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The Classical Interpretation of Convergence in the Space of Entire Functions of Seremeta Order not Exceeding ρ is Obtained

Klasyczna interpretacja zbieżności w pewnej przestrzeni Frécheta

Классическая интерпретация сходимости в некотором пространстве Фрешета

1. Introduction. Let Λ^0 denote the class of all functions α defined over (a,∞), $-\infty \le a$ satisfying:

$$\alpha \text{ is differentiable on } (a,\infty)\ , \tag{1.1}$$

$$\alpha \text{ increases strictly monotonically}\ , \tag{1.2}$$

$$\alpha(x) \to +\infty \text{ as } x \to +\infty\ , \tag{1.3}$$

$$\alpha(x) \to -\infty \text{ as } x \to a^+ \text{ and}\ , \tag{1.4}$$

$$\lim_{x\to\infty} \frac{\alpha(kx)}{\alpha(x)} = 1 \tag{1.5}$$

for all k, $0 < k < \infty$, the convergence being uniform on every finite interval of $(0,\infty)$. Let L_b^0 represent the class of all functions β defined on (b,∞), $-\infty \le b$ satisfying

$$\beta \text{ is differentiable on } (b,\infty)\ , \tag{1.6}$$

$$\beta \text{ increases strictly monotonically}\ , \tag{1.7}$$

$$\beta(x) \to +\infty \text{ as } x \to +\infty\ , \tag{1.8}$$

$$\beta(x) \to -\infty \text{ as } x \to b^+ \text{ and}\ , \tag{1.9}$$

$$\lim_{x\to\infty} \frac{\beta[(1+\nu(x))x]}{\beta(x)} = 1 \tag{1.10}$$

for every function ν such that $\nu(x) \to 0$ as $x \to \infty$.

A positive function h defined on $[a, \infty]$ belongs to the class L^0 if it satisfies (1.6), (1.7), (1.8) and (1.10). Similarly a function $h \in \Lambda^0$ if $h \in L^0$ and satisfies (1.5) in place of (1.10).

Let $\alpha_1 \in \Lambda^0$, $\beta_1 \in L^0$. If

$$\frac{d\{\beta_1^{-1}(k\alpha_1(x))\}}{d(\log x)} = O(1) \quad \text{as } x \to \infty$$

for all k, $0 < k < \infty$, then Šeremeta order ρ [8] of an entire function f, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is defined by:

$$\rho \equiv \rho(\alpha, \beta) = \limsup_{r \to \infty} \frac{\alpha_1(\log(M(r, f))}{\beta_1(\log r)} = \limsup_{n \to \infty} \frac{\alpha_1(n)}{\beta_1\left(\frac{1}{n} \log |a_n|^{-1}\right)} \tag{1.11}$$

where

$$M(r, f) = \max_{|z|=r} |f(z)|$$

Obviously, if $\alpha \in \Lambda_a^0$, $\beta \in L_b^0$ then Šeremeta order ρ of the entire function f is given by the same formula (1.11).

Let $\Gamma(\alpha, \beta, \rho)$ denote the class of all entire functions f, including $f \equiv 0$, whose Šeremeta order does not exceed ρ where $\alpha \in \Lambda_a^0$, and $\beta \in L_b^0$. It is simple to verify that $\Gamma(\alpha, \beta, \rho)$ is a linear space over the field of complex numbers C with usual addition and scalar multiplication. Futher, any element f, $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma(\alpha, \beta, \rho)$ is characterized by the equation:

$$\limsup_{r \to \infty} \frac{\alpha(\log(M(r, f))}{\beta(\log r)} \le \rho\,. \tag{1.12}$$

Or equivalently:

$$|a_n|^{\frac{1}{n}} \exp\left\{\beta^{-1}\left(\frac{\alpha(n)}{\rho + \delta}\right)\right\} \to 0 \quad \text{as } n \to \infty \tag{1.13}$$

for each $\delta > 0$.

Define, for $f \in \Gamma(\alpha, \beta, \rho)$

$$\| f; \alpha, \beta, \rho + \delta \| = |a_0| + \sum_{n=1}^{\infty} |a_n| \exp\left\{n\beta^{-1}\left(\frac{\alpha(\mu_n)}{\rho + \delta}\right)\right\} \tag{1.14}$$

where

$$\mu_n = \begin{cases} N^*, & 0 \le n \le N^* \\ n & \text{for } n > N^* \end{cases}$$

