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# Fredholm Eigenvalues and Complementary Hardy Spaces 

Wartości wlasne Fredholma i komplementarne przestrzenie Hardy'ego


#### Abstract

Suppose $\Gamma$ is a chord-arc (or Lavrentiev) curve in the finite plane $\mathbf{C}$ and $H^{P}\left(D_{k}\right)$, $p>1, k=1,2$ are complementary (generalized) Hardy spaces on the components $D_{k}$ of $\mathbf{C} \backslash \Gamma$, $\left(\infty \in D_{2}\right)$.

If $(B w)(z)=(2 \pi i)^{-1} \int_{\Gamma}(\zeta-z)^{-1} \overline{w(\zeta)} \overline{d \zeta}, w \in H^{2}\left(D_{1}\right)$, then the eigenvalues of the operator $B$ coincide with the eigenvalues of generalized Neumann-Poincare operator $C$; corresponding to absolutely continuous eigenfunctions. The converse is also true.

Let $\mathcal{A}(D)$ denote the Hilbert space of functions $f$ analytic in the domain $D$ with the norm $\|f\|=\left(\iint_{D}|f(z)|^{2} d x d y\right)^{1 / 2}$.

If $\Gamma$ is chord-arc then the operator $L$ defined by the formulas (2.1)-(2.3) is bounded on $\mathcal{A}\left(D_{1}\right)$, $H^{2}\left(D_{1}\right) \subset \mathcal{A}\left(D_{1}\right)$ and $L w=B w$ for any $w \in H^{2}\left(D_{1}\right)$. Moreover, if a constant $d_{k}$ in the formula (2.2) is positive, then $\lambda_{k}=1 / d_{k}$ is an eigenvalue of the operators $C_{1}^{\Gamma}, B$ and $L$ We have $d_{k}=0$ for all $k \in \mathbf{N}$ if and only if $\Gamma$ is a circle.


1. Introduction. Statement of results. Let $\Gamma$ be a rectifiable Jordan curve in the finite plane $\mathbf{C}$. Many important problems in conformal mapping and the potential theory can be reduced to the solution of a linear integral equation of Fredholm type: $u(z)=\int_{\Gamma} k(z, t) u(t) d s_{\imath}=v(z)$ with the Neumann-Poincare kernel $k(z, t)=-(\pi)^{-1} \frac{\partial}{\partial n_{z}} \log |z-t|, z, t \in \Gamma$. If $\Gamma$ is in $C^{3}$ and $\kappa(t)$ denotes the curvature of $\Gamma$ at $t \in \Gamma$ then putting $2 \pi k(t, t)=\kappa(t)$ we obtain a kernel continuously differentiable w.r.t. the arc length $s$ on $\Gamma$. The eigenvalues of $k$, i.e. the real numbers $\lambda$ such that the homogeneous integral equation $u(z)=\lambda \int_{\Gamma} k(z, t) u(t) d s_{\mathrm{l}}, z, t \in \Gamma$, has a non-trivial real-valued solution $u$ are called Fredholm eigenvalues of $\Gamma$.

A satisfactory theory of Fredholm eigenvalues of $\Gamma$ in the $C^{3}$ case has been created by Schiffer [S1], [S2]. However, confining oneself to curves with continuous curvature excludes even polygonal lines from applications and actually various modifications of the kernel $k$, e.g. admitting curves with corners, could be made, cf. [G]. Nevertheless, no unified approach involving a suitable operator acting on an appropriate class of functions associated with the curve $\Gamma$ was proposed so far. Recent
results of French mathematicians [D], [Z] enable us to extend Schiffer's approach on a fairly general class of curves.

In sect. 2 we introduce a bounded antilinear operator $L$ acting on the class $\mathcal{A}(D)$ of functions $f$ holomorphic in a quasidisk $D$, with the norm $\|f\|=\left(\iint_{D}|f|^{2} d \sigma\right)^{1 / 2}$, where $d \sigma$ is the area element. We prove (Theorem 2.1) that $\|L\| \leq \kappa<1$ and show that for $D$ not being a disk the set of eigenvalues of $L$ is not empty. In sect. 3 we introduce an operator $B$ from $L^{2}(\Gamma)$ to the complementary Hardy spaces $H^{2}\left(D_{k}\right)$, $k=1,2$, where $D_{k}$ are components of $\mathbf{C} \backslash \Gamma$. The operator $B$ is bounded if and only if $\Gamma$ is $A D$-regular. If $\Gamma$ is an $A D$-regular quasicircle, i.e. if $\Gamma$ is a chord-arc curve, cf. [Z], then we have $H^{2}\left(D_{1}\right) \subset \mathcal{A}\left(D_{1}\right)$ and also $B w=L w, w \in H^{2}\left(D_{1}\right)$. In sect. 4 we find a relation between the generalized Neumann-Poincaré operator $C_{\mathrm{F}}^{\Gamma}$ and the operator $B$ for $\Gamma$ being a chord-arc curve. We arrive to the conclusion: If $\Gamma$ is a chord-arc curve but not a circle the eigenvalues of $C_{1}^{r}, B, L$ coincide and the set of eigenvalues is not empty.

Finally, we show (sect.5) that the eigenvalues in the sense of a definition given in [K1], [Kü2] correspond to the eigenvalues of an operator $P$ acting on $L_{\mathbf{R}}^{P}(\Gamma)$ with $\Gamma$ also being chord-arc.
2. The operator $L$ and its eigenvalues. Let $\mathcal{A}(D)$ denote the Hilbert space of functions $f$ holomorphic in a domain $D$ with the inner product $(f \mid g)=\iint_{D} f \bar{g} d \sigma$, where $d \sigma$ is the area element.

We have the following
Theorem 2.1. Suppose $D$ is a quasidisk and $\varphi$ maps $D$ conformally onto the unit disk $\Delta$. Then the function

$$
\begin{equation*}
l(z, t)=\frac{1}{\pi}\left[\frac{\varphi^{\prime}(z) \varphi^{\prime}(t)}{(\varphi(z)-\varphi(t))^{2}}-\frac{1}{(z-t)^{2}}\right], \quad z, t \in D \tag{2.1}
\end{equation*}
$$

is analytic in $D \times D$, does not depend on a particular choice of $\varphi$ and vanishes identically if and only if $D$ is a disk. If $D$ is not a disk, there exist a constant $\kappa \in(0 ; 1)$ and a sequence $\left(d_{n}\right), 0 \leq d_{n} \leq \kappa(n \in \mathbb{N})$ such that

$$
\begin{equation*}
l(z, t)=\sum_{n=1}^{\infty} d_{n} \varphi_{n}(z) \varphi_{n}(t), \quad z, t \in D \tag{2.2}
\end{equation*}
$$

where $\left\{\varphi_{n}(z)\right\}$ is a complete orthonormal system in $\mathcal{A}(D)$ and not all $d_{n}$ vanish.
Moreover,

