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# The Frame Field Along a Curve in the Space $P(p,q)$ of $p$-dimensional Planes in the $(p+q)$-dimensional Euclidean Space

Pole reperów wzdłuż krzywej w przestrzeni $P(p,q)$ płaszczyzn $p$-wymiarowych w $(p+q)$-wymiarowej przestrzeni euklidesowej

Поле реперов вдоль кривой в пространстве $P(p,q)$ $p$-мерных плоскостей $(p+q)$-мерного эвклидового пространства

**1. On the metod applied in the paper.** To construct a frame field along a curve in the space $P(p,q)$ and to obtain differential equations of it we used the method due to K.Radziszewski in [7]. The method is different from the classic Cartan's one, moreover, we have introduced a certain modification of it. That is why the applied construction should be explained in a few words.

Let $(M,G)$ be a homogeneous space, where $M$ is a manifold and $G$ is a Lie group of transformations of $M$. Given a surface $X : R^n \to M : u \mapsto X(u)$ in $M$. Let $p_0$ be an arbitrary point of $M$. Any function of the form $A : R^n \to G : u \mapsto A(u)$ such that $X(u) = A(u) \cdot p_0$ is called a representation of the surface $X$ in a Lie group $G$ with respect to the point $p_0$.

Let $R_0$ be a fixed frame at $p_0$. The formula $R(u) = A(u) \cdot R_0$ determines one-to-one correspondence between the representations of $X$ with respect to $p_0$ and the frame fields along $X$. So the construction of a frame field $u \mapsto R(u)$ along $X$ can be replaced by the construction of a representation $u \mapsto A(u)$ of $X$ in $G$. K.Radziszewski has proposed in [7] the following construction of the representation $A$ of $X$ in $G$. We take an arbitrary representation $A_0$ of $X$ with respect to the fixed point $p_0$. Each representation $A$ of $X$ with respect to $p_0$ is of the form $A(u) = A_0(u) \cdot h(u)$

for a certain function $h : R^n \to H_0 \subset G$ taking its values in the isotropy group $H_0$ of $p_0$.

Let us consider the Maurer-Cartan form $A^{-1}dA$ for the representation of the form $A = A_0h$. Let $m_0$ be a linear subspace of a Lie algebra $\bar{g}$ of $G$ such that $\bar{g} = \bar{h}_0 \oplus m_0$, where $\bar{h}_0$ is a Lie algebra of $H_0$, and let pr $_{m_0}$ denotes the projection onto $m_0$. Then

$$\mathrm{pr}_{m_0} A^{-1}dA = \mathrm{pr}_{m_0} h^{-1}A_0^{-1}dA_0h.$$

We want the representation $A$ to give the form $A^{-1}dA$ that satisfies a condition:

$$\mathrm{pr}_{m_0} A^{-1}dA \subset P_0 \qquad (*,1)$$

for a certain linear subspace $P_0 \subset m_0$. We choose such $m_0$ and $P_0$ that make the condition $(*,1)$ as strong restriction for $A$ as possible. When there exists one and only one representation $\bar{A} = A_0h_0$ that satisfies $(*,1)$, then the construction is finished. If not, we repeat the above process taking any representation $A_1$ of the property $(*,1)$ instead of $A_0$, the group $H_1 = \{h \in H_0,\ \mathrm{pr}_{m_0}(h^{-1}P_0h) \subset P_0\}$ instead of $H_0$ and an adequate condition $(*,2)$ instead of $(*,1)$. The process is finished when after a finite number $k$ of steps we obtain the only representation $\bar{A} = A_k = A_0h_0h_1\ldots h_{k-1}$ that satisfies a condition:

$$\mathrm{pr}_{m_{k-1}} A_k^{-1}dA_k \subset P_{k-1}. \qquad (*.k)$$

The form $\Phi = \bar{A}^{-1}d\bar{A}$ is invariant for $X$ up to the transformations from $G$. So the coefficients of $\Phi$ in any basis of a Lie algebra $\bar{g}$ form the complete system of invariants of $X$. The equation $d\bar{A} = \bar{A}\Phi$ is the equation of the constructed frame field along $X$.

The author of the present paper has introduced a certain modification of the above method. In the first step a general form of the representation $A_0$ of $X$ was taken and then the properties of $A_0$ that make it satisfying the condition $(*,1)$ were found. So was done in the next steps. This way turns out to be very convenient in a practical applications.

**2. The space $P(p,q)$ and curves in it.** Given the Euclidean space $E^{p+q} = (R^{p+q}, I^{p+q}, (\ ,\ ))$, where $I^{p+q}$ is the group of isometries of $R^{p+q}$ and $(\ ,\ )$ is the scalar product.

**Definition 1.** A $p$-dimensional, passing the point $x$ and spaned by the vectors $e_1, \ldots, e_p$ plane in $E^{p+q}$ is an image set $[Y] := Y(R^p)$ of a function:

$$Y : R^p \to R^{p+q} : (\lambda^i) \mapsto x + \lambda^i e_i\ ,\quad i = 1, \ldots, p. \qquad (1)$$

It can be assumed that $(e_i, e_j) = \delta_{ij}$.

Let $P(p,q)$ be a set of all $p$-dimensional planes in $E^{p+q}$. $I^{p+q}$ acts on $P(p,q)$ by the rule:

$$g([Y]) = [gY], \quad g \in I^{p+q}, \quad [Y] \in P(p,q)\,,$$

where $gY$ is the usual composition of $Y$ and $g$.

**Lemma 1.** *$(P(p,q), I^{p+q})$ is a homogeneous space.*

**Proof.** It is obvious that $I^{p+q}$ acts on $P(p,q)$ transitively, so we have only to show that the action is effective. Let $f$ and $g$ be two different isometries of $E^{p+q}$. There exists such a point $x_0 \in R^{p+q}$ that $f(x_0) \neq g(x_0)$. Let $[Z] \in P(p,q)$ be any plain passing the point $g(x_0)$ that does not contain $f(x_0)$ and let $[Y] = g^{-1}([Z])$. Then $f([Y]) \neq g([Y])$, so $f$ and $g$ are different transformations of $P(p,q)$.

**Definition 2.**

(a) A parametrized curve (we shall write p.c.) in $P(p,q)$ is a function:

$$\Sigma : R \to P(p,q) : t \mapsto [Y(t)] \tag{2}$$

where $Y(t) : R^p \to R^{p+q} : (\lambda^i) \mapsto x(t) + \lambda^i e_i(t)$

(b) A curve in $P(p,q)$ is an image set $[\Sigma] := \Sigma(R)$

(c) Given a p.c. $\Sigma$ in $P(p,q)$ as in (a). Any curve in $E^{p+q}$ of the form $y(t) = x(t) + \lambda^i(t)e_i(t)$ is called the generating line both of the p.c. $\Sigma$ and the curve $[\Sigma]$.

