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# Univalent Logharmonic Mappings

Odwzorowania jednolistne log-harmoniczne

Однолистные лог-гармонические отображения

**1. Introduction.** This presentation is essentially a brief survey on univalent orientation-preserving mappings $f$ defined on the unit disc $U \subset C$ whose image is in $C$, and which are of the form

$$f(z) = z|z|^{2\beta} h(z)\, \overline{g(z)} \tag{1.1}$$

where

(1.1.a) $\operatorname{Re}\{\beta\} > -1/2$
(1.1.b) $h$ and $g$ are nonvanishing analytic functions on $U$
(1.1.c) $g(0) = 1$ .

We shall call such mappings to be univalent logharmonic on $U$ vanishing at the origin. They can be characterized as univalent solutions of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar z}(z)} = a(z) \left[\overline{f(z)}/f(z)\right] f_z(z) \quad ; \quad f(0) = 0 \tag{1.2}$$

where $a(z)$ belongs to the class $B$ of all analytic functions on $U$ having the property that $|a(z)| < 1$ for all $z \in U$. Therefore a univalent logharmonic mapping on $U$ is locally quasiconformal; however the dilatation $K(z) = (1+|a(z)|)/(1-|a(z)|)$ may go to infinity as $z$ approaches $\partial U$. The exponent $\beta$ in (1.1.a) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)}\,(1+a(0))/(1-|a(0)|^2) \quad . \tag{1.3}$$

Note that all univalent conformal mappings on $U$ are logharmonic ($a \equiv 0$). The composition of a conformal premapping with a logharmonic mapping is again logharmonic. However the composition of a logharmonic mapping with a conformal postmapping is in general not logharmonic. In particular translations $f - w_0$ of the image of a logharmonic mapping $f$ are in general not logharmonic. Furthemore, the inverse $f^{-1}$ of a univalent logharmonic mapping $f$ does not inherit the property of logharmonicity.

Let $f = z|z|^{2\beta} h \cdot \overline{g}$ be a univalent logharmonic mapping on $U$ such that $f(0) = 0$. Then $F(\zeta) = \log f(e^{\zeta})$ is a univalent harmonic mapping on $\{\zeta : \operatorname{Im}(\zeta) < 0\}$. Such mappings are closely related to the theory of minimal surfaces and have been studied by several authors.

## 2. Mapping Problems.

2.a) Let $\Omega$ be a simply connected domain of $C$ which contains the origin. Given $a \in B$, is there a univalent solution $f$ of (1.2) such that $f(U) = \Omega$ and $f(0) = 0$. Unfortunatly the answer is no. In particular there is no logharmonic univalent mapping from $U$ onto $C \setminus (\infty, -d]$, $d > 0$, with $a(z) = -z$

(see [2]). But there is a weaker form of the Riemann Mapping Theorem.

Theorem 2.1. [2]. Let $\Omega$ be a bounded simply connected domain of $\mathbb{C}$ which contains the origin and whose boundary $\partial\Omega$ is locally connected (i.e. each prime end is a singleton). Then there is for each $a \in B$ a univalent solution $f$ of 1,2 such that

2.1.1) $f(U) \subset \Omega$ .

2.1.2) $f(z) = c\, z|z|^{2\beta}(1 + o(1))$ if $z \longrightarrow 0$ and $c > 0$ , ( $\beta$ as in (1.3)).

2.1.3) $\lim\limits_{z \to e^{it}} f(z) = \hat{f}(e^{it})$ exists and is in $\partial\Omega$ on $\partial U \setminus E$ ; $E$ is countable.

2.1.4) For each $e^{it_0} \in \partial U$ , we have that

$$f_*(e^{it_0}) = \operatorname{ess}\lim_{t \uparrow t_0} \hat{f}(e^{it}) \quad \text{and} \quad f^*(e^{it_0}) = \operatorname{ess}\lim_{t \downarrow t_0} \hat{f}(e^{it})$$

exists and are in $\partial\Omega$ .