$$
\begin{equation*}
(L w)(z):=\iint_{D} l(z, t) \overline{w(t)} d \sigma_{t} \tag{2.3}
\end{equation*}
$$

is a bounded antilinear openator on $\mathcal{A}(D)$ and

$$
\begin{equation*}
\|L\| \leq \kappa<1 \tag{2.4}
\end{equation*}
$$

Proof. We have

$$
\begin{equation*}
\lim _{t \rightarrow x} l(z, t)=(6 \pi)^{-1}\{\varphi, z\}, \tag{2.5}
\end{equation*}
$$

where $\{\varphi, z\}$ denotes the Schwarzian derivative. Hence the analyticity of $l(z, t)$ in $D \times D$, as well as $l(z, t) \not \equiv 0$ for $D$ not being a disk, immediately follow. If $h(z)=e^{i \beta}(\varphi(z)-a)(1-\bar{a} \varphi(z))^{-1}, \beta \in \mathbf{R}, a \in \Delta$, then $h^{\prime}(z) h^{\prime}(t)(h(z)-h(t))^{-2} \equiv$ $\varphi^{\prime}(z) \varphi^{\prime}(t)(\varphi(z)-\varphi(t))^{-2}$, so $l(z, t)$ does not depend on a special choice of $\varphi$.

We may assume without loss of generality that $D=f(\Delta)$ with $f$ belonging to the familiar class $S$. Then the Grunsky coefficients $b_{m n}$ are defined by the equality

$$
G(w, \omega):=-\log \frac{f(w)-f(\omega)}{w-\omega}=\sum_{m, n=0}^{\infty} b_{m n} w^{m} \omega^{n} ; \quad w, \omega \in \Delta
$$

Hence

$$
\frac{\partial^{2} G}{\partial w \partial \omega}=\sum_{m, n=1}^{\infty} m n b_{m n} w^{m-1} \omega^{n-1}=\frac{1}{(w-\omega)^{2}}-\frac{f^{\prime}(w) f^{\prime}(\omega)}{(f(w)-f(\omega))^{2}}
$$

and the equalities $w=\varphi(z), \omega=\varphi(t), f(w)=z, f(\omega)=t$ imply

$$
\begin{equation*}
l(z, t)=\sum_{m, n=1}^{\infty} \frac{m n}{\pi} b_{m n} \varphi^{m-1}(z) \varphi^{\prime}(z) \varphi^{n-1}(t) \varphi^{\prime}(t) \tag{2.6}
\end{equation*}
$$

If we introduce modified Grunsky coefficients $c_{m n}=\sqrt{m n} b_{m n}$ then (2.6) takes the form

$$
\begin{equation*}
l(z, t)=\sum_{m, n=1}^{\infty} c_{m n} p_{m}(z) p_{n}(t) \tag{2.7}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{n}(z)=\sqrt{\frac{n}{\pi}} \varphi^{n-1}(z) \varphi^{\prime}(z), \quad n=1,2, \ldots \tag{2.8}
\end{equation*}
$$

It is easily verified that $\left\{p_{n}(z)\right\}$ is a complete orthonormal $(=\mathrm{CON})$ system in $\mathcal{A}(D)$. In fact, if $w=\varphi(z)$, then the relation $F(w)=f(z) / \varphi^{\prime}(z)$ establishes an isometry between $\mathcal{A}(\Delta)$ and $\mathcal{A}(D)$. Now, $\left\{\sqrt{n / \pi} w^{n-1}\right\}$ is a CON-system in $\mathcal{A}(\Delta)$ and consequently $\left\{p_{n}(z)\right\}$ is a CON-system in $\mathcal{A}(D)$. For $D$ being a quasidisk the Grunsky inequality in a sharper form due to $\mathrm{K} u \mathrm{hn}$ au [ Ku 1$]$, [ $\mathrm{Kü} 3]$, also of. [ P$]$, takes place, i.e. there exists $\kappa \in(0 ; 1)$ such that

$$
\left|\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m n} x_{m} x_{n}\right| \leq \kappa\|x\|^{2}, \quad x=\left(x_{1}, x_{2}, \ldots, x_{n}, \ldots\right)=:\left(x_{n}\right) \in l^{2}
$$

Since $c_{n}$ is symmetric, an equivalent inequality for the associated bilinear form is true:

$$
\begin{equation*}
\left|\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m n} x_{m} y_{n}\right| \leq \kappa\|x\|\|y\| ; \quad x, y=\left(y_{n}\right) \in l^{2} . \tag{2.9}
\end{equation*}
$$

Suppose now that $g=\sum_{k=1}^{\infty} \alpha_{k} p_{k}(z), h=\sum_{k=1}^{\infty} \beta_{k} p_{k}(z)$ belong to $\mathcal{A}(D)$ which implies that $\left(\alpha_{n}\right),\left(\beta_{n}\right) \in l^{2}$. Then we have by (2.7) and (2.9):

$$
\begin{aligned}
& \| \iint_{D \times D} l(z, t) \overline{g(z)} \overline{h(t)} d \sigma_{z} d \sigma_{t}\left|=\left|\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m n} \bar{\alpha}_{m} \bar{\beta}_{n}\right|\right. \\
& \leq \kappa\left\|\left(\alpha_{n}\right)\right\|\left\|\left(\beta_{n}\right)\right\|=\kappa\|g\|\|h\| .
\end{aligned}
$$

This implies that the symmetric antilinear operator (2.3) is bounded in $\mathcal{A}(D)$ and its norm satisfies (2.4). Suppose now that $D$ is not a disk and so $0<\kappa<1$.

Consider the function

$$
\begin{equation*}
l_{N}(z, t)=\sum_{m=1}^{N} \sum_{n=1}^{N} c_{m n} p_{m}(z) p_{n}(t)=\left[P_{N}(z)\right]^{\mathbf{T}} C_{N}\left[P_{N}(t)\right] \tag{2.10}
\end{equation*}
$$

where

$$
\left[P_{N}(z)\right]=\left[\begin{array}{c}
p_{1}(z) \\
\vdots \\
p_{N}(z)
\end{array}\right] \quad, \quad C_{N}=\left[c_{m n}\right]_{1 \leq m, n \leq N}
$$

and the superscript $\mathbf{T}$ denotes the transpose. We shall prove that

$$
\begin{equation*}
\lim _{N \rightarrow \infty} l_{N}(z, t)=l(z, t) . \tag{2.11}
\end{equation*}
$$

Putting for short $x_{n}=p_{n}(z), y_{n}=p_{n}(t)$ we have to prove that

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \sum_{m=1}^{N} \sum_{n=1}^{N} c_{m n} x_{m} y_{n}=\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{m n} x_{m} y_{n} \tag{2.12}
\end{equation*}
$$

In view of (2.9) we have for $x, y \in l^{2}$

$$
\begin{aligned}
& \left|\sum_{n=1}^{\infty} \sum_{m=N+1}^{\infty} c_{m n} x_{m} y_{n}\right| \leq \kappa\left\|\left(0,0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots\right)\right\| \cdot\|y\| \\
& =\kappa\|y\|\left(\sum_{m=N+1}^{\infty}\left|x_{m}\right|^{2}\right)^{1 / 2} \rightarrow 0 \text { as } N \rightarrow+\infty
\end{aligned}
$$