We shall use the notation $\dot{F} = \dfrac{dF}{dt}$ for a derivative of a function $F$ of one variable. It can be easy proved the following (see [9]):

**Lemma 2.** *Given $Y(t) : R^p \to R^{p+q} : (\lambda^i) \to x(t) + \lambda^i e_i(t)$ and $Z(t) : R^p \to R^{p+q} : (\mu^i) \to y(t) + \mu^i E_i(t)$ . If $[Y(t)] = [Z(t)]$ for any $t$ then:*

$$\operatorname{lin}(e_1,\ldots,e_p,\dot{e}_1,\ldots,\dot{e}_p) = \operatorname{lin}(E_1,\ldots,E_p,\dot{E}_1,\ldots,\dot{E}_p)$$

*where* $\operatorname{lin}(v_1,\ldots,v_r)$ *denotes a vector space spaned by the vectors* $v_1,\ldots,v_r$.

**Definition 3.** A p.c. in $P(p,q)$ is called admissible (we shall write a.p.c.) if:

$$\dim(\operatorname{lin}(e_1,\ldots,e_p,\dot{e}_1,\ldots,\dot{e}_p)) = p+1 \ . \tag{3}$$

We shall prove an important property of an a.p.c. in $P(p,q)$.

**Lemma 3.** *The planes $[Y(t)]$ of an a.p.c. $\Sigma : t \to [Y(t)]$ in $P(p,q)$ can be spaned by vector fields $e_1,\ldots,e_p$ of the following property:*

$$\dim(\operatorname{lin}(e_1,\ldots,e_p,\dot{e}_2,\ldots,\dot{e}_p)) = p \ . \tag{4}$$

**Proof.** Let an a.p.c. $\Sigma$ be given in the form: $\Sigma : t \to [Y(t) : (\lambda^i) \mapsto x(t) + \lambda^i E_i(t)]$. Since $\Sigma$ is admissible then there exist a non vanishing, orthogonal to $\operatorname{lin}(E_1,\ldots,E_p)$ vector field $v$ and scalar functions $f_i$ such that the projections of $\dot{E}_i$ onto the orthogonal complement of $(E_1,\ldots,E_p)$ can be expressed in the form:

$$\dot{E}_i - \sum_{k=1}^{p}(\dot{E}_i, E_k)E_k = f_i v \ , \quad \sum_{i=1}^{p}(f_i)^2 > 0 \ .$$

Any vector fields $e_1, \ldots, e_p$ spaning the planes $[Y(t)]$ are of the form $e_i = b_i^j E_j$ where $[b_i^j]$ is an orthogonal matrix. Therefore $\dot{e}_i = \dot{b}_i^j E_j + b_i^j \dot{E}_j$, and

$$\dot{e}_i - \sum_{k=1}^{p} (\dot{e}_i, e_k) e_k = \dot{e}_i - \sum_{k=1}^{p} (\dot{e}_i, E_k) E_k =$$

$$= \dot{b}_i^j E_j + b_i^j \dot{E}_j - \dot{b}_i^k E_k - b_i^j \sum_{k=1}^{p} (\dot{E}_j, E_k) E_k =$$

$$= b_i^j \left( \dot{E}_j - \sum_{k=1}^{p} (\dot{E}_j, E_k) E_k \right) = b_i^j f_j v \ .$$

If we set $b_1^j = \left[ \sum_{i=1}^{p} (f_i)^2 \right]^{-1/2}$, then:

$$\dot{e}_i - \sum_{k=1}^{p} (\dot{e}_i, e_k) e_k = \begin{cases} \left[ \sum_{k=1}^{p} (f_k)^2 \right]^{-1/2} v \ , & \text{for } i = 1 \\ 0 \ , & \text{for } i > 1 \end{cases}$$

i.e. $\dot{e}_i \in \text{lin}\,(e_1, \ldots, e_p)$ for $i > 1$.

It is obvious that in the mentioned in Lemma 3 situation we have:

$$\dim\,(\,\text{lin}\,(e_1, \ldots, e_p, \dot{e}_1)) = p + 1 \ . \tag{5}$$

A vector field $e_1$ of the property (4) is determined for an a.p.c. up to the sign.

**Definition 4.** A vector field $e_1$ mentioned in Lemma 3 (as well as $-e_1$) is called the directional vector of an a.p.c. in $P(p,q)$.

**Definition 5.** An a.p.c. in $P(p,q)$ is oriented (we shall write a.o.p.c.) if one of two directional vectors of it is marked out. The orienting directional vector will be denoted by $e_*$.

**Definition 6.** Given an p.c. $\Sigma : t \mapsto [Y(t) : (\lambda^i) \mapsto x(t) + \lambda^i e_i(t)]$ in $P(p,q)$. A curve:

$$\Sigma^{0y} : t \mapsto [Y^{oy}(t) : (\mu^\alpha) \mapsto y(t) + \mu^\alpha a_\alpha(t)] \ , \quad \alpha = 1, \ldots, q \tag{6}$$

in $P(q,p)$ such that $(a_\alpha, e_i) = 0$ for any $\alpha$ and $i$ (i.e. $[Y(t)]$ and $[Y^{oy}(t)]$ are orthogonal planes for any $t$) is called an orthogonal to $\Sigma$ p.c. relative to the curve $y : t \mapsto y(t)$ in $E^{p+q}$.

**Lemma 4.** *If $\Sigma$ is an a.p.c. in $P(p,q)$ so is any orthogonal to $\Sigma$ p.c. $\Sigma^{0y}$ in $P(q,p)$. Moreover, if $e_1$ is the directional vector of $\Sigma$ then the normalized projection of $\dot{e}_1$ onto the linear orthogonal complement of* lin $(e_1, \ldots, e_p)$ *is the directional vector of $\Sigma^{0y}$.*

**Proof.** In virtue of Lemma 3 it can be assumed that: $\dot{e}_1 - \sum_{k=1}^{p}(\dot{e}_1, e_k)e_k = 0$ and $\dot{e}_i - \sum_{k=1}^{p}(\dot{e}_i, e_k)e_k = 0$ for $i > 1$ i.e. $e_1$ is the directional vector of $\Sigma$. Let $\Sigma^{0v}$ in $P(q,p)$ be given in the form:

$$\Sigma^{0v} : t \mapsto [Y(t) : (\mu^\alpha) \mapsto y(t) + \mu^\alpha a_\alpha(t)] \ .$$

Then $\sum_{\alpha=1}^{q}(\dot{e}_1, a_\alpha)a_\alpha \neq 0$ and $\sum_{\alpha=1}^{q}(\dot{e}_i, a_\alpha)a_\alpha \neq 0$ for $i > 1$ therefore $\sum_{\alpha=1}^{q}(\dot{a}_\alpha, e_1)a_\alpha = 0$ and $\sum_{\alpha=1}^{q}(\dot{a}_\alpha, e_i)a_\alpha = 0$ for $i > 1$ what implies that for any $\alpha = 1, \dots, q$

$$\dot{a}_\alpha - \sum_{\beta=1}^{q}(\dot{a}_\alpha, a_\beta)a_\beta = (\dot{a}_\alpha e_1)e_1$$

and that these vector fields do not vanish simultaneously, so $\Sigma^{0v}$ is admissible.