2.1.5) For $e^{it_0} \in E$ , the cluster set of $f$ at $e^{it_0}$ lies on a helix joining the point $f^*(e^{it_0})$ to the point $f_*(e^{it_0})$

Remarks.

1) If $|a(z)| \leq k < 1$ for all $z \in U$ , then $f(U) = \Omega$ .

2) If $e^{it_0} \in E$ and $f^*(e^{it_0}) = f_*(e^{it_0})$ then the cluster set of $f$ at $e^{it_0}$ is the circle $|w| = |f^*(e^{it_0})|$ .

3) If $e^{it_0} \in E$ and $A_1 = f_*(e^{it_0}) \neq f^*(e^{it_0}) = A_2$ , then there are infinitely many helices from $A_1$ to $A_2$ . The claim 2.1.5) states that the cluster set lies on one of them. Thus, for example, the cluster set of

$$f(z) = z\left[(1-\bar{z})/(1-z)\right]\cdot \exp\left\{-2\arg\left[(1-iz)/(1-z)\right]\right\}$$

at $z = 1$ lies on the helix, $\gamma(\tau) = \exp(-\tau + i(\pi/2+\tau))$ joining the points $f^*(1) = -e^{-\pi/2}$ and $f_*(1) = -e^{3\pi/2}$, whereas the cluster set of $f$ at $z = -i$ is the straight line segment from $f_*(-i) = -e^{-\pi/2}$ to $f^*(-i) = -e^{3\pi/2}$.

4) If $\Omega$ is strictly starlike then $f$ is uniquely determined.

Outline of the proof.

a) Without loss of generality we may assume that $a(0) = 0$. Indeed, if not, then consider the domain $\tilde{\Omega} = \left\{ w|w|^{2\gamma} ; w \in \Omega \right\}$, where $\tilde{a}(z) = \left[(1+\overline{a(0)})(a(z)-a(0))\right] / \left[(1+a(0))(1-\overline{a(0)}a(z))\right]$. If $\tilde{f}$ is the desired mapping for $\tilde{\Omega}$ and $\tilde{a}$ then $f = \tilde{f}|\tilde{f}|^{2\delta}$, $\delta = \overline{a(0)}(1+a(0)) / (1-|a(0)|^2)$ satisfies Theorem 2.1.

b) Let $\Phi$ be the conformal mapping from $U$ onto $\Omega$ normalized by $\Phi(0) = 0$, $\Phi'(0) > 0$. Put $r_n = (1-1/n)$ and $\Omega_n = \Phi(|z| < r_n)$. Then there is a mapping $f_n$ from $U$ onto $\Omega_n$ satisfying Theorem 2.1 with respect to $a_n(z) = a(r_n z)$ (see [2]). Since $\operatorname{dist}(0, \partial\Omega_1) \leq (f_n)_z(0) \leq 16 \operatorname{dist}(0, \partial\Omega)$, there is a subsequence of $f_n$ which converges locally uniformlly to a univalent solution $f$ of (1.2). Finally, the Poisson integral applied to $\log f/z$ gives the required properties.

2.b) Let $D$ be an arbitrary domain of $\overline{C}$ which contains infinity. We are interested in conditions such that $D$ can be mapped by univalent logharmonic mappings $f$, $f(\infty) = \infty$ onto a canonical domains.