Similarly

$$
\left|\sum_{n=N+1}^{\infty} \sum_{m=1}^{N} c_{m n} x_{m} y_{n}\right| \leq \kappa\|x\|\left(\sum_{n=N+1}^{\infty}\left|y_{n}\right|^{2}\right)^{1 / 2} \rightarrow 0
$$

as $N \rightarrow+\infty$ and this proves (2.12) and also (2.11).
Now, in view of a theorem due to Schur [Sch], $[P]$, we have for a symmetric matrix $C_{N}$ the decomposition $C_{N}=U_{N}^{T} D_{N} U_{N}$, where $U_{N}$ is unitary and $D_{N}=$ $\left[d_{1 N}, d_{2 N}, \ldots, d_{N N}\right]$ is diagonal with $0 \leq d_{k N} \leq \kappa$. Hence (2.10) takes the form

$$
\begin{equation*}
l_{N}(z, t)=\left[U_{N} P_{N}(z)\right]^{\mathbf{T}} D_{N}\left[U_{N} P_{N}(t)\right] . \tag{2.13}
\end{equation*}
$$

Unitary transformation $U_{N}$, as applied to $P_{N}(z)$, results in a vector $\Phi_{N}(z)$ with coordinates

$$
\varphi_{k N}(z)=\alpha_{k 1} p_{1}(z)+\alpha_{k 2} p_{2}(z)+\cdots+\alpha_{k N} p_{N}(z)
$$

We have $\left\|\varphi_{k N}\right\|=1$ for $k=1,2, \ldots, N,\left(\varphi_{k N} \mid \varphi_{I N}\right)=0$ for $k \neq l$ and (2.13) can be written as

$$
\begin{equation*}
\iota_{N}(z, t)=\sum_{k=1}^{N} d_{k N} \varphi_{k N}(z) \varphi_{k N}(t), \quad 0 \leq d_{k N} \leq \kappa \tag{2.14}
\end{equation*}
$$

We first observe that $\left\{l_{N}(z, t): N \in \mathbf{N}\right\}$ is a normal family in $D \times D$. In fact, for every compact subset $F$ of $D \times D$ there exists a positive number $\delta$ such that for any $(z, t) \in F$ and $\delta_{w}=\operatorname{dist}(w, C \backslash D)$ we have $\min \left\{\delta_{z}, \delta_{\ell}\right\} \geq \delta$. Putting $\varphi_{k N}(z)=p_{k}(z)$ for $k \geq N+1$ we obtain a CON-system $\left\{\varphi_{k} N(z)\right\}$ for any fixed $N$. We have

$$
\left|l_{N}(z, t)\right|^{2} \leq\left(\max _{1 \leq k \leq N} d_{k N}\right)^{2} \sum_{k=1}^{\infty}\left|\varphi_{k N}(z)\right|^{2} \sum_{k=1}^{\infty}\left|\varphi_{k N}(t)\right|^{2} \leq\left(\frac{\kappa}{\pi \delta^{2}}\right)^{2}
$$

for any $(z, t) \in F$ and our assertion follows by Stieltjes-Osgood theorem. Hence $\left\{l_{N}\right\}$ contains a subsequence $\left\{l_{N_{j}}\right\}$ converging uniformly on $F$ to $l(z, t)$. By the usual diagonal process we obtain a subsequence of $\left\{N_{j}\right\}$, say $\left\{\tilde{N}_{j}\right\}$, such that the limits $\lim _{j \rightarrow \infty} d_{k \tilde{N}_{j}}=d_{k}, \lim _{j \rightarrow \infty} \varphi_{k} \tilde{N}_{j}=\varphi_{k}$, exist. Obviously, for $k \neq l$ we have $\left(\varphi_{k N} \mid \varphi_{I N}\right)=0$ and hence $\left(\varphi_{k} \mid \varphi_{l}\right)=0$ for any pair $k, l ; k \neq l$. Moreover, $0 \leq d_{k} \tilde{N}_{j} \leq \kappa$ and this implies $0 \leq d_{k} \leq \kappa$. Since $l(z, t) \not \equiv 0$, we have $d_{k}\left\|\varphi_{k}\right\|>0$ on a non-empty set of integers and then $0<\left\|\varphi_{k}\right\| \leq 1$. If $\left\|\varphi_{k}\right\|<1$, put $\tilde{\varphi}_{k}=\varphi_{k} /\left\|\varphi_{k}\right\|, \tilde{d}_{k}=d_{k}\left\|\varphi_{k}\right\|$. Then $\tilde{d}_{k} \tilde{\varphi}_{k}=d_{k} \varphi_{k}$ with $\left\|\tilde{\varphi}_{k}\right\|=1$ and $0<\tilde{d}_{k} \leq \kappa$. This verifies the decomposition (2.2) and ends the proof of our theorem.

Corollary 2.2. If $f \in \mathcal{A}(D)$ then $f(z)=\sum_{n=1}^{\infty} \alpha_{n} \varphi_{n}(z)$ with $\sum_{n=1}^{\infty}\left|\alpha_{n}\right|^{2}=$ $\|f\|^{2}$ and

$$
\begin{equation*}
(L f)(z)=\sum_{n=1}^{\infty} d_{n} \bar{\alpha}_{n} \varphi_{n}(z) \tag{2.17}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\|L f\|^{2}=\sum_{n=1}^{\infty} d_{n}^{2}\left|\alpha_{n}\right|^{2} \tag{2.18}
\end{equation*}
$$

and for $d_{k}>0$ we have $L \varphi_{k}=d_{k} \varphi_{k}$ with $\left\|\varphi_{k}\right\|=1$. Consequently, $\lambda_{k}=1 / d_{k}$ is an eigenvalue of $L$ and so the spectrum of $L$ is not empty.

Corollary 2.3. As observed by K. Samotij (oral communication) L is generally not of Hilbert-Schmidt type. E.g. if $D$ is the map of $\Delta$ under $f(w)=(w+1)^{n}$, $0<\alpha<2, \alpha \neq 1$, then $\iint_{D}|l(z, t)|^{2} d \sigma_{z} d \sigma_{t}=+\infty$. Obvinusly $D$ is. a quasidish: and $\partial D \backslash\{0\}$ is an open analytic are.