It can be assumed that $a_1$ is a directional vector of $\Sigma^{0v}$ from now on. Then

$$\dot{a}_1 - \sum_{\beta=1}^{q}(\dot{a}_1, a_\beta)a_\beta \neq 0 \text{ and } \dot{a}_\alpha - \sum_{\beta=1}^{q}(\dot{a}_\alpha, a_\beta)a_\beta = 0 \text{ for } \alpha > 1 \ .$$

Consequently

$$\underset{\alpha_0}{\exists}(\dot{e}_1, a_{\alpha_0}) \neq 0 \ , \quad \underset{i_0}{\exists}(\dot{a}_1, e_{i_0}) \neq 0 \ ,$$

$$\underset{i>1}{\forall}\underset{\alpha}{\forall}(\dot{e}_i, a_\alpha) = 0 \ , \quad \underset{\alpha>1}{\forall}\underset{i}{\forall}(\dot{a}_\alpha, e_i) = 0 \ ,$$

so $(\dot{e}_1, a_1) \neq 0$ and $(\dot{e}_i, a_\alpha) = 0$ when $i > 1$ or $\alpha > 1$ i.e. $\dot{e}_1 - \sum_{k=1}^{p}(\dot{e}_1, e_k)e_k$ and $a_1$ have the same direction.

The above lemma allows us to introduce the following definition.

**Definition 7.** Given an a.o.p.c. $\Sigma : t \mapsto [Y(t) : (\lambda^i) \mapsto x(t) + \lambda^i e_i(t)]$ in $P(p,q)$ oriented with $e_* = e_1$. The induced orientation of an orthogonal to $\Sigma$ a.p.c. $\Sigma^{0v}$ in $P(q,p)$ is an orientation determined by:

$$a_* = \left[(\dot{e}_*)^2 - \sum_{k=2}^{p}(\dot{e}_*, e_k)^2\right]^{-1/2} \left(\dot{e}_* - \sum_{k=2}^{p}(\dot{e}_*, e_k)e_k\right) \ . \tag{7}$$

$a_*$ is called the normal orienting directional vector of $\Sigma$.

We can introduce a certain generalization of a notion of the cylindric ruled surface in $E^3$.

**Definition 8.** A parametrized curve $\Sigma : t \mapsto [Y(t)]$ in $P(q,p)$ is called $r$-cylindric ($1 \leq r \leq p$) if there exists an $r$-dimensional constant plane parallel to every plane $[Y(t)]$ and the dimension $r$ is greatest possible. A p.c. $\Sigma$ in $P(p,q)$ is called orthogonally $s$-cylindric ($1 \leq s \leq q$) if any orthogonal to $\Sigma$ p.c. $\Sigma^{0s}$ is $s$-cylindric.

It can be shown the following (see [9]):

**Lemma 5.** *A p.c. $\Sigma : t \mapsto [Y(t)]$ in $P(q,p)$ is $r$-cylindric iff the planes $[Y(t)]$ can be spaned by vectors $e_1,\ldots,e_p$ of the property*

$$\dim\left(\ \mathrm{lin}\,(e_1,\ldots,e_r,\dot{e}_1,\ldots,\dot{e}_r)\right) = r \tag{8}$$

*where $r$ is greatest possible.*

Note that a p.c. in $P(p,q)$ that is not $p$-cylindric need not be admissible.

**3. The construction of a frame field along a curve in P(p, q).** We shall use the following notations:

$$i,j,k = 1,\ldots,p\ ; \quad \alpha,\beta,\gamma = 1,\ldots,q\ ; \quad I,J,K = 1,\ldots,p+q\ .$$

$(\varepsilon_I, 0)$ is an orthonormal system of coordinates in $E^{p+q}$. $C^I_J\ ,C^I\ \ (I<J)$ are the matrices that form a basis of Lie algebra $\bar{I}^{p+q}$ of $I^{p+q}$, where:

$$[C_J]^a_b = \begin{cases} 1 & \text{if } a = I\ ,\ b = J \\ -1 & \text{if } a = J\ ,\ b = I \\ 0 & \text{for remaining } a\ ,\ b \text{ from } 1,\ldots,p+q+1 \end{cases}$$

$$[C^I]^a_b = \begin{cases} 0 & \text{if } a\ ,\ b = 1,\ldots,p+q \text{ or } a = p+q+1 \\ \delta^{Ia} & \text{if } b = p+q+1 \end{cases}$$

Given a p.c. $\Sigma$ in $P(p,q)$

$$\Sigma : t \mapsto [Y(t) : (\lambda^i) \mapsto x(t) + \lambda^i e_i(t)]\ , \tag{9}$$

$$x(t) = 0 + x^K(t)\varepsilon_K\ \ ,\quad e_i(t) = e^K_i(t)\varepsilon_K$$

and a plane $[Y_0]$ in $P(p,q)$

$$[Y_0 : (\lambda^i) \mapsto 0 + \lambda^i \varepsilon_{q+i}]\ . \tag{10}$$

The isotropy group $H_0$ of $[Y_0]$ consists of the matrices of the form:

$$\begin{bmatrix} h^\alpha_\beta & 0 & 0 \\ 0 & k^i_j & k^i \\ 0 & 0 & 1 \end{bmatrix},$$

where matrices $\left[h^\alpha_\beta\right]$ and $\left[k^i_j\right]$ are orthogonal.