Theorem 2.2. [1]. Let $D$ be a domain of arbitrary connectivity containing the point at infinity. Fix $z_0 \in D$ and let $a$ be in $H(D)$, $|a(z)| < 1$ for all $z \in D$, $a(z_0) = m_0/(1+m_0)$ and $a(\infty) = m/(1+m)$ where $m_0$ and $m$ are nonnegative integers. Denote by $\Phi$ a conformal mapping of $D$ onto a radial slit domain normalized by $\Phi(z_0) = 0$ and $\Phi(z) = z + O(1)$ as $z \to \infty$. If

$$(2.1) \qquad \operatorname{Re}\left\{\mu_1(z)dz\right\} = \operatorname{Re}\left\{\left[\frac{1+a(z)}{1-a(z)}\,\frac{\Phi'(z)}{\Phi(z)} - \frac{1+a(z_0)}{1-a(z_0)}\,\frac{1}{z-z_0}\right]dz\right\}$$

defines an exact differential on $D \setminus \{\infty\}$, then there is a univalent logharmonic function which maps $D$ onto a radial slit domain and is normalized by

$$(2.2) \qquad f(z_0) = 0 \quad \text{and} \quad f(z) = z|z|^{2m}(1 + o(1)) \quad \text{as} \quad z \to \infty \; .$$

Furthemore, if $D$ has finitely many boundary components, then $f$ is uniquely determined.

Remark. If $D$ is simply connected then the condition on (2.1) is not active since it is satisfied whenever $a(\infty)$ is real.

Theorem 2.3. [1]. Let $D$ be as in Theorem 2.2 with $a(\infty) = a(z_0) = m/(1+m)$, $m \in N \cup \{0\}$, and let $\psi$ be a conformal mapping of $D$ onto circular slit domain normalized by

$\psi(z_0) = 0$ <u>and</u> $\psi(z) = z + O(1)$ <u>as</u> $z \longrightarrow \infty$ . <u>If</u>

$$(2.3) \qquad \operatorname{Im}\left\{\mu_2(z)dz\right\} = \operatorname{Im}\left\{\left[\frac{1-a(z)}{1+a(z)}\,\frac{\psi'(z)}{\psi(z)} - \frac{1-a(z_0)}{1+a(z_0)} - \frac{1}{z-z_0}\right]dz\right\}$$

<u>defines an exact differential on</u> $D\setminus\{\infty\}$ , <u>then there is a univalent logharmonic function that maps</u> $D$ <u>onto a circular slit domain and is normalized by</u>

$$(2.4) \qquad f(z_0) = 0 \quad \underline{\text{and}} \quad f(z) = z|z|^{2m}(1 + o(1)) \quad \underline{\text{as}} \quad z \longrightarrow \infty \ .$$

<u>Furthermore, if</u> $D$ <u>has finitely many boundary components, then</u> $f$ <u>is uniquely determined.</u>

<u>Remark.</u> If $D$ is simply connected, then the condition on (2.3) is not active since it is satisfied, whenever $a(\infty) = a(z_0) \in \mathbb{R}$ .

<u>3. Univalent starlike logharmonic mappings.</u> Let $\Omega$ be a simply connected domain of $\mathbb{C}$ which contains the origin. We say that $\Omega$ is $\alpha$-spirallike, $-\pi/2 < \alpha < \pi/2$ , if $w_0 \in \Omega$ implies that $w_0\exp(-te^{i\alpha}) \in \Omega$ for all $t \geq 0$ . If $\alpha = 0$ , the domain is called starlike (w. r. to the origin). Let $S^{\alpha}_{Lh}$ be the set of all univalent logharmonic mappings $f$ on $U$ such that $f(0) = 0$ , $g(0) = h(0) = 1$ , and $f(U)$ is $\alpha$-spirallike domain,

$$S^{*}_{Lh} = S^{\alpha=0}_{Lh} \ ,$$

$$S^{\alpha} = \left\{f \in S^{\alpha}_{Lh} \cap H(U)\right\} \ , \quad \text{and} \quad S^{*} = \left\{f \in S^{*}_{Lh} \cap H(U)\right\} \ .$$

Whenever we use the representation $f(z) = z|z|^{2\beta} h(z)\overline{g(z)}$ for a univalent logharmonic mapping on $U$ we mean that $h$ and $g$ are nonvanishing analytic functions on $U$ normalized by $g(0) = 1$.