In view of (2.2) we have $l(\zeta, t)=\sum_{n=1}^{\infty} d_{n} \varphi_{n}(\zeta) \varphi_{n}(t)$ and hence $\sum_{n=1}^{\infty} d_{n}^{2}\left|\varphi_{n}(\zeta)\right|^{2}=$ $\|l(\zeta, \cdot)\|^{2} \leq \kappa^{2} /\left(\pi \delta_{\zeta}^{2}\right)$ so that $l(\zeta, \cdot) \in \mathcal{A}(D)$ for any fixed $\zeta \in D$. Thus by (2.17)

$$
\begin{equation*}
(L I(\zeta, \cdot))(z)=\iint_{D} l(z, t) \overline{l(\zeta, t)} d \sigma_{t}=\sum_{n=1}^{\infty} d_{n}^{2} \overline{\varphi_{n}(\zeta)} \varphi_{n}(z) . \tag{2.19}
\end{equation*}
$$

In particular, for $\zeta=z \in D$ we obtain

$$
\begin{equation*}
\iint_{D}|l(z, t)|^{2} d \sigma_{\ell}=\sum_{n=1}^{\infty} d_{n}^{2}\left|\varphi_{n}(z)\right|^{2} \leq \kappa^{2} /\left(\pi \delta_{x}^{2}\right) \tag{2.20}
\end{equation*}
$$

Corollary 2.4. Theorem 2.1 remains true if the quasidisk $D^{*}$ contains $\infty$ as an interior point and $\psi$ maps $D^{*}$ conformally onto $\Delta^{*}=\{w:|w|>1\}$, so that $\psi(\infty)=\infty$. Again $F(w)=f(z) / \psi^{\prime}(z)$ establishes an isometry between $\mathcal{A}\left(D^{*}\right)$ and $\mathcal{A}\left(\Delta^{*}\right)$.
3. Schiffer's identity for chord-arc curves. In this section we show that the identity (3.6) obtained by Schiffer holds under much weaker assumptions. This generalization was possible in view of some remarkable results due to G. David [D] and M. Zinsmeister [Z].

Devid was able to give a complete characterization of locally rectifiable curves $\Gamma$ and exponents $p$ for which Cauchy singular integral is a bounded operator on the space $L^{p}(\Gamma)$ of complex-valued functions $h$ on $\Gamma$ that satisfy $\int_{\Gamma}|h(z)|^{p}|d z|<+\infty$.

A locally rectifiable curve $\Gamma$ is called regular in the sense of Ahlfors-David, or AD-regular, if there exists a positive constant $M$ such that for any disk $\Delta(a ; R)$ the arc-length measure $|\Gamma \cap \Delta(a ; R)| \leq M R$. The Cauchy singular integral operator $C^{\Gamma}$ is defined as

$$
\begin{align*}
\left(C^{\Gamma} h\right)\left(z_{0}\right) & =C h(z)=\frac{1}{\pi i} P \cdot V \cdot \int_{\Gamma} \frac{h(z) d z}{z-z_{0}}  \tag{3.1}\\
& =\frac{1}{\pi i} \lim _{\varepsilon \rightarrow 0} \int_{\Gamma \backslash \Gamma \varepsilon} \frac{h(z) d z}{z-z_{0}}, \quad z_{0} \in \Gamma
\end{align*}
$$

where $\Gamma_{e}$ is a subarc of $\Gamma$ of length $2 \varepsilon$ bisected by $z_{0}$.
According to David the operator $h \mapsto C^{\Gamma} h, h \in L^{p}(\Gamma)$, is bounded on a locally rectifiable curve $\Gamma$ for some $p>1$, if and only if $\Gamma$ is $A D$-regular. Then it is also bounded for all $p>1$.

If $\Gamma$ is an $A D$-regular Jordan curve we may consider, following David, complementary (generalized) Hardy spaces $H^{p}\left(D_{k}\right)(k=1,2 ; p>1)$ on complementary domains $D_{1}, D_{2} \ni \infty$ of $\Gamma$, assuming for $g \in H^{p}\left(D_{2}\right)$ the normalization $g(\infty)=0$. These classes coincide with the familiar classes $E^{p}\left(D_{k}\right)$ for AD -regular $\Gamma$ in the finite plane, cf. [D], [Du]. Any $f \in H^{p}\left(D_{1}\right)$ has non-tangential limiting values a.e. on $\Gamma$ and $\int_{\Gamma}|f(z)|^{p}|d z|<+\infty$. Since the functions $f, g \in H^{p}\left(D_{k}\right)$ can be recovered from their boundary values by the Cauchy integral formula, we may consider $H^{p}\left(D_{k}\right)$ as subspaces of $I^{p}(\Gamma)$.

As shown by David, $D_{1}$ and $D_{2}$ are domains of Smirnov type, i.e. $H^{P}\left(D_{1}\right)$, $H^{P}\left(D_{2}\right)$ are $L^{p}(\Gamma)$-closures of polynomials, or polynominls in $z^{-1}$, resp. A locally
rectifiable closed curve $\Gamma$ in the extended plane $\overline{\mathbf{C}}$ is called a chord-arc (or Lavrentiev) curve, iff there exists a positive constant $K$ such that for any pair $\Gamma_{1}, \Gamma_{2}$ of complementary subarcs of $\Gamma$ with common end-points $z_{1}, z_{2}$ we have $\min \left\{\left|\Gamma_{1}\right|,\left|\Gamma_{2}\right|\right\} \leq K\left|z_{1}-z_{2}\right|$, where $\left|\Gamma_{k}\right|$ denotes the length of $\Gamma_{k}, k=1,2$. Note that a chord-arc curve is necessarily Jordan. Zinsmeister characterized chord-arc curves as AD-regular quasicircles, [Z].

In what follows we shall consider an operator $B$ which is bounded on $H^{2}\left(D_{k}\right)$ and connect it with the operator $L$ acting on $\mathcal{A}\left(D_{k}\right)$. To this end we need the following

Lemma 3.1. Suppose $\Gamma$ is a chord-arc curve of length $\gamma$ and $D_{1}, D_{2} \ni \infty$ are the components of $\overline{\mathbf{C}} \backslash \Gamma$. For $f \in H^{2}\left(D_{1}\right)$ put $\|f\|_{H^{2}}^{2}=\gamma^{-1} \int_{\Gamma}|f(\zeta)|^{2}|d \zeta|$ and $\|f\|_{A}^{2}=\iint_{D_{1}}|f(z)|^{2} d x d y$. Then

$$
\begin{equation*}
\|f\|_{A} \leq \gamma\|f\|_{H^{2}} \tag{3.2}
\end{equation*}
$$

If $g \in H^{2}\left(D_{2}\right)$ and $g(z)=O\left(z^{-2}\right)$ as $z \rightarrow \infty$ then an analogous inequality for $g$ is also true.