A Lie algebra $\bar{h}_0$ of $H_0$ is spaned by $C^\alpha_\beta$, $C^{q+i}_{q+j}$, $C^{q+i}$. For any vector fields $t \mapsto a_\alpha(t) = a^K_\alpha(t)\varepsilon_K$ in $E^{p+q}$ such that $(a_\alpha(t), a_\beta(t)) = \delta_{\alpha\beta}$, $(a_\alpha(t), e_i(t)) = 0$, a function

$$A : R \to I^{p+q} : t \mapsto \begin{bmatrix} a^K_\alpha(t) & e^K_i(t) & x^K(t) \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

is a representation of the p.c. $\Sigma$ (9) in a Lie group $I^{p+q}$ with respect to the point $[Y_0]$ (10), since

$$[Y(t)] = A(t)[Y_0] \tag{12}$$

**The 1-th step.** Let $A_0$ be a representation of $\Sigma$ of the form (11) and let $h_{(0)}$ be a function of the form $h_{(0)} : R \to H_0$. We set

$$m_o = \text{ lin } \left(C^{\alpha}_{q+j}, C^{\alpha}\right) \ , \quad I^{p+q} = h_0 \oplus m_0 \tag{13}$$

and

$$P_o = \text{ lin } \left(C^1_{q+1}, C^{\alpha} \ \ \alpha > 1\right) \ , \quad P_0 \subset m_0 \ . \tag{14}$$

Then for the representation $A(t) = A_0(t) \cdot h_{(0)}(t)$ we have:

$$\text{pr}_{\ m_0} A^{-1} dA = h^{\beta}_{\alpha} k^k_i (a_\beta, de_k) C^{\alpha}_{q+1} + [(h_\alpha, de_k)k^k + (h_\alpha, dx)]C^{\alpha} \ , \tag{15}$$

where $h_\alpha = h^{\beta}_{\alpha} a_\beta$. The representation $A$ satisfies a condition:

$$\text{pr}_{\ m_0} A^{-1} dA \subset P_0 \tag{*,1}$$

if and only if

$$h^{\beta}_{\alpha} k^k_i (a_\beta, de_k) = 0 \text{ for } \alpha > 1 \text{ or } i > 1 \tag{16}$$

and

$$(h_1, de_k)k^k + (h_1, dx) = 0 \ . \tag{17}$$

Easy calculation shows (see [9]) that $h^{\beta}_{\alpha}$ and $k^k_i$ satisfy (16) iff $h^{\beta}_1$ and $k^k_1$ are solutions of the following system of equations:

$$\begin{array}{lc}
1 & h^{\beta}_1 k^k_1 = \left[\sum\limits_{\alpha, i} (a_\alpha, \dot{e}_i)^2\right]^{-1/2} (a_\beta, \dot{e}_k) \ , \\
2 & \sum\limits_{\beta} \left(h^{\beta}_1\right)^2 = 1 \ , \\
3 & \sum\limits_{k} \left(k^k_1\right)^2 = 1 \ .
\end{array} \tag{18}$$

**Lemma 6.** (a) *The system of equations* (18) *is solvable iff the* p.c. $\Sigma$ (9) *is admissible.*

(b) *The solutions of* (18) *are of the form*

$$k^k_1 = \delta^k_1 \ , \quad h^{\beta}_1 = \delta^{\beta}_1 \ \textit{or} \ k^k_1 = -\delta^k_1 \ , \quad h^{\beta}_1 = -\delta^{\beta}_1$$

*iff* $e_1$ *is one of two directional vectors and* $a_1$ *is one of two normal directional vectors of* $\Sigma$.

**Proof.** (a) When the system (18) is assumed to be solvable then $\sum_{\alpha,i}(a_\alpha, \dot{e}_i)^2 > 0$. Let $h_1^\beta$, $k_1^k$ be a solution of (18). For any $k$ we have:

$$\sum_\beta h_1^\beta k_1^k a_\beta = \frac{\sum_\beta (\dot{e}_k, a_\beta) a_\beta}{\sqrt{\sum_{\alpha,i}(a_\alpha, \dot{e}_i)^2}} = \frac{\left(\dot{e}_k - \sum_i (\dot{e}_k, e_i) e_i\right)}{\sqrt{\sum_{\alpha,i}(a_\alpha, \dot{e}_i)^2}} .$$

On the other hand: $\sum_\beta h_1^\beta k_1^k a_\beta = k_1^k \sum_\beta h_1^\beta a_\beta = k_1^k h_1$, where $h_1 := \sum_\beta h_1^\beta a_\beta$ is a non vanishing vector field in virtue of (18) (2). So the vector fields $\dot{e}_k - \sum_i (\dot{e}_k, e_i) e_i$ are parallel to $h_1$ and they do not vanish simultaneously, i.e. $\Sigma$ is admissible.

Let $\Sigma$ be assumed to be admissible now. Let $e_1$ be a directional vector of $\Sigma$ and $a_1$ a normal directional vector of $\Sigma$. Then $(\dot{e}_1, a_1) \neq 0$ and $(\dot{e}_i, a_\alpha) = 0$ when $\alpha > 1$ or $i > 1$. In this situation the system (18) takes a form:

$$h_1^1 k_1^1 = \frac{(a_1, \dot{e}_1)}{|(a_1, \dot{e}_1)|}$$

$$h_1^\beta k_1^k = 0 \quad \text{for } \beta > 1 \text{ or } k > 1 \tag{18'}$$

$$\sum_\beta (h_1^\beta)^2 = 1 \ ; \quad \sum_k (k_1^k)^2 = 1 \ .$$

It is clear that the system (18') is solvable and its only solutions are of the form:

$$k_1^k = \delta_1^k \ , \quad h_1^\beta = \delta_1^\beta \text{ and } k_1^k = -\delta_1^k \ , \quad h_1^\beta = -\delta_1^\beta \ .$$

(b) It remains to note that putting the solutions of the above form into (18) (1) we obtain the conditions on $e_1$ and $a_1$ to be the directional vector and the normal directional vector of $\Sigma$ respectively.

Because of the above lemma, the considered p.c. $\Sigma$ must be assumed to be admissible from now on. Moreover, when the representation $A_0$ itself satisfies (16) ($h_{(0)} =$ id ) it has determined two columns up to the sign. Its $(q+1)$-th column is formed by the coordinates of one of two directional vectors of $\Sigma$ and its 1-th column is formed by the coordinates of one of two normal directional vectors of $\Sigma$. If a.p.c. is oriented then the orienting directional vector $e_*$ and the normal directional vector $a_*$ are chosen. So the mentioned columns can be exactly determined. Let us consider the condition (17) after the assumption that the representation $A_0$ satisfies (16). Since $h_1 = a_1$ and $(a_1, de_k) = 0$ for $k > 1$ then (17) takes the form:

$$(a_1, de_1) k^1 + (a_1, dx) = 0 \tag{19}$$

Thus the coefficient $k^1$ of $h_{(0)}$ is determined by the formula

$$k^1 = -\frac{(a_1, \dot{x})}{(a_1, \dot{e}_1)} . \tag{20}$$

When the representation $A_0$ itself satisfies (16) and (19) ($h_{(0)} =$ id i.e. $k^1 = 0$) it has partially determined the last column by the condition:

$$(a_1, \dot{x}) = 0 \tag{21}$$

Finally, the representation $A_0$ of an a.o.p.c. satisfies (*,1) iff

$$\bar{e}_1 = e_* \ , \ \bar{a}_1 = a_* \ , \ (\bar{a}_1, dx) = 0 \tag{22}$$

where bars over $e_1$ and $a_1$ denote that these vectors are already determined.