For each $f = z|z|^{2\beta} h\cdot\overline{g} \in S^*_{Lh}$, we associate the function $\psi(z) = zh(z)/g(z) \in H(U)$. The first result is:

Theorem 3.1. [4].

a) If $f = z|z|^{2\beta} h\overline{g} \in S^*_{Lh}$, then $\Phi = zh/g \in S^*$.

b) Conversly, if $\Phi \in S^*$ and $a \in B$, then there is a unique couple $(h,g)$ of nonvanishing analytic functions on $U$ such that $\Phi = zh/g$ and $f = z|z|^{2\beta} h\overline{g}$ is a univalent solution of (1.2) in $S^*_{Lh}$.

Outline of the proof.

a) Let $f \in S^*_{Lh}$ and $\gamma = -\beta/(1+\beta+\overline{\beta})$. Then $\tilde{f} = f|f|^{2\gamma} \in S^{\alpha}_{Lh}$ where $\alpha = -\arg(1+2\gamma) \in (-\pi/2, \pi/2)$. The corresponding dilatation function $\tilde{a}$ vanishes at the origin and therefore $\tilde{f} = z\tilde{h}\cdot\overline{\tilde{g}}$, $\tilde{h}(0) = \tilde{g}(0) = 1$. Put $\psi(z) = z\tilde{h}(z)\left[\tilde{g}(z)\right]^{-e^{2i\alpha}}$. Then $\psi$ is in $S^{\alpha}$. Finally $\Phi = z\left[\tilde{h}/(\tilde{g})^{-e^{2i\alpha}}\right]^{e^{-i\alpha}/\cos(\alpha)} = zh/g \in S^*$.

b) Let $\Phi \in S^*$ and $a \in B$ be given. Put

$$g(z) = \exp\int_0^z \frac{sa(s)\Phi'(s) + a(s)\beta\Phi(s) - \overline{\beta}\Phi(s)}{s\Phi(s)(1-a(s))}\, ds ,$$

$h(z) = \Phi(z)g(z)/z$ and $f = z|z|^{2\beta}\cdot h\overline{g}$, where $\beta$ is as in (1.3). Then $h$ and $g$ are nonvanishing and analytic on $U$,

$h(0) = g(0) = 1$ , and $f$ is a solution of (1.2). Following backforwards the first part of the proof one concludes that $f$ is the desired solution.

Remark. There is no similar result for the family of convex mappings. Indeed, $\psi(z) = z$ is a convex mapping, $a(z) = z^4 \in B$ , but $f(z) = z/|1-z^4|^{1/2}$ is not a convex mapping.

An immediate consequence of Theorem 3.1 is:

Corollary 3.2. [4]. *If $f \in S^*_{Lh}$ then $f(rz)/r \in S^*_{Lh}$ for all $r \in (0,1)$ .*

However, Corollary 3.2 may fail whenever $f(0) \neq 0$ . Indeed, for each $z_0 \in U\setminus\{0\}$ , one can give an example of univalent logharmonic mapping $f$ such that $f(z_0) = 0$ , and $f(U)$ is starlike but no level set $f(|z|<r)$ , $|z_0|<r<1$ , is a starlike domain (see [4]).

For the first application let us consider the problem

$$\operatorname{Min} \int_{f(U)} |w|^p \, dudv \ ; \quad p \geqslant 0 \text{ given},$$

over all solutions of (1.2) whose function $a(z)$ vanishes at the origin and for which $f_z(0) = 1$ . The optimal solution is

$$f(z) = z\, \overline{(1+(p+2)z/(p+4))}^{(p+4)/(p+2)} / (1+(p+2)z/(p+4))$$

which by Theorem 3.1 is starlike univalent. In the case of the minimal area problem $(p=0)$ the extremal function is not a convex mapping.