Proof. Obviously $f$ has a primitive $F$ in $D_{1}$ and we can take $F(z)=\int_{\gamma\left(x_{0}, z\right)} f(w) d w$, where $\gamma\left(z_{0}, z\right)$ is an arc in $D_{1}$ with end-points $z_{0}, z$. Since $\Gamma$ is chord-arc, any point of $\Gamma$ can be approached non-tangentially from $D_{1}$ (and also from $D_{2}, c f$. [J-K]), so we can take $z_{0}=\zeta_{0} \in \Gamma$. Then $F(z)=\int_{\gamma\left(\zeta_{0,2}\right)} f(w) d w$ and making $z$ tend to $\zeta \in \Gamma$ non-tangentially we obtain $F(\zeta)=\int_{\gamma\left(\zeta_{0}, \zeta\right)} f(w) d w$. We may also take $\gamma\left(\zeta_{0}, \zeta\right)$ to be a subarc of $\gamma$ and this implies absolute continuity of $F$ on $\gamma$. If $F(z)=u(z)+i v(z)$ then $f=u_{z}-i u_{y}$ and

$$
\begin{align*}
|F(\zeta)| & \leq \int_{\gamma\left(\zeta_{0}, \zeta\right)}\left|u_{x}-i u_{y}\right| d s \leq \int_{\Gamma}\left(u_{x}^{2}+u_{y}^{2}\right)^{1 / 2} d s  \tag{3.3}\\
& \leq\left[\int_{\Gamma}\left(u_{x}^{2}+u_{y}^{2}\right) d s \int_{\Gamma} 1 \cdot d s\right]^{1 / 2}=\gamma\|f\|_{H^{2}}, \quad \zeta \in \Gamma,
\end{align*}
$$

and hence

$$
\begin{equation*}
|u(\zeta)| \leq \gamma\|f\|_{H^{2}} \quad, \quad \zeta \in \Gamma . \tag{3.4}
\end{equation*}
$$

For the inner product $a * b$ in $\mathbf{R}^{2}$ we have $a * b=\operatorname{Re}(\bar{a} b)$ and with this notation

$$
\frac{\partial u}{\partial n}:=\lim _{x \rightarrow \zeta(s) \in \Gamma}\left(\operatorname{grad} u(z) * i \zeta^{\prime}(s)\right)=\operatorname{Re}\left(u_{z}-i u_{y}\right) i \zeta^{\prime}(s)
$$

a.e. on $\Gamma$ and hence

$$
\begin{equation*}
\left|\frac{\partial u}{\partial n}\right| \leq\left|u_{z}-i u_{y}\right|=|f(\zeta)| \quad \text { a.e. on } \Gamma \text {. } \tag{3.5}
\end{equation*}
$$

If we apply the Green formula to the level lines of $D$ we obtain after passing to the limit

$$
\begin{aligned}
& \|f\|_{A}^{2}=\iint_{D}\left(u_{z}^{2}+u_{y}^{2}\right) d \sigma=-\int_{\Gamma} u \frac{\partial u}{\partial n} d s \\
& \leq \gamma\|f\|_{H^{2}} \int_{\Gamma}\left|\frac{\partial u}{\partial n}\right| d s \leq \gamma\|f\|_{H^{2}}\left\{\int_{\Gamma}\left(u_{x}^{2}+u_{y}^{2}\right) d s \cdot \int_{\Gamma} 1 \cdot d s\right\}^{1 / 2}=\gamma^{2}\|f\|_{H^{2}}^{2}
\end{aligned}
$$

in view of (3.3), (3.4) and this proves (3.1).
The case of $g \in H^{2}\left(D_{2}\right), g(z)=O\left(z^{-2}\right)$, can be treated similarly by considering instead of $D_{2}$ a ring domain with boundary $\Gamma \cup\{z:|z|=R\}$ and letting $R$ tend to infinity.

We now introduce for an AD -regular curve $\Gamma$ in the finite plane the operator

$$
\begin{equation*}
(B w)(z):=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\overline{w(\zeta) \zeta^{\prime}(s)^{2}}}{\zeta-z} d \zeta, \tag{3.6}
\end{equation*}
$$

where $w \in H^{2}\left(D_{k}\right), k=1,2$. Since $\overline{w(\zeta) \zeta^{\prime}(s)^{2}} \in L^{2}(\Gamma)$ and $z \notin \Gamma$, the formula (3.6) defines actually four bounded antilinear operators $B_{k l}$ with the domain $H^{2}\left(D_{k}\right)$ and the range $H^{2}\left(D_{l}\right) ; k, l=1,2$. However, in what follows we assume $B=B_{11}$. Then we have the following result (obtained by Schiffer [S2] for $\Gamma$ in $C^{3}$ and $f$ having a continuous extension on $\bar{D}_{1}$ )

Theorem 3.2. If $\Gamma$ is a chord-arc curve in the finite plane then for any $w \in H^{2}\left(D_{1}\right)$ we have

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}=\iint_{D_{1}} l(z, t) \overline{w(t)} d \sigma_{\mathrm{t}}, \quad z \in D_{1} \tag{3.7}
\end{equation*}
$$

Proof. Since $D_{1}$ is a domain of Smirnov type, cf. [D], it is sufficient, in view of (3.2), to prove (3.7) for polynomials. Let $D_{r}, 0<r<1$, denote the preinage of $\Delta_{r}=\{z:|z|<r\}$ under $\varphi$ and let $\Gamma_{r}=\partial \Delta_{r}$ be the level line of $D$. If $\varphi_{r}$ maps $D_{r}$ onto $\Delta$ conformally then obviously $\varphi_{r}=r^{-1} \varphi$ and hence the function $l_{r}$ being an analogue of $l$ for the domain $D_{r}(c f .(2.1))$ satisfies $l_{r}=l$. Therefore, as shown by Schiffer [S2], (3.7) holds for $D=D_{r}, \Gamma=\Gamma_{r}$ and $w$ being a polynomial and we have.

$$
f_{r}(z):=\frac{1}{2 \pi i} \int_{\Gamma_{r}} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}=\iint_{D_{r}} l(z, t) \overline{w(t)} d \sigma_{t}, \quad z \in D_{r}
$$

If $\Phi=\varphi^{-1}$ then for a fixed $z \in D_{r}$

$$
\begin{aligned}
f_{r}(z) & =-\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{\overline{w\left(\Phi\left(r e^{i \theta}\right)\right)} r e^{-i \theta} \overline{\Phi^{\prime}\left(r e^{i \theta}\right)} d \theta}{\Phi\left(r e^{i \theta}\right)-z} \rightarrow \\
& \rightarrow \frac{1}{2 \pi i} \int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}=f_{1}(z) \quad \text { as } r \rightarrow 1^{-}
\end{aligned}
$$

On the other hand

$$
f_{r}(z)=\iint_{D_{r}} l(z, t) \overline{w(t)} d \sigma_{\ell} \rightarrow \iint_{D_{1}} l(z, t) \overline{w(t)} d \sigma_{\ell} \quad \text { as } r \rightarrow 1^{-}
$$

and this ends the proof.

As an immediate consequence we obtain
Corollary 3.3. If $\partial D_{1}$ is chord-arc then the eigenvalues of operators $B$ and $L$ acting on $H^{2}\left(D_{1}\right)$ and $\mathcal{A}\left(D_{1}\right)$, resp., coincide.