The form $(\bar{a}_1, d\bar{e}_1) = (a_*, de_*) \neq 0$ is invariant for a.o.p.c. We shall use the notation:

$$\Pi \ := \ (a_*, de_*) \neq 0 \tag{23}$$

It can be easy verified that the group $H_1 = \{h \in H_0 \ , \ pr_{m_0}(h^{-1}P_0 h) \subset P_0\}$ is of the form $H_1 = \left\{h \in H_0 \ , \ h_1^\beta = \delta_1^\beta \ , \ k_1^k = \delta_1^k \ , \ k^1 = 0\right\}$, what finishes the 1-st step.

It is clear, that the representation of $\Sigma$ is completely determined in the first step, when $\Sigma$ is an a.o.p.c. in $P(1,1)$ or $P(1,2)$. When $p > 1$ or $q > 2$ the process must be continued.

**The 2-nd step.** Let $A_1$ be an assumed to satisfy (*,1) representation of the a.o.p.c. $\Sigma$ and let $h_{(1)}$ be a function of the form $h_{(1)} \ : \ R \to H_1$. We set:

$$m_1 = \ \mathrm{lin} \ \left(C^\alpha_{q+j}, C^\alpha, C^1_\beta, C^{q+1}_{q+j}, C^{q+1}\right) \tag{24}$$

and

$$P_1 = \ \mathrm{lin} \ \left(C^1_{q+1}, \ C^\alpha \ \ \alpha > 1 \ , \ C^1_2, \ C^{q+1}_{q+2}\right) \tag{25}$$

For the representation $A(t) = A_1(t)h_{(1)}(t)$ we have:

$$\begin{aligned} \mathrm{pr}_{\ m_1} A^{-1}dA = h^\beta_\alpha(\bar{a}_1, da_\beta)C^1_\alpha + k^k_l(\bar{e}_1, de_k)C^{q+1}_{q+l} + \Pi C^1_{q+1} + \\ +(h_\alpha, dx)C^\alpha + [(\bar{e}_1, de_k)k^k + (\bar{e}_1, dx)]C^{q+1} \ \ .\alpha, \beta, k, l > 1 \end{aligned} \tag{26}$$

Then $A$ satisfies the condition:

$$\mathrm{pr}_{\ m_1} A^{-1}dA \subset P_1 \tag{*,2}$$

if and only if

$$\sum_{\beta > 1} h^\beta_\alpha(\bar{a}_1, da_\beta) = 0 \quad \text{for } \alpha > 2 \tag{27,a}$$

$$\sum_{k>1} k_l^k(\bar{e}_1, de_k) = 0 \quad \text{for } l > 2 \tag{27,e}$$

$$\sum_{k>1}(\bar{e}_1, de_k)k^k + (\bar{e}_1, dx) = 0 \tag{27,x}$$

Let us introduce the notations:

$$< d\bar{a}_1 >= \sum_{\alpha>1}(d\bar{a}_1, a_\alpha)a_\alpha \quad , \tag{28,a}$$

$$< d\bar{e}_1 >= \sum_{i>1}(d\bar{e}_1, e_i)e_i \quad , \tag{28,e}$$

and $h_\alpha = h_\alpha^\beta a_\beta$, $k_l = k_l^k e_k$. Then (27,a) and (27,e) can be written down in the following short form:

$$(h_\alpha, < d\bar{a}_1 >) = 0 \quad \text{for } \alpha > 2 \ , \tag{29,a}$$

$$(k_l, < d\bar{e}_1 >) = 0 \quad \text{for } l > 2 \ , \tag{29,e}$$

If an a.o.p.c. $\Sigma$ is assumed to satisfy:

$$\begin{aligned} &< d\bar{a}_1 > \neq 0 \quad (\text{ i.e. } < da_* > \neq 0) \ , \\ &< d\bar{e}_1 > \neq 0 \quad (\text{ i.e. } < de_* > \neq 0) \ , \end{aligned} \tag{30}$$

then (29,a) and (29,e) determine vector fields $h_2$ and $k_2$

$$h_2 = \frac{< \dot{\bar{a}}_1 >}{| < \dot{\bar{a}}_1 > |} \tag{31,a}$$

$$k_2 = \frac{< \dot{\bar{e}}_1 >}{| < \dot{\bar{e}}_1 > |} \tag{31,e}$$

A geometrical sense of the assumption (30) is explained by the following lemma (proved in [9])

**Lemma 7.** *Given an* a.p.c. $\Sigma$ *in* $P(p,q)$ *with the directional vector* $e_*$ *and the normal directional vector* $a_*$. *Then*

$$< de_* >= 0 \ \textit{iff}\ \Sigma \textit{ is } (p-1)\textit{-cylindric}$$

$$< da_* >= 0 \ \textit{iff}\ \Sigma \textit{ is orthogonally } (q-1)\textit{-cylindric} \ .$$

Thus (30) means that $\Sigma$ is neither $(p-1)$-cylindric nor orthogonally $(q-1)$-cylindric.

