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Meromorphic Starlike Univalent Functions with Positive Coefficients

Funkcje meromorficzne, gwiaździste i jednolistne o dodatnich współczynnikach

Мероморфические звездочные и однолистные функции с позитивными коэффициентами

1. Let S denote the class of functions of the form: $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ that are analytic in $U = \{z : |z| < 1\}$. Denote by $S^*(\alpha)$ and $K(\alpha)$, $(0 \le \alpha < 1)$ the subclasses of functions g in S that satisfy respectively the conditions: $\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \alpha$ and $\mathrm{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} > \alpha$ for $z \in U$.

Let T denote the subclass of functions in S of the form: $g(z) = z - \sum_{n=1}^{\infty} b_n z^n$, $b_n \ge 0$. Also set $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$. The classes $T^*(\alpha)$ and $C(\alpha)$ possess some interesting properties and have been recently studied in detail by Silverman and others (See, e.g., [10] to [14]).

Let Σ denote the class of functions of the form:

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are regular in $D = \{z : 0 < |z| < 1\}$, having a simple pole at the origin. Let Σ_S denote the class of functions in Σ which are univalent in D and $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$ $(0 \le \alpha \le 1)$ be the subclasses of functions $f(z)$ in Σ satisfying respectively the conditions:

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

and

$$\operatorname{Re}\left\{-(1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad \text{for } z \in U, \tag{1.3}$$

it being understood that if $\alpha = 1$ then $f(z) = 1/z$ is the only function which is in $\Sigma^*(1)$ and $\Sigma_K(1)$. Functions in $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$ are called meromorphically starlike functions of order α and meromorphically convex functions of order α respectively.

The classes $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$ have been extensively studied by Pommerenke [7], Clunie [2], Kaczmarski [5], Royster [9] and others.

Since to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, one is tempted to search for a class of functions which are regular in D with simple pole at the origin having properties analogous to those of $T^*(\alpha)$. To this end we introduce in this paper such a class of functions which are regular in D and which demonstrate properties similar to those of $T^*(\alpha)$.

Let Σ_M denote the subclass of functions in Σ_S of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad \text{with } a_n \ge 0 \tag{1.4}$$

and let

$$\Sigma_M^*(\alpha) = \Sigma_M \cap \Sigma^*(\alpha).$$

In Section 2 we find necessary and sufficient condition in terms of a'_n for a function f in Σ_M to be in $\Sigma_M^*(\alpha)$. Sharp coefficient estimates are obtained, these bounds being sharper then those obtained by Pommerenke [7] and Clunie [2]. Section 3 is devoted to obtain distortion properties and radius of meromorphic convexity of order δ $(0 \le \delta < 1)$ for functions in $\Sigma_M^*(\alpha)$. In Section 4 we study integral transforms of functions in $\Sigma_M^*(\alpha)$. In Section 5 it is shown that the class $\Sigma_M^*(\alpha)$ is closed under convex linear combinations. The last section deals with certain convolution properties of functions in $\Sigma_M^*(\alpha)$.

2. Coefficient inequalities for the class $\Sigma_M^*(\alpha)$. We first obtain a sufficient condition for a function $f(z)$ in Σ to be in $\Sigma^*(\alpha)$.

Theorem 1 *Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ be regular in D. If*

$$\sum_{n=1}^{\infty} (n + \alpha)|a_n| \le 1 - \alpha \quad (0 \le \alpha \le 1) \tag{2.1}$$

then $f(z) \in \Sigma^*(\alpha)$.

Proof: It is easy to see that (2.1) implies $f(z) \neq 0$ in D. Suppose (2.1) holds. If $\alpha = 1$ then (2.1) gives $a_n = 0$ for $n = 1, 2, \ldots$ and so $f(z) = 1/z$ is in $\Sigma^*(1)$. For $0 \leq \alpha < 1$, consider the expression:

$$H(f, f') = |zf'(z) + f(z)| - |zf'(z) + (2\alpha - 1)f(z)| \tag{2.2}$$

Replacing f and f' by their series expansions we have, for $0 < |z| = r < 1$:

$$H(f, f') = \left|\sum_{n=1}^{\infty}(n+1)a_n z^n\right| - \left|2(\alpha - 1)\frac{1}{z} + \sum_{n=1}^{\infty}(n + 2\alpha - 1)a_n z^n\right| \tag{2.3}$$

or

$$\begin{aligned} rH(f, f') &\leq \sum_{n=1}^{\infty}(n+1)\,|a_n|\,r^{n+1} - 2(1-\alpha) + \sum_{n=1}^{\infty}(n + 2\alpha - 1)\,|a_n r^{n+1}| \\ &= \sum_{n=1}^{\infty} 2(n+\alpha)\,|a_n|\,r^{n+1} - 2(1-\alpha). \end{aligned}$$