Proof. Suppose that $w=\lambda L w$ holds for some real $\lambda, w \in \mathcal{A}\left(D_{1}\right),\|w\|_{\mathcal{A}}>0$. Then for $w(z)=\sum_{n=1}^{\infty} \alpha_{n} \varphi_{n}(z)$ we have by Corollary 2.2 : $\lambda L w(z)=\lambda \sum_{n=1}^{\infty} d_{n} \bar{\alpha}_{n} \varphi_{n}(z)=w(z)$ and this holds if and only if there exists $k \in \mathbf{N}$ such that $d_{k}>0, \lambda=\lambda_{k}=1 / d_{k}$. Then we can take $\alpha_{k} \in \mathbf{R}, \alpha_{k} \neq 0$, and $\alpha_{n}=0$ for $n \neq k$. Suppose now that $w=\lambda B w$ holds for some $w \in H^{2}\left(D_{1}\right),\|w\|_{H^{2}}>0$ and $\lambda \in \mathbf{R}$. Then by Theorem $3.2 w=\lambda L w$. However, Lemma 3.1 implies $w \in \mathcal{A}\left(D_{1}\right)$ with $\|w\|_{\mathcal{A}}>0$ and this is the case already considered.
4. Generalized Neumann-Poincaré operator $C_{1}^{\Gamma}$ and its eigenvalues. As pointed out in sect.3, the operator $C^{\Gamma}$ defined by (3.1) is bounded on $L^{p}(\Gamma)$ for $p>1$ and $\Gamma$ being AD-regular. If $L_{\mathbf{R}}^{p}(\Gamma)=\left\{h \in L^{p}(\Gamma): h\right.$ is real-valued $\}$, then we can split $C^{\Gamma}$ acting on $L_{\mathrm{R}}^{P}(\Gamma)$ into its real and imaginary parts: $C^{\Gamma} h=C_{1}^{\Gamma} h+i C_{2}^{\Gamma} h$. This way we obtain bounded linear operators $C_{k}^{\Gamma}$ on $L_{\mathbf{R}}^{p}(\Gamma), k=1,2$, and $C_{1}^{\Gamma}$ shows to be identical with the classical Neumann-Poincaré operator for $\Gamma$ being $C^{3}$, cf. [K2]. Thus $\lambda \in \mathbf{R}$ satisfying $w=\lambda C_{1}^{\Gamma} w$ with $0 \neq w \in L_{\mathbf{R}}^{p}(\Gamma)$ are a natural generalization of Fredholm eigenvalues for AD-regular $\Gamma$.

We shall now state a theorem establishing a relation between the eigenvalues of the operators $C_{1}^{\Gamma}$ and $B$ in the case $p=2$.

Theorem 4.1. Let $\Gamma$ be chord-arc and $D$ the bounded component of $\mathbf{C} \backslash \Gamma$. If $\lambda \in \mathbf{R}$ satisfies

$$
\begin{equation*}
\rho(\zeta)=\lambda\left(C_{1}^{\Gamma} \rho\right)(\zeta), \quad \zeta \in \Gamma \tag{4.1}
\end{equation*}
$$

and $\rho \neq$ const is absolutely continuous on $\Gamma$ with $d \rho / d s \in L_{\mathbf{R}}^{2}(\Gamma)$ then the function $f=u+i v$ generated by $\lambda \rho$ :

$$
\begin{equation*}
f(z)=\frac{\lambda}{2 \pi i} \int_{\Gamma} \frac{\rho(\zeta) d \zeta}{\zeta-z}, \quad z \in D \tag{4.2}
\end{equation*}
$$

has the derivative $w \in H^{2}(D)$ that satisfies

$$
\begin{equation*}
w(z)=\frac{\lambda}{2 \pi i} \int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}, \quad z \in D \tag{4.3}
\end{equation*}
$$

Conversely, if $w \in H^{2}(D), w \neq$ const, and (4.3) holds for some $\lambda \in \mathbf{R}$, then $w$ has a primitive $f$ which is generated by $\lambda \rho(\zeta)$ according to (4.2) and $\lambda, \rho$ satisfy (4.1); Moreover, $\rho$ is absolutely continuous on $\Gamma$ and $d \rho / d s \in L_{\mathbb{R}}^{2}(\Gamma)$.

Proof. It follows from (4.2) that $f$ has non-tangential limiting values a.e. on $\Gamma$ given by Plemelj's formula: $f(\zeta)=\frac{1}{2}\left[\lambda \rho+C^{\Gamma}(\lambda \rho)\right]=\frac{1}{2}\left[\lambda \rho+\lambda\left(C_{1}^{\Gamma} \rho+i C_{2}^{\Gamma} \rho\right)\right]$ and hence, due to (4.1)

$$
\begin{equation*}
u(\zeta)=\operatorname{Re} f(\zeta)=\frac{1}{2}(1+\lambda) \rho(\zeta) \quad \text { a.e. on } \Gamma \tag{4.4}
\end{equation*}
$$

On the other hand, $f(z)=\lambda(2 \pi i)^{-1} \int_{\Gamma} \rho(\zeta) d_{\zeta} \log (\zeta-z)$ and this implies
(4.5) $u(z)=\operatorname{Re} f(z)=\frac{\lambda}{2 \pi} \int_{\Gamma} \rho(\zeta) \frac{\partial}{\partial s} \arg (\zeta-z) d s$

$$
=\frac{\lambda}{2 \pi} \int_{\Gamma} \rho(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|} d s=\frac{\lambda}{2 \pi} \int_{\Gamma} \frac{2 u(\zeta)}{1+\lambda} \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|} d s .
$$

Thus

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{\Gamma} u(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|} d s=\frac{1+\lambda}{2 \lambda} u(z) \tag{4.6}
\end{equation*}
$$

The Green formula for $D \backslash \Delta(z ; r)$ gives for $r \rightarrow 0$

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{\Gamma}\left[u(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|}-\log \frac{1}{|\zeta-z|} \frac{\partial u}{\partial n_{\zeta}}\right] d s=u(z) \tag{4.7}
\end{equation*}
$$

and hence, in view of (4.6)

$$
\begin{equation*}
u(z)=\frac{\lambda}{\pi(1-\lambda)} \int_{r} \log \frac{1}{|\zeta-z|} \frac{\partial u}{\partial n_{\zeta}} d s \tag{4.8}
\end{equation*}
$$