When the representation $A_1$ itself satisfies (29,a) and (29,e) (i.e. $h_{(1)} =$ id ) then $h_2 = a_2$ and $k_2 = e_2$. So $a_2$ and $e_2$ (i.e. the 2-nd and the (q+2)-th columns of $A_1$) are determined

$$\bar{a}_2 = \frac{< \dot{\bar{a}}_1}{| < \dot{\bar{a}}_1 > |} \tag{32,a}$$

$$\bar{e}_2 = \frac{<\dot{\bar{e}}_1>}{|<\dot{\bar{e}}_1>|} \qquad (32,e)$$

Since $(da_\alpha, \bar{e}_1) = 0$ for $\alpha > 1$, $(de_k, \bar{a}_1) = 0$ for $k > 1$ and because of (23), (32,a), (32,e) we have:

$$d\bar{a}_1 = -\Pi\bar{e}_1 + \omega_1^2 a_2 \ , \ \omega_1^2 = (d\bar{a}_1, \bar{a}_2 \neq 0 \ , \qquad (33,a)$$

$$d\bar{e}_1 = \Pi\bar{a}_1 + \Omega_1^2 e_2 \ , \ \Omega_1^2 = (d\bar{e}_1, \bar{e}_2 = 0 \ , \qquad (33,e)$$

and

$$\bar{a}_2 = \frac{(\dot{\bar{a}}_1 - (\dot{\bar{a}}_1, \bar{e}_1)\,\bar{e}_1)}{\sqrt{(\dot{\bar{a}}_1)^2 - (\dot{\bar{a}}_1, \bar{e}_1)^2}} \qquad (34,a)$$

$$\bar{e}_2 = \frac{(\dot{\bar{e}}_1 - (\dot{\bar{e}}_1, \bar{a}_1)\,\bar{a}_1)}{\sqrt{(\dot{\bar{e}}_1)^2 - (\dot{\bar{e}}_1, \bar{a}_1)^2}} \qquad (34,e)$$

Let us consider the condition (27,x) for $A$ after the assumption that the representation $A_1$ satisfies (27,a) and (27,e). Then the equation (27,x) can be written down in the form:

$$\Omega_1^2 k^2 = (\bar{e}_1, dx) \ . \qquad (35,x)$$

So the coefficient $k^2$ of $h_{(1)}$ is determined by the formula:

$$k^2 = \frac{(\bar{e}_1, \dot{x})}{(F_1^2)} \ , \quad \Omega_1^2 = F_1^2 dt \ . \qquad (36,x)$$

When the representation $A_1$ itself satisfies (27,x) then

$$(\bar{e}_1, \dot{x}) = 0 \qquad (37,x)$$

It is the next after (21) condition, that partially determines the last column of $A_1$. Finally, the representation of an a.o.p.c. $\Sigma$ satisfies (*,2) if and only if it satisfies (22), (37,x) and its 2-nd and (q+2)-th columns are determined by (34,a) and (34,e) respectively. Note that the forms $\omega_1^2$ and $\Omega_1^2$ in (33,a) and (33,e) and the form $(h_2, dx) = (\bar{a}_2, dx)$ (see (26)) are invariant forms for $\Sigma$. We shall use the notation:

$$\omega^2 = (\bar{a}_2, dx) \qquad (38)$$

The group $H_2$ obtained as the reduction of $H_1$ in the 2-nd step is of the form $H_2 = \{h \in H_0 \ , \ h_s^\beta = \delta_s^\beta \ , \ k_s^k = \delta_s^k \ , \ k^s = 0 \ , \ s = 1 \ , \ 2\}$ . After the 2-nd step the representation of $\Sigma$ is completely determined when $\Sigma$ is a neither $(p-1)$-cylindric nor orthogonally $(q-1)$-cylindric a.o.p.c. in $P(p,q)$ for $p = 1$, $q = 3$ ; $p = 2$, $q = 2$ ; $p = 2$, $q = 3$ .

Note that the written down in the formulas with symbols (a) and (e) parts of considerations in the 2-nd step were led independently. They have determined the vectors $\bar{a}_2$ and $\bar{e}_2$ respectively (i.e. the 2-nd and the (q+2)-th columns of $\bar{A}$) and have given us the formulas on $d\bar{a}_1$ and $d\bar{e}_1$. The results of the parts (a) and (e) were used in the part denoted with (x) to obtain the conditions for the generating line. When $p > 2$ or $q > 3$ than the process should be continued in the steps analogous to the 2-nd one. After the assumption that the considered a.o.p.c. $\Sigma$ is neither $r$-cylindric for any $r = 1,\ldots,p-1$ nor orthogonally $s$-cylindric for any $s = 1,\ldots,p-1$, the part (a) finishes after $q-1$ steps and the part (e) after $p-1$ steps. These parts determine vectors $\bar{a}_1,\ldots,\bar{a}_q$ , $\bar{e}_1,\ldots,\bar{e}_p$ that form the columns of the representation $\bar{A}$ of $\Sigma$ except the last one. The last column of $\bar{A}$ is completely determined in the part denoted with ($x$), that finishes after $p$ steps. The part ($x$) gives the sequence of equations on the only generating line of $\Sigma$. The mentioned equations are following:

$$(\bar{a}_1, d\bar{x}) = 0\,, \quad (\bar{e}_r, d\bar{x}) = 0\,, \quad 1 \leq r < p\,. \tag{39}$$

**Definition 9.** The unique satisfying (39) generating line $\bar{x}$ of an assumed to be neither $r$-cylindric nor orthogonally $s$-cylindric for any $r = 1,\ldots,p-1$, $s = 1,\ldots,q-1$, a.o.p.c. $\Sigma$ in $P(p,q)$ is called the striction line of $\Sigma$.

It can be easy verified that if x is an arbitrary generating line of $\Sigma$ then the striction line $\bar{x}$ of $\Sigma$ can be obtained in the form:

$$\bar{x}(t) = x(t) + \lambda^i(t)\bar{e}_i(t) \tag{40}$$

where $\lambda^i$ are given by the following formulas:

$$\begin{aligned} \lambda^1 &= -\frac{(\dot{x},\bar{a}_1)}{(\dot{\bar{e}}_1,\bar{a}_1)} \\ \lambda^2 &= -\frac{(\dot{x},\bar{e}_1)+\dot{\lambda}^1}{(\dot{\bar{e}}_2,\bar{e}_1)} \\ \lambda^{k+1} &= -\frac{(\dot{x},\bar{e}_k)+\dot{\lambda}^k+\lambda^{k-1}(\dot{\bar{e}}_{k-1},\bar{e}_k)}{(\dot{\bar{e}}_{k+1},\bar{e}_k)} \qquad k = 2,\ldots,p-1 \ \ \text{(see [9])} \end{aligned} \tag{41}$$

Note that the constructed representation $\bar{A}$ of $\Sigma$ can be understood as a linear orthonormal frame field in $E^{p+q}$ alond the striction line of $\Sigma$. A linear orthonormal frame is not a frame in $P(p,q)$ in the sense of Cartan but it is much more convenient in the considerations than a frame understood as a sequence of planes in $P(p,q)$. That is why we shall call the representation $\bar{A}$ itself a frame field along $\Sigma$.

The main results of the paper can be formulated in the following theorem.