N^* being the least positive integer such that $\alpha(N^*) \geq 0$. Obviously, for each $\delta > 0$, the expression (1.14) gives a norm on $\Gamma(\alpha,\beta,\rho)$. Denote the corresponding normed space by $(\Gamma(\alpha,\beta,\rho)\ ,\ \|\cdot\|_\delta)$. Let $d_{\alpha,\beta}$ be the metric topology on $\Gamma(\alpha,\beta,\rho)$ generated by these family of norms i.e. for f and $g \in \Gamma(\alpha,\beta,\rho)$ define

$$d_{\alpha,\beta}(f,g) = \sum_{p=1}^{\infty} \frac{1}{2^p} \ \frac{\|f-g\,;\alpha\,,\beta\,,\rho+\frac{1}{p}\|}{1+\|f-g\,;\alpha\,,\beta\,,\rho+\frac{1}{p}\|} \tag{1.15}$$

It is easy to see that $(\Gamma(\alpha,\beta,\rho)\ ,\ d_{\alpha,\beta})$ is a Fréchet space (for the definition of Fréchet space see Rudin[7]). Several authors including Iyer [2], [3], Krishnamurthy [5], [6], Srivastava [9] have obtained the classical interpretation of convergence in various subspaces of analytic functions in general. Krishnamurthy [5], [6], studied the convergence criteria in the spaces $\Gamma(\rho)$ and $\Gamma(0)$ where $\Gamma(\rho)$ denotes the space of all entire functions whose order does not exceed ρ while $\Gamma(0)$ contains entire functions of order zero. Recently, Juneja and Srivastava [9] have obtained the classical interpretation of convergence in the space $\Gamma_{p,q}(\rho)$ of all entire functions whose index pair does not exceed (p,q) and whose (p,q) order does not exceed ρ if of index pair (p,q).

In this paper we obtain a classical interpretation of convergence in $(\Gamma(\alpha,\beta,\rho)\ ,\ d_{\alpha,\beta})$ which includes the corresponding results of Krishnamurthy [5], [6], and Juneja and Srivastava [4].

2. Preliminary results. In this section we state a few lemma which are used in the proof of our main theorem. These lemmas are either well known or can be easily proved on the lines adopted by Krishnamurthy [5].

Lemma 2.1. *If $d_{\alpha,\beta}(f,0) \geq l$ $(0 < l < 2)$, then*

$$\|f;\alpha,\beta,\rho+\delta\| \geq \frac{l}{2-l} \text{ for some } \delta = \delta_0 \text{ where } 0 < \delta_0 < 1\,,$$

and therefore for all values of $\delta \leq \delta_0$.

Remark. A consequence of this lemma is that if a series converges in $(\Gamma(\alpha,\beta,\rho)\ ,\|\cdot\|_\delta)$ for each $\delta > 0$, then it converges in $(\Gamma(\alpha,\beta,\rho)\ ,\ d_{\alpha,\beta})$, converse is also true.

Lemma 2.2. (Dunford and Schwartz [1], p.58). *If a vector space X is a complete linear metric space under each of the two invariant metrices d_1 and d_2 and if one of corresponding topologies contains the other, the two topologies are equal.*

Lemma 2.3. (Rudin [7], p.4). *If a topology τ is induced by a metric d which is derived from a norm, then it is easy to verify that the vector space operations are continuous in τ.*

3. The main theorem. We are now ready to prove our main theorem.

Theorem 3.1. *Let $\{f_n\}$ be a sequence of elements of $\Gamma(\alpha,\beta,\rho)$. The statement $f_n \to f$ in $(\Gamma(\alpha,\beta,\rho)\ ,\ d_{\alpha,\beta})$ is equivalent to the statement that for each $\delta > 0$,*

$\{f_n\}$ converges uniformly to f in $D_{a_1^0} = \{z \ : \ |z| > a_1^0\}$ relative to the function:

$$\exp\left\{\int_{a_1^0}^{|z|} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t}dt\right\}$$

where

$$a_1^0 > \max\left[\exp\left\{\beta^{-1}\left(\frac{\alpha(a+)}{\rho+\delta}\right)\right\}, \exp(b)\right] \ .$$

Proof. Define

$$\|f\ ;\ \alpha,\beta,\rho+\delta\|_1 = \max_{z\in D_{a_1^0}}\left[\exp\left\{-\int_{a_1^0}^{|z|} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t}dt\right\}\right]|f(z)| \quad (3.1)$$

clearly, for each $\delta > 0$ and $f \in \Gamma(\alpha,\beta,\rho)$ the expression (3.1) gives a norm on $\Gamma(\alpha,\beta,\rho)$. Denote the corresponding normed space by $(\Gamma(\alpha,\beta,\rho)\ ,\ \|\cdot\|_\delta, 1)$. Let $d'_{\alpha,\beta}$ be the metric defined by:

$$d'_{\alpha,\beta}(f,g) = \sum_{p=1}^{\infty}\frac{1}{2^p}\ \frac{\|\ f-g\ ;\alpha\ ,\beta\ ,\rho+\frac{1}{p}\ \|_1}{1+\|\ f-g\ ;\alpha\ ,\beta\ .\rho+\frac{1}{p}\ \|_1} \quad (3.2)$$

where $f, g \in \Gamma(\alpha,\beta,\rho)$.

