,M) = e^{i\psi}\frac{M}{i}\,\frac{e^{-i\theta}H(z) - e^{i\theta}}{1 + H(z)}$$

(3.4) $$H(z) = e^{2i\theta}\left(\frac{1 - e^{i(\gamma-\alpha)}z}{1 - e^{i(\gamma+\alpha)}z}\right)^{\frac{2\theta}{\alpha}},$$

$$\alpha = \frac{4\pi\theta}{\pi + 2\theta}, \quad 0 < \theta < \frac{\pi}{2}, \ -\pi \leqslant \varphi \leqslant \pi, \ -\pi \leqslant \psi \leqslant \pi.$$

The function of the form (3.4) belongs to the class $C(M, z_0)$ if the following condition holds:

(3.5) $$f_0(z_0, M) = z_0 \quad .$$

Alas because of calculation difficulties we have not discovered the final form of the function which has the type $f_0(z,M)$, giving estimation for $|f|$ in $C(M,z_0)$ where $|z|$ is fixed.

4. Let $f \in C(M,z_0)$, $\eta = f(z)$ and $G = f(K)$, then $r(\eta,G) = (1-|z|^2)|f'(z)|$. It is known that $\gamma(\eta,G) = \log r(\eta,G)$, where $\gamma(\eta,G)$ denotes the Robin's constant and $r(\eta,G)$ conformal inner radius in $\eta$ of the domain $G$. We can write Hadamard's formula for the Robin's constant in the form:

$$\delta\gamma(\eta,G) = \frac{1}{2\pi}\int_{\Gamma} |h'(w)|^2 H_\eta^2(w)\,\delta n(s)\,ds \quad . \tag{4.1}$$

The function $H_\eta^2(w)$ is monotonic as $H_\eta(w)$ is. Hence, in an analogical manner as in theorem 1 we obtain estimation for the conformal inner radius $r(\eta,G)$. Maximum and also minimum (for $|\eta|<\delta(M,z_0)$ ) of the value $r(\eta,G)$, $G \in U$ are attained for the domain of the type $G_0$.

We can also note that:

$$r(0,G) = |f'(0)| = |a_1(f)| \quad ,$$

where $f \in C(M,z_0)$, $f(z) = a_1(f)z + a_2(f)z^2 + \dots$ .
Hence by the continuity:

$$|f'(0)| \leqslant |f_0'(0)| \quad , \quad f \in C(M,z_0)$$

and $f_0(z,M)$ is defined by (3.4) and satisfies (3.5).

REFERENCES

[1] Krzyż, J., Distortion theorems for bounded convex functions II, Ann. Univ. Mariae Curie-Skłodowska, Sect.A, 14 (1960), 7-18.

[2] Nehari, Z., Conformal Mapping, New York-Toronto-London 1952.

STRESZCZENIE

Niech $C(M,z_0)$ będzie klasą funkcji $f$ jednolistnych i wypukłych w kole jednostkowym $K$, takich, że $f(0) = 0$, $f(z_0) = z_0$, $0 \neq z_0 \in K$, $|f(z)| < M$ dla $z \in K$. W pracy otrzymano za pomocą metody wariacyjnej podanej przez J. Krzyża dokładne oszacowanie w klasie $C(M, z_0)$ funkcjonałów $|f(z)|$, $(1 - |z|^2)|f'(z)|$, $|f'(0)|$.

РЕЗЮМЕ

Пусть $C(M, z_0)$ класс выпуклых и однолистных в единичном круге $K$ функций $f$, таких что $f(0) = 0$, $f(z_0) = z_0$, $0 \neq z_0 \in K$, $|f(z)| < M$ для $z \in K$. В этой работе получены вариационным методом данным Я.Крижиком, точные оценки в классе $C(M, z_0)$ для функционалов $|f(z)|$, $(1-|z|^2)|f'(z)|$, $|f'(0)|$.