**Theorem 1.** *Given an* a.o.p.c. $\Sigma$ *in* $P(p,q)$

$$\Sigma \,:\, t \mapsto \left[Y(t) \,:\, (\lambda^i) \mapsto x(t) + \lambda^i e_i(t)\right]$$

*that is assumed to be neither $r$-cylindric for any $r = 1,\ldots,p-1$ nor orthogonally $s$-cylindric for any $s = 1,\ldots,q-1$. There exists a linear orthonormal frame field in $E^{p+q}$*

$$\bar{A}(s) = [\bar{a}_1(s),\ldots,\bar{a}_q(s)\,,\ \bar{e}_1(s),\ldots,\bar{e}_p(s)\,,\ \bar{x}(s)]$$

*such that*

$$\text{A p.c. } \Sigma \,:\, s \mapsto \left[\bar{Y}(s) \,:\, (\lambda^i) \mapsto \bar{x}(s) + \lambda^i \bar{e}_i(s)\right] \tag{T1,1}$$

*and $\Sigma$ determine the same curve $[\bar{\Sigma}] = [\Sigma]$ in $P(p,q)$.*

$$
\begin{aligned}
&\bar{e}_1 = e_\circ \text{ is the orienting directional vector of } \Sigma\\
&\bar{a}_1 = a_\circ \text{ is the normal orienting directional vector of } \Sigma\\
&\bar{e}_2 = \frac{\dot{\bar{e}}_1 - (\dot{\bar{e}}_1, \bar{a}_1)\bar{a}_1}{\sqrt{(\dot{\bar{e}}_1)^2 - (\dot{\bar{e}}_1, \bar{a}_1)^2}}\\
&\bar{a}_2 = \frac{\dot{\bar{a}}_1 - (\dot{\bar{a}}_1, \bar{e}_1)\bar{e}_1}{\sqrt{(\dot{\bar{a}}_1)^2 - (\dot{\bar{a}}_1, \bar{e}_1)^2}}\\
&\bar{e}_i = \frac{\dot{\bar{e}}_{i-1} - (\dot{\bar{e}}_{i-1}, \bar{e}_{i-2})\bar{e}_{i-2}}{\sqrt{(\dot{\bar{e}}_{i-1})^2 - (\dot{\bar{e}}_{i-1}, \bar{e}_{i-2})^2}}, \quad i = 3, \ldots, p\\
&\bar{a}_\alpha = \frac{\dot{\bar{a}}_{\alpha-1} - (\dot{\bar{a}}_{\alpha-1}, \bar{a}_{\alpha-2})\bar{a}_{\alpha-2}}{\sqrt{(\dot{\bar{a}}_{\alpha-1})^2 - (\dot{\bar{a}}_{\alpha-1}, \bar{a}_{\alpha-2})^2}}, \quad \alpha = 3, \ldots, q
\end{aligned}
\qquad (T1,2)
$$

*$\bar{x}$ is the striction line of $\Sigma$.*

*The equations of the linear orthonormal frame field $\bar{A}$ are of the form:*

$$
\begin{aligned}
&(e)\quad
\begin{cases}
\dot{\bar{e}}_1 = \bar{a}_1 + F_1^2 \bar{e}_2\\
\dot{\bar{e}}_k = F_k^{k-1}\bar{e}_{k-1} + F_k^{k+1}\bar{e}_{k+1}, \quad k = 2, \ldots, p-1\\
\dot{\bar{e}}_p = F_p^{p-1}\bar{e}_{p-1}
\end{cases}\\
&\qquad\qquad F_k^{k+1} = -F_{k+1}^k\\
&(a)\quad
\begin{cases}
\dot{\bar{a}}_1 = -\bar{e}_1 + f_1^2 \bar{a}_2\\
\dot{\bar{a}}_\beta = f_\beta^{\beta-1}\bar{a}_{\beta-1} + f_\beta^{\beta+1}\bar{a}_{\beta+1}, \quad \beta = 2, \ldots, q-1\\
\dot{\bar{a}}_q = f_q^{q-1}\bar{a}_{q-1}
\end{cases}\\
&\qquad\qquad f_\beta^{\beta+1} = -f_{\beta+1}^\beta\\
&(x)\quad
\begin{cases}
\dot{\bar{x}} = f^\alpha \bar{a}_\alpha + F\bar{e}_p, \quad \alpha = 2, \ldots, q\\
(\dot{\bar{x}} = F\bar{e}_p \text{ when } q = 1)
\end{cases}
\end{aligned}
\qquad (T1,3)
$$

*where*

$$(d\bar{e}_1, \bar{a}_1) = \Pi =: ds \quad (\text{i.e. } (\dot{\bar{e}}_1(s), \bar{a}_1(s)) = 1)$$

$$F_i^{i+1}(s) = -F_{i+1}^i(s) := (\dot{\bar{e}}_i(s), \bar{e}_{i+1}(s)), \quad i = 1, \ldots, p-1$$

$$F_i^{i+1}(s) > 0 \text{ for } i = 1, \ldots, p-2$$

$$f_\gamma^{\gamma+1}(s) = -f_{\gamma+1}^\gamma(s) := (\dot{\bar{a}}_\gamma(s), \bar{a}_{\gamma+1}(s)), \quad \gamma = 1, \ldots, q-1$$

$$f_\gamma^{\gamma+1}(s) > 0 \text{ for } \gamma = 1, \ldots, q-2$$

$$f^\gamma(s) := (\dot{\bar{x}}(s), \bar{a}_\gamma(s)), \quad \gamma = 2, \ldots, q$$

$$F(s) := (\dot{\bar{x}}(s), \bar{e}_p(s)).$$

*are invariants of* $\Sigma$.

**Definition 10.** (a) The vectors $\bar{e}_i$ $(i = 1,\ldots,p)$ and $\bar{a}_\alpha$ $(\alpha = 1,\ldots,q)$ determined in (T1,2) we shall call the $i$-th directional vector and the $\alpha$-th normal directional vector of $\Sigma$ respectively.

(b) The appearing in (T1,3) scalar functions we shall call as follows:

$F_i^{i+1} > 0$ $(i = 1,\ldots,p-2)$ the $i$-th curvature of $\Sigma$

$F_{p-1}^p$ the torsion of $\Sigma$

$f_\gamma^{\gamma+1} > 0$ $(\gamma = 1,\ldots,q-2)$ the $\gamma$-th normal curvature of $\Sigma$

$f_{q-1}^q$ the normal torsion of $\Sigma$

$f^\gamma$ $(\gamma = 2,\ldots,q)$ the $\gamma$-th strictional curvature of $\Sigma$

$F$ the strictional torsion of $\Sigma$.