Now, (4.8) can be written as

$$
u(z)=\frac{-\lambda}{2 \pi(1-\lambda)} \int_{\Gamma}[\log (\zeta-z)+\log (\zeta-\bar{z})] \frac{\partial u}{\partial n_{\zeta}} d s
$$

and the formula $2 \frac{\partial u}{\partial z}=u_{x}-i u_{y}=f^{\prime}(z)=: w(z)$ implies

$$
\begin{equation*}
w(z)=\frac{\lambda}{\pi(1-\lambda)} \int_{\Gamma} \frac{1}{\zeta-z} \frac{\partial u}{\partial n_{\zeta}} d s \tag{4.9}
\end{equation*}
$$

 since $D$ is a Neumann domain (cf. [Z]), we have $v_{z}-i v_{y}=-i u \in H^{2}(D)$, and also $w \in H^{2}(D)$. Therefore

$$
\begin{equation*}
\frac{\partial u}{\partial n_{\zeta}}=-\operatorname{Im} w(\zeta) \zeta^{\prime}(s) \quad \text { a.e. on } \Gamma \tag{4.10}
\end{equation*}
$$

and hence

$$
\left.\frac{\partial u}{\partial n_{\zeta}} d s=\frac{1}{2 i}[\overline{w(\zeta)} \overline{d \zeta})-w(\zeta) d \zeta\right]
$$

so that (4.9) takes the form

$$
\begin{aligned}
w(z) & =\frac{\lambda}{2 \pi i(1-\lambda)}\left\{\int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}-\int_{\Gamma} \frac{w(\zeta) d \zeta}{\zeta-z}\right\} \\
& =\frac{\lambda}{2 \pi i(1-\lambda)} \int_{\Gamma} \frac{\overline{w(\zeta)} \overline{d \zeta}}{\zeta-z}-\frac{\lambda}{1-\lambda} w(z)
\end{aligned}
$$

and this implies (4.3).
Suppose now that (4.3) holds for some $w \in H^{2}(D), u \neq c o n s t$ and $\lambda \in \mathbf{R}$. Then we determine $\frac{\partial u}{\partial n_{6}}$ from (4.10) and also $u(z)$ from (4.8).

Then (4.7) and (4.8) imply

$$
\begin{equation*}
\frac{1}{2}(1+\lambda) u(z)=\frac{\lambda}{2 \pi} \int_{\Gamma} u(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|} d s \tag{4.11}
\end{equation*}
$$

Consider now, similarly' as before the function $F(z)$ generated by $\lambda u(\zeta)$. Then we have, as in (4.5)

$$
\begin{equation*}
\operatorname{Re} F(z)=\operatorname{Re} \frac{\lambda}{2 \pi i} \int_{\Gamma} \frac{u(\zeta) d \zeta}{\zeta-z}=\frac{\lambda}{2 \pi} \int_{\Gamma} u(\zeta) \frac{\partial}{\partial n_{\zeta}} \log \frac{1}{|\zeta-z|} d s \tag{4.12}
\end{equation*}
$$

As $z \rightarrow \zeta \in \Gamma$ non-tangentially, we obtain from (4.11) and (4.12)

$$
\frac{1}{2}(1+\lambda) u(\zeta)=\frac{1}{2}\left[\lambda u(\zeta)+\left(C_{1}^{\Gamma} \lambda u\right)(\zeta)\right]
$$

and this implies (4.1) with $\rho(\zeta)=u(\zeta)$ a.e. on $\Gamma$. Since $\Gamma$ is chord-arc, the function $\int_{6}^{z} w(t) d t, \zeta_{0} \in \Gamma$, is holomorphic in $D$, continuous in $\bar{D}$ and absolutely continuous on $\Gamma$ and so is $u(\zeta)$. Morenver, $w \in H^{2}(D)$ implies $d u / d s \in L^{2}(\Gamma)$.

Corollary 4.2. If $\Gamma$ is chord-arc, then the eigenvalues of the operator $C_{1}^{\Gamma}$ associated uith absolutely continuous eigenfunctions coincide with eigenvalues of $B$ acting on $H^{2}(D)$ and also with eigenvalues of $L$ acting on $\mathcal{A}(D)$.

Corollary 4.3. If the chord-arc curve $\Gamma$ is not a circle then the set of eigenvalues of $C_{1}^{\Gamma}$ associated with non-constant absolutely continuous eigenfunctions is not empty.

If we replace in the formula (2.2) the CON -system $\left\{\varphi_{n}(z)\right\}$ by another CON system $\left.\psi_{n}(z)\right\}$, where $\varphi_{n}(z)=i \psi_{n}(z)$ then the equality (2.2) takes the form

$$
l(z, t)=\sum_{n=1}^{\infty}\left(-d_{n}\right) \psi_{n}(z) \psi_{n}(t), \quad 0 \leq d_{n} \leq \kappa<1
$$

This implies
Corollary 4.4. If $\lambda$ is an eigenvalue of $C_{1}^{\Gamma}$ for chord-arc $\Gamma$ corresponding to a non-constant absolutely continuous eigenfunction $\rho$ with $d \rho / d s \in L_{\mathrm{R}}^{2}(\Gamma)$, so is $-\lambda$.
5. Neumann domains and Fredholm eigenvalues. The present author proposed a definition of Fredholm eigenvalues of a Jordan curve $\Gamma$ that does not involve an operator, cf. [K1]. An equivalent but formally different notion can be also found in an carlier paper by Kühnau [Kï2, Theorem 5). This definition can be restated os follows.

Suppose $\Gamma$ is a rectifiable Jordan curve and $D_{1}, D_{2} \ni \infty$ are components of $\mathrm{C} \backslash \Gamma$. Two non constant functions $f$, $!$ holornorphic in $D_{1}$ and $D_{2}$, resp., are said to

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be conjugate holomorphic eigenfunctions w.r.t. $\Gamma$, if their non-tangential boundary values exist a.e. on $\Gamma$ and there exists a real number $\lambda$ such that

$$
\begin{equation*}
f(\zeta)=(1+\lambda) u(\zeta)+i v(\zeta) \quad, \quad g(\zeta)=(1-\lambda) u(\zeta)+i v(\zeta), \quad \zeta \in \Gamma \tag{5.1}
\end{equation*}
$$

The numbers $\lambda$ so defined have most of the propertics of Fredholm eigenvalues of $\Gamma$ in the $C^{3}$-case as established by Schiffer.

Recent results due to Zinsmeister [Z] enable us to determine a suitable class of curves and a bounded operator whose eigenvalues intervene in the equations (5.1).

The most natural assumption on $f$ and $g$ which secures the recovering of functions from their boundary values is that they range over the complementary Hardy spaces $H^{p}\left(D_{1}\right), H^{p}\left(D_{2}\right)$, resp., for some $p>1$. This means that $\Gamma$ should be AD-regular. Another natural assumption is that any $v \in L_{\mathbf{R}}^{p}(\Gamma)$ determines up to a real constant isomorphically the function $f \in H^{p}\left(D_{1}\right)$ such that $\operatorname{Im} f(\zeta)=v(\zeta)$ a.e. on $\Gamma$ and this means that $D_{1}$ is a Neumann domain. The important notion of a Neumann domain and its characterizations (analytic and geometric) are due to Zinsmeister [Z]. On the other hand, $v \in L_{\mathbb{R}}^{p}(\Gamma)$ should also determine $g \in H^{p}\left(D_{2}\right)$ isomorphically so that $D_{2}$ has to be a Neumann domain, too. It was shown by Zinsmeister that, if $D_{1}$ and $D_{2}$ are Neumann domains then $\Gamma$ is a chord-arc curve. Put $L_{0}^{p}=L_{0}^{p}(\Gamma)=L_{\mathbf{R}}^{p}(\Gamma) /\{$ const $\}$. We may also assume, if convenient, that $L_{0}^{p}$ denotes the set of representatives of equivalence classes, e.g. $\left\{h \in L_{\mathbf{R}}^{p}(\Gamma): \int_{\Gamma} h(\zeta)|d \zeta|=0\right\}$. The following lemma is an immediate consequence of Zinsmeister's characterization of chord-arc curves.