Since this holds for all r, $0 < r < 1$, making $r \longrightarrow 1$, we have:

$$H(f, f') \leq \sum_{n=1}^{\infty} 2(n+\alpha)|a_n| - 2(1-\alpha) \leq 0 \tag{2.4}$$

in view of (2.1). From (2,2), we thus have:

$$\left|\left[\frac{zf'(z)}{f(z)} + 1\right]\left[\frac{zf'(z)}{f(z)} + (2\alpha - 1)\right]^{-1}\right| \leq 1$$

or

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha$$

Hence $f(z) \in \Sigma^*(\alpha)$.

Theorem 2. *Let* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ *be regular in* D. *Then* $f(z) \in \Sigma_M^*(\alpha)$ *if and only if* (2.1) *is satisfied.*

Proof. In view of Theorem 1 it is sufficient to show the "only if" part. Let us assume that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ is in $\Sigma_M^*(\alpha)$, i.e.,

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\alpha, \quad z \in D. \tag{2.5}$$

Replacing $f(z)$ and $f'(z)$ in (2.5) by their series expansions we have:

$$\text{Re}\left\{\frac{-\frac{1}{z} + \sum_{n=1}^{\infty} na_n z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n}\right\} < -\alpha \quad z \in D \tag{2.6}$$

When z is real $\frac{zf'(z)}{f(z)}$ is real and since $a_n \geq 0$, making $z \longrightarrow 1$ through positive values, (2.6) becomes:

$$\frac{-1+\sum_{n=1}^{\infty} na_n}{1+\sum_{n=1}^{\infty} a_n} \leq -\alpha \tag{2.7}$$

Since $\left(1+\sum_{n=1}^{\infty} a_n\right) > 0$, from (2.7) we have:

$$\sum_{n=1}^{\infty}(n+\alpha)a_n \leq 1-\alpha$$

Hence the result follows.

Corollary 1. *If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$ is in $\Sigma_M^*(\alpha)$, then:*

$$|a_n| \leq \frac{1-\alpha}{n+\alpha}, \quad n = 1, 2, \ldots \tag{2.8}$$

with equality, for each n, for functions of the form:

$$f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n+\alpha} z^n. \tag{2.9}$$

Remark: It was shown by Pommerenke[7] that for $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ in $\Sigma^*(\alpha)$, one has $|a_n| \leq \frac{2(1-\alpha)}{n+1}$. The coefficient estimates obtained in (2.8) are sharper then the above estimates.

3. Distortion properties and radius of convexity estimates.

Theorem 3. *If $f(z)$ is in $\Sigma_M^*(\alpha)$, then:*

$$\frac{1}{r} - \frac{1-\alpha}{1+\alpha} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{1+\alpha} r \quad \text{for} \quad 0 < |z| = r < 1. \tag{3.1}$$

Equality holds for the function:

$$f_1(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z \quad \text{at} \quad z = ir, r. \tag{3.2}$$

Proof. Since $f(z)$ in $\Sigma_M^*(\alpha)$ implies that $\sum_{n=1}^{\infty}(n+\alpha)a_n \leq 1-\alpha$ one has:

$$(1+\alpha)\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty}(n+\alpha)a_n \leq 1-\alpha,$$

or

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{1+\alpha}. \tag{3.3}$$

Now

$$|f(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right| \leq \left|\frac{1}{z}\right| + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \leq$$

$$\leq \frac{1}{r} + \frac{1-\alpha}{1+\alpha},$$

by (3.3) for $0 < |z| < 1$. This gives the right side of (3.1). Also

$$|f(z)| \geq \frac{1}{r} - \sum_{n=1}^{\infty} a_n r^n \geq \frac{1}{r} - r\sum_{n=1}^{\infty} a_n \geq \frac{1}{r} - r\frac{1-\alpha}{1+\alpha}$$

which is the left side of (3.1). It can be easily seen that the function $f_1(z)$ defined by (3.2) is extremal for the theorem.