**Theorem 2.** *Given scalar functions (of the class* $C^\infty$*) of the parameter* $s \in R$

$$F_i^{i+1} : s \mapsto F_i^{i+1}(s)\ , \quad i = 1,\ldots,p-1$$

$$F_j^{j+1}(s) > 0 \quad \text{for } j = 1,\ldots,p-2$$

$$f_\alpha^{\alpha+1} : s \mapsto f_\alpha^{\alpha+1}(s)\ , \quad \alpha = 1,\ldots,q-1$$

$$f_\beta^{\beta+1}(s) > 0 \quad \text{for } \beta = 1,\ldots,q-2$$

$$f^\gamma : s \mapsto f^\gamma(s)\ , \quad \gamma = 2,\ldots,q$$

$$F : s \mapsto F(s)$$

*Then the system of equations of the form* (e) (a) (x) *from* (T1,3) *determines:*
(T2,1): *A linear orthonormal frame field in* $E^{p+q}$

$$A(s) = [a_1(s),\ldots,a_q(s)\ , e_1(s),\ldots,e_p(s)\ , x(s)]$$

*up to the isometry in* $E^{p+q}$.
(T2,2): *An* a.o.p.c. $\Sigma : s \mapsto [Y(s) : (\lambda^i) \mapsto x(s) + \lambda^i e_i(s)]$ *in* $P(p,q)$ *(up to the isometry) that is neither* $r$*-cylindric for each* $r = 1,\ldots,p-1$ *nor orthogonally* $s$*-cylindric for each* $s = 1,\ldots,q-1$, *and*

1. $e_1$ *is the* 1*-th orienting directional vector of* $\Sigma$

2. $e_j$ $(j = 2,\ldots,p)$ *is the* $j$*-th directional vector of* $\Sigma$

3. $a_1$ *is the* 1*-th normal orienting directional vector of* $\Sigma$

4. $a_\beta$ $(\beta = 2,\ldots,q)$ *is the* $\beta$*-th normal directional vector of* $\Sigma$

5. $x$ *is the striction line of* $\Sigma$

6. $F_i^{i+1}$ $(i = 1,\ldots,p-2)$ *is the* $i$*-th curvature of* $\Sigma$

7. $F^{p}_{p-1}$ *is the torsion of* $\Sigma$

8. $f^{\alpha+1}_{\alpha}$ $(\alpha = 1,\dots,q-2)$ *is the $\alpha$-th normal curvature of* $\Sigma$

9. $f^{q}_{q-1}$ *is the normal torsion of* $\Sigma$

10. $f^{\gamma}$ $(\gamma = 2,\dots,q)$ *is the $\gamma$-th strictional curvature of* $\Sigma$

11. $F$ *is the strictional torsion of* $\Sigma$.

**Proof.** It is well known that the system of differential equations (T1,3) has the unique solution $A(s) = [a_1(s),\dots,a_q(s)\ ,\ e_1(s),\dots,e_p(s)\ ,\ x(s)]$ with the initial condition $A(s_0) = A^0 = [a^0_1,\dots,a^0_q\ ,\ e^0_1,\dots,e^0_p\ ,\ x^0]$. Because of the skewsymmetricity of the matrices of parts (e) and (a) of the system (T1,3), if $A^0$ is an orthonormal frame in $E^{p+q}$ so is $A(s)$ for each $s$. Since $A^0$ is an arbitrary orthonormal frame then $A$ is determined up to the isometry. (T2,2) is a direct consequence of the properties of solutions of (T1,3). Thus $\dim(\operatorname{lin}(e_1,\dots,e_p\ ,\ \dot e_1,\dots,\dot e_p)) = p+1$ and $\dim(\operatorname{lin}(e_1,\dots,e_p\ ,\ \dot e_2,\dots,\dot e_p)) = p$ then $\Sigma$ is admissible and $e_1$ is the 1-th orienting directional vector of $\Sigma$. Moreover, $\dot e_1 - \sum_{i=2}^{p}(\dot e_1, e_i)e_i = a_i$ than $a_i$ is the 1-th normal orienting directional vector of $\Sigma$. Easy calculations show that solutions $e_2,\dots,e_p\ ,\ a_2,\dots,a_q$ of (e) and (a) must be of the form (T1,2) then they are succesive the directional vectors and the normal directional vectors of $\Sigma$ respectively. Since $F^{k+1}_k \neq 0$ for $k = 1,\dots,p-1$ and $f^{\beta+1}_{\beta} \neq 0$ for $\beta = 1,\dots,q-1$ $\Sigma$ is neither $k$-cylindric nor orthogonally $\beta$-cylindric. Because of (T1,3) (x) we have $(\dot x, a_1) = 0$ and $(\dot x, e_i) = 0$ for $i = 1,\dots,p-1$ so $x$ is the striction line of $\Sigma$. The names of the functions $F^{i+1}_i$, $f^{\alpha+1}_{\alpha}$, $f^{\gamma}$, $F$ are suitable because of Definition 10.

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## STRESZCZENIE

Niech $P(p,q)$ będzie przestrzenią jednorodną płaszczyzn $p$-wymiarowych w $p+q$-wymiarowej przestrzeni euklidesowej $E^{p+q}$. W pracy zdefiniowano pojęcie krzywej w $P(p,q)$ oraz wyróżniono specjalne typy krzywych: krzywe dopuszczalne, krzywe $r$-walcowe i krzywe ortogonalnie $s$-walcowe. Wprowadzone zostały pojęcia wektora kierunkowego krzywej dopuszczalnej. Następnie, przy użyciu metody opracowanej przez K.Radziszewskiego, skonstruowano pole ortonormalnych reperów liniowych dla szerokiej klasy krzywych w $P(p,q)$. Uzyskano zupełny układ niezmienników krzywej w $P(p,q)$ oraz równanie różniczkowe skonstruowanego pola reperów. Pracę kończy twierdzenie o wyznaczaniu krzywej w $P(p,q)$ przez zupełny układ jej niezmienników.

## РЕЗЮМЕ

Пусть $P(p,q)$ обозначает однородное пространство $p$-мерных плоскостей в $p+q$-мерном эвклидовом пространстве $E^{p+q}$. В работе определено понятие кривой в $P(p,q)$ и выделены специальные типы кривых: допустимые кривые, $r$-цилиндрические кривые и ортогонольно $s$-цилиндрические кривые. Введены понятия направляющего вектора и нормального нопраалеющего вектора допустимой кривой. Затем, пользуясь методом разработанным К.Радзишевским было сконструировано поле ортонормальных линейных реперов для широкого класса кривых в $P(p,q)$. Получена полная система инвариантов кривой в $P(p,q)$ и дифференциальное уравнение сконструированного поля реперов. Работу кончает теорема об определению кривой в $P(p,q)$ полной системой ее инвариантов.