Lemma 5.1, [K2]. Suppose $\Gamma$ is an $A D$-regular Jordan curve. Then the following are equivalent:
(i) $\Gamma$ is a chord-arc curve;
(ii) $\mp 1$ are regular values of the operator $C_{1}^{\Gamma}$ acting on $L_{0}^{p}(\Gamma)$ for some $p>1$;
(iii) $C_{2}^{\Gamma}$ is an isomorphism of $L_{0}^{p}(\Gamma)$ for some $p>1$.

It follows easily from (ii) that for all $p>1$ the operator $\frac{1}{2}\left(I+C_{1}^{\Gamma}\right)$ generates an isomorphism of $L_{\mathbf{R}}^{P}(\Gamma)$ which leaves constant functions unchanged; here $I$ denotes the identity operator.

Suppose that $\Gamma$ is a chord-arc curve and $f=\varphi+i v \in H^{p}\left(D_{1}\right), g=\psi+i v \in$ $H^{p}\left(D_{2}\right)$ for some $p>1$. Given an arbitrary $\varphi \in L_{\mathbb{R}}^{p}(\Gamma)$ we may find a unique $y \in L_{\mathbf{R}}^{p}(\Gamma)$ such that $2 \varphi=\left(I+C_{1}^{\Gamma}\right) y$, i.e. $y=2\left(I+C_{1}^{\Gamma}\right)^{-1} \varphi$. Then $v=\frac{1}{2} C_{2}^{\Gamma} y$ and $\psi=\frac{1}{2}\left(-y+C_{1}^{\Gamma} y\right)=-\left(I-C_{1}^{\Gamma}\right)\left(I+C_{1}^{\Gamma}\right)^{-1} \varphi=: P \varphi$. The operator $P$ is closely related to the equations (5.1) which is evident from the following

Theorem 5.2. If $\Gamma$ is a chord-arc curve in the finite plane then the operator

$$
\begin{equation*}
P=-\left(I-C_{1}^{\Gamma}\right)\left(I+C_{1}^{\Gamma}\right)^{-1} \tag{5.2}
\end{equation*}
$$

defines an isomorphism of $L_{\mathbf{R}}^{P}(\Gamma)\left(p>1\right.$, arbitrary) onto $\left\{u(\zeta)=\operatorname{Re} g(\zeta): g \in H^{p}\left(D_{2}\right)\right\}$.

## If

$$
\begin{equation*}
\mu=(1+\lambda)(1-\lambda)^{-1} \tag{5.3}
\end{equation*}
$$

is a regular (singular) value, or eigenvalue of $P$ then $\lambda$ is an analogous value for the operator $C_{1}^{\Gamma}$.

Proof. We have

$$
\begin{align*}
I-\mu P & =\left[I+C_{1}^{\Gamma}+\mu\left(I-C_{1}^{\Gamma}\right)\right]\left(I+C_{1}^{\Gamma}\right)^{-1}  \tag{5.4}\\
& =(1+\mu)\left(I-\lambda C_{1}^{\Gamma}\right)\left(I+C_{1}^{\Gamma}\right)^{-1} .
\end{align*}
$$

Since $\left(I+C_{1}^{\Gamma}\right)^{-1}, I+C_{1}^{\Gamma}$ are bounded on $L_{\mathbf{R}}^{p}(\Gamma), I-\mu P$ has a bounded inverse if and only if $I-\lambda C_{1}^{\Gamma}$ has one. Hence the sets of regular values correspond under (5.3). The same is obviously true for singular values. The equality $(I-\mu P) u=0$ is equivalent to $\left(I-\lambda C_{1}^{\Gamma}\right) v=0$ for $u=\left(I+C_{1}^{\Gamma}\right) v$ and this implies the correspondence of eigenvalues.

Suppose now that (5.1) holds with $\|u\|>0$. Then we have $(1-\lambda) u=P(1+\lambda) u$, or $u=\mu P u$, where $\mu$ satisfies (5.3).

In the case $p=2$ and the operator $C_{1}^{\Gamma}$ acting on $L_{0}^{2}(\Gamma)=\left\{u \in L_{R}^{2}(\Gamma)\right.$ : $\left.\int_{\Gamma} u(\zeta)|d \zeta|=0\right\}$ the inequality $\left\|C_{1}^{\Gamma}\right\|<1$ means that $\Gamma$ is chord-arc and $-P$ is a positive operator, cf. [K2]. It is an open question, whether $\Gamma$ being chord-arc implies $\left\|C_{1}^{\Gamma}\right\|<1$ on $L_{0}^{2}(\Gamma)$.

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## STRESZCZENIE

Niech $\Gamma$ będzie krzywą luk-cį̣ciwa (czyli krzywą Lawrentiewa) w plaszczyźnie otwartej $\mathbf{C}$ i niech $H^{p}\left(D_{k}\right), p>1, k=1,2$ bede komplementarnymi priestrzeniami Hardy'ego w skladowych $D_{k}$ zbioru $C \backslash \Gamma,\left(\infty \in D_{2}\right)$.

Jésli $(B w)(z)=(2 \pi i)^{-1} \int_{\Gamma}(\zeta-z)^{-1} \overline{w(\zeta)} \overline{d \zeta}, w \in H^{2}\left(D_{1}\right), z \in D_{1}$, to wartości wlasne operatora $B$ są identyczne z wartościami wlasnymi uogólnionego operatora Neumana-Poincarêgo $C_{1}^{[ }$odpowiadajacymi funkcjom wlasnym absolutnie ciaglym. Równiez twierdzenie odwrotne jest prawdziwe.

Niech $\mathcal{A}(D)$ oznacza przestrzeń Hilberta funkcji $f$ analitycznych w obszarze $D$ z norma $\|f\|=\left(\iint_{D}|f(z)|^{2} d x d y\right)^{1 / 2}$.

Jeáli $\Gamma$ jest krzywą luk-cięciwa, to operator $L$ określony wzorami (2.1)-(2.3) jest ograniczony w $\mathcal{A}\left(D_{1}\right), H^{2}\left(D_{1}\right) \subset \mathcal{A}\left(D_{1}\right)$ oraz $L w=B w$ dla $w \in H^{2}\left(D_{1}\right)$. Ponadto, jeáli stala $d_{k}$ we wzorze (2.2) jest dodatnia, to $\lambda_{k}=1 / d_{k}$ jest wartością wlasną operatorów $C_{1}^{\Gamma}, B, L$. Równość $d_{k}=0$ dla każdego $k \in \mathbf{N}$ ma miejsce wtedy i tylko wtedy, gdy $\Gamma$ jest okrggiem.