Now, we first show that $(\Gamma(\alpha,\beta,\rho\ ,\ d'_{\alpha,\beta})$ is complete. For this, consider $f_p(z) = \sum_{n=0}^{\infty} a_n^{(p)} z^n$ as Cauchy sequence in $(\Gamma(\alpha,\beta,\rho)\ ,d'_{\alpha,\beta})$. Then it is Cauchy in each of the normed space $(\Gamma(\alpha,\beta,\rho)\ ,\|\cdot\|_\delta)$. So, given $\varepsilon > 0$ these exists $p_0(\varepsilon)$ such that for $p\ ,\ q \geq p_0(\varepsilon)$

$$\max_{z\in D_{a_1^0}}\left[\exp\left\{-\int_{a_1^0}^{|z|} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t}dt\right\}\right]|f_p(z) - f_q(z)| \leq \varepsilon \quad (3.3)$$

Let S_0 be an arbitrary compact subset of $D_{a_1^0}$ and let $r = \sup_{z\in S_0}|z|$. Then for each $z \in S_0$ and $p\ ,\ q\ \geq\ p_0(\varepsilon)$, (3.3) gives:

$$|f_p(z) - f_q(z)| \leq \varepsilon \exp\left\{\int_{a_1^0}^{r} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t}dt\right\} \quad (3.4)$$

It shows that $\{f_p\}$ converges uniformly on S_0. Using Weierstrass theorem, it is clear from (3.4) that $\{f_p\}$ converges uniformly to a regular function f in $D_{a_1^0}$. Now consider a circle $W' = \{z\ :\ |z| = r_0 > a_1^0\}$. Since $\{f_p\}$ is a sequence of entire functions converging uniformly on W, it also converges uniformly inside W' to a function g regular inside W. But f and g coincide in the region $\{z\ :a_1^0 < |z| \leq r_0\}$.

So f can be analytically continued over the finite complex plane. Hence f is entire. Further, $f_p \to f$ uniformly in $D_{s_1^0}$ so $|f_p(z) - f(z)| \le \varepsilon$ for every z belonging to a compact subset of $D_{s_1^0}$ and $p \ge p_0(\varepsilon)$. This gives $M(r, f) \le M(r, f_p) + \varepsilon$. Using the fact that $f_p \in \Gamma(\alpha, \beta, \rho)$ it is easy to see that $f \in \Gamma(\alpha, \beta, \rho)$.

From (3.3) it follows that $f_p \to f$ in $(\Gamma(\alpha, \beta, \rho), \|\cdot\|_\delta, 1)$ for each $\delta > 0$. Hence $f_p \to f$ in $(\Gamma(\alpha, \beta, \rho), d'_{\alpha,\beta})$. Thus $(\Gamma(\alpha, \beta, \rho), d'_{\alpha,\beta})$ is complete. Because of Lemma 2.3, it is also linear metric space. We now show that $d'_{\alpha,\beta}$ is compared with $d_{\alpha,\beta}$. For this, consider

$$I_n \equiv \max_{s_1^0 < r < \infty} |a_n| r^n \left[\exp\left\{-\int_{s_1^0}^{r} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt\right\}\right] . \tag{3.5}$$

Let

$$P(r) = \int_{s_1^0}^{r} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt - n \log r .$$

It is easy to see that for

$$r = \exp\left\{\beta^{-1}\left(\frac{\alpha(n)}{\rho+\delta}\right)\right\} \equiv r_n ,$$

say, the function $P(r)$ has a minimum value.