Another consequence of Theorem 3.1 is the following integral representation for mappings in $S^*_{Lh}$ :

$$(3.1) \qquad S^*_{Lh} = \left\{ \int\int_{\partial U \times \partial U} K(z, \eta, \xi)\, d\mu(\eta) d\nu(\xi) \right\}_{\mu,\nu}$$

where $K$ is a fixed kernel function and where $\mu$ and $\nu$ are probability measures on the Borel $\sigma$-algebra of $\partial U$ . Even if $S^*_{Lh}$ is not compact the relation (3.1) may be used for optimization problems over subclasses of starlike univalent logharmonic mappings $f = z|z|^{2\beta} h \cdot \overline{g}$ having fixed exponent $\beta$ . In particular, one gets for $a(0) = 0$ and $f \in S^*_{Lh}$

$$r \exp\{-4r/(1+r)\} \leqslant |f(z)| \leqslant r \exp\{4r/(1-r)\} \quad .$$

The inequalities are sharp [3].

4. Automorphisms of logharmonic mappings. In this section we are concerned with univalent logharmonic mappings from $U$ onto $U$ . With no loss of generality we shall assume that $f(0) = 0$ and $h(0) > 0$ . Otherwise, we consider an appropriate Moebius transformation of the preimage. Let $AUT_{Lh}(U)$ denote the class of such mappings.

The first theorem characterizes completely mappings in $AUT_{Lh}(U)$ .

Theorem 4.1. [4]. Let $h$ and $g$ be two nonvanishing analytic functions on $U$ . Then $f(z) = z|z|^{2\beta} h(z) \cdot \overline{g(z)}$ is in $AUT_{Lh}(U)$ satisfying $h(0) > 0$ and $g(0) = 1$ if and only if

$g = 1/h$ , $\mathrm{Re}\{\beta\} > -1/2$ and $\mathrm{Re}\{zh'/h\} > -1/2$ on $U$ .

We now associate to each $f(z) = z|z|^{2\beta} h(z)/h(z)$ in $AUT_{Lh}(U)$ , the mapping $\Phi(z) = z(h(z))^2 \in S^*$ .

Theorem 4.2. [4].

a) For each $\Phi \in S^*$ and for each $\beta$ , $\mathrm{Re}\{\beta\} > -1/2$ , there is one and only one $f \in AUT_{Lh}(U)$ such that $f(z)/(\Phi(z)|z|^{2\beta}) > 0$ for every $z \in U$ and $h(0) = 1$ .

b) For each $a \in B$ , there is a unique solution of (1.2) which is in $AUT_{Lh}(U)$ .

Remarks.

1. Part a) of Theorem 4.2 is quite surprising. Indeed, consider $\Phi(z) = z/(1-z)^2$ and $\beta = 0$ . Then $\arg f(e^{it}) = \arg \Phi(e^{it}) = \pm\pi$ , almost everywhere; however $f(U) = U$ . To be more precise, the corresponding mapping is $f(z) = z(1-\bar{z})/(1-z)$ satisfying $f(e^{it}) = -1$ for all $0 < |t| \leq \pi$ , and where the cluster set of $f$ at the point $1$ is the unit circle.

2. Part b) of Theorem 4.2 states that 2.1.1) and 2.1.3) in Theorem 2.1 can be replaced by $f(U) = U$ .

## REFERENCES

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## STRESZCZENIE

Podano przegląd najważniejszych własności odwzorowań log-harmonicznych, tzn. różnowartościowych, lokalnie quasikonforemnych odwzorowań $f$ koła jednostkowego w płaszczyznę zespoloną, mających postać $f(z) = z|z|^{2\beta} h(z)\overline{g(z)}$ , gdzie $\beta$ , h, g spełniają warunki (1.1a)-(1.1.c), względnie równoważny warunek (1.2).

## РЕЗЮМЕ

Представленный обзор самых важных свойств лог-гармонических отображений или однолистных, локально квазиконформных отображений $f$ единичного круга в плоскости вида $f(z) = z|z|^{2\beta} h(z)\overline{g(z)}$ где $h, g, \beta$ удовлетворяют условиям (1.1.a) - (1.1.c) или эквивалентному условию (1.2) .