Case I : for $n > N^*$,

$$I_n < |a_n| \exp\{n \log r_n\} = |a_n| \exp\left\{n\beta^{-1}\left(\frac{\alpha(n)}{\rho+\delta}\right)\right\} .$$

Case II : for n, $0 \le n < N_0$ where $N_0 = [a] + 1$ if $a > 0$ and $N_0 = 1$ if $a < 1$.
(a) if $r \ge r_0$ where

$$r_0 = \exp\left\{\beta^{-1}\left(\frac{\alpha(N_0)}{\rho+\delta}\right)\right\}$$

then $P(r)$ is an increasing function of r. So we have:

$$\max_{r_0 \le r \le \infty} |a_n| \left[\exp\left\{-\int_{s_1^0}^{r} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt + n \log r\right\}\right] =$$

$$= |a_n| \exp\left[-\int_{s_1^0}^{r_0} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt + n \log r_0\right] <$$

$$< |a_n| \exp\left[n\beta^{-1}\left(\frac{\alpha(N_0)}{\rho+\delta}\right)\right] \le$$

$$\le |a_n| \exp\left[n\beta^{-1}\left(\frac{\alpha(N^*)}{\rho+\delta}\right)\right] \quad \text{as } N_0 \le N^* .$$

(b) If $r < r_0$ then

$$\max_{e_1^0 < r < r_0} |a_n| \exp\left\{ -\int_{e_1^0}^{r} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt + n \log r \right\} <$$

$$< |a_n| \exp(n \log r_n) < |a_n| \exp(n \log r_0) \text{ as } r_n < r_0 <$$

$$|a_n| \exp\left[n\beta^{-1}\left(\frac{\alpha(N^*)}{\rho+\delta}\right)\right] \text{ as } N_0 \leq N^* .$$

Case III: for n, $N_0 \leq n \leq N_1$

$$I_n < |a_n| \exp\left\{ n\beta^{-1}\left(\frac{\alpha(n)}{\rho+\delta}\right)\right\} < |a_n| \exp\left\{ n\beta^{-1}\left(\frac{\alpha(N^*)}{\rho+\delta}\right)\right\} \quad \text{as} \quad n \leq N^*$$

Combining all cases together, we have

$$I_n < |a_n| \exp\left\{ n\beta^{-1}\left(\frac{\alpha(\mu_n)}{\rho+\delta}\right)\right\} \quad \text{for each } n \ .$$

Hence $d_{\alpha,\beta}$ and $d'_{\alpha,\beta}$ are comparable.

So, by Lemma 2.2, $d_{\alpha,\beta}$ and $d'_{\alpha,\beta}$ are equal. Therefore, $f_n \to f$ in $d_{\alpha,\beta}$ implies $f_n \to f$ in $d'_{\alpha,\beta}$ that is for each $\delta > 0$, $f_n \to f$ uniformly in $D_{e_1^0}$ with respect to

$$\exp\left\{ \int_{e_1^0}^{z} \frac{\alpha^{-1}((\rho+\delta)\beta(\log t))}{t} dt \right\}$$

this completes the proof.

Remark. On setting $\alpha(x) = \log x$, $\beta(x) = x$, the above theorem leads to analogous result for the spaces $\Gamma(\rho)$ and $\Gamma(0)$ studied by Krishnamurthy [5], [6]. Further, for $\alpha(x) = \log^{[p-1]} x$, $\beta(x) = \log^{[q-1]} x$, analogous result for the space $\Gamma_{(p,q)}(\rho)$, $p \geq 2$, $q \geq 1$ studied by Juneja and Srivastava [4] can also be obtained.

REFERENCES

[1] Dunford, N., Schwartz, J.T., *Linear Operator I*, Interscience Publ. Inc., New York 1957.

[2] Iyer, V.G., *On the space of integral functions I*, J.Indian Math. Soc. 12 (1948), 13-30.

[3] Iyer, V.G., *On the space of integral functions I*, J.Indian Math. Soc. 24 (1960), 269-278.

[4] Juneja, O.P., Srivastava, P.D., *On the space of entire functions of (p, q) order*, Comment. Math. Univ. St.Pauli, 27 (1) (1978) 71-79.

[5] Krishnamurthy, V., *On the spaces of certain classes of entire functions*, J.Austral. Math. Soc. 1 (1960), 147-170.

[6] Krishnamurthy, V., *On the continuous endomorphisms in the space of classes of entire functions*, Proc. Nat. Acad. Sci. India, Sect. A, 26 (1960), 642-655.

[7] Rudin, W., *Functional Analysis*, Tata MacGraw Hill Pub.Co., New Delhi 1976.

[8] Seremeta,M.N. , *On the connection between the growth of the maximum moduli of the coefficients of its power series expansion*, Amer. Math. Soc. Transl. (2) , (88) (1970), 291-301.

[9] Srivastava,P.D. , *On the space of certain class of analytic functions*, Indian J. Pure Appl. Math. 10 (1) (1979) , 84-93.

STRESZCZENIE

W pracy podano klasyczną interpretację zbieżności w przestrzeni funkcji całkowitych, których rząd Szeremiety nie przekracza ρ.

РЕЗЮМЕ

В данной работе представлена классическая интерпретация сходимости в пространстве целых функций , которых ряд Шереметы не провосходит ρ.