Theorem 4. *If $f(z)$ is in $\Sigma_M^*(\alpha)$, then $f(z)$ is meromorphically convex of order*

$$\delta\ (0 \leq \delta < 1)\ \textit{in}\ |z| < \gamma = \gamma(\alpha, \delta) = \inf_n \left[\frac{(n+\alpha)(1-\delta)}{n(n+2-\delta)(1-\alpha)}\right]^{\frac{1}{n+1}}$$

and the result is sharp for each n for functions of the form (2.9).

Proof. Let $f(z) \in \Sigma_M^*$. In view of (1.3) it is sufficient to show that:

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| \leq 1-\delta \quad \text{for} \quad |z| < \gamma(\alpha, \beta) \tag{3.4}$$

where $\gamma(\alpha, \beta)$ is as specified in the statement of the theorem; or equivalently, to show that:

$$\left|\frac{f'(z) + (zf'(z))'}{f'(z)}\right| \leq 1-\delta \quad \text{for} \quad |z| < \gamma(\alpha, \beta). \tag{3.5}$$

Substituting the series for $f'(z)$, $(zf'(z))'$ in the left side of (3.5) we have:

$$\left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} + \sum_{n=1}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} na_n |z|^{n+1}}$$

This will be bounded by $1-\delta$ if :

$$\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1} \leq (1-\delta)\left(1 - \sum_{n=1}^{\infty} na_n |z|^{n+1}\right)$$

or

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \le 1 . \tag{3.6}$$

Since for $f(z) \in \Sigma_M^*(\alpha)$ we have:

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n \le 1$$

(3.6) will be satisfied if

$$\frac{n(n+2-\delta)}{1-\delta} |z|^{n+1} \le \frac{n+\alpha}{1-\alpha}$$

for each n, or

$$|z| \le \inf_n \left[\frac{(n+\alpha)(1-\delta)}{n(n+2-\delta)(1-\alpha)} \right]^{\frac{1}{n+1}} = \gamma(\alpha, \beta).$$

Sharpness can be verified easily.

4. Integral transforms. In this section we consider integral transforms of functions in $\Sigma_M^*(\alpha)$ of the type considered by Bajpai [1] and Goel and Sohi [3].

Theorem 5. *If $f(z) \in \Sigma_M^*(\alpha)$, then the integral transform*

$$F(z) = c \int_0^1 u^c f(uz) du , \quad \text{for} \quad 0 < c < \infty \tag{4.1}$$

is in $\Sigma_M^*(\beta)$, where

$$\beta = (\alpha, c) = \frac{(1+\alpha)(2+c) - c(1-\alpha)}{(1+\alpha)(2+c) + c(1+\alpha)} .$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z .$$

Proof. Suppose

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_M^*(\alpha) ,$$

then

$$F(z) = c \int_0^1 u^c f(uz) du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c a_n}{n+c+1} z^n$$

It is sufficient to show that:

$$\sum_{n=1}^{\infty} \frac{(n+\beta) c a_n}{(1-\beta)(n+c+1)} \le 1. \tag{4.2}$$

Since $f(z) \in \Sigma^*_M(\alpha)$ implies that $\sum_{n=1}^{\infty} \frac{(n+\alpha)}{1-\alpha} a_n \leq 1$, (4.2) will be satisfied it

$$\frac{c(n+\beta)}{(1-\beta)(n+c+1)} \leq \frac{n+\alpha}{1-\alpha} \text{ for each } n \text{ or}$$

$$\beta \leq \frac{(n+\alpha)(n+c+1) - cn(1-\alpha)}{(n+\alpha)(n+c+1) + c(1-\alpha)} \tag{4.3}$$

The right side of (4.3) is an increasing function of n, therefore putting $n = 1$ in (4.3) we get:

$$\beta \leq \frac{(1+\alpha)(2+c) - c(1-\alpha)}{(1+\alpha)(2+c) + c(1-\alpha)}.$$

Hence the theorem.

Remark. It is interesting to note that for $c = 1$ and $\alpha = 0$ Theorem 5 gives that if $f(z) \in \Sigma^*_M(\alpha)$ then $F(z) = \int_0^1 uf(uz)du$ is in $\Sigma^*_M(\frac{1}{2})$.

5. Convex linear combinations. In this section we shall prove that the class $\Sigma^*_M(\alpha)$ is closed under convex linear combinations.

Theorem 6. *Let*

$$f_0(z) = \frac{1}{z}$$

$$f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n+\alpha} z^n \ (0 \leq \alpha \leq 1),\ n = 1, 2, \ldots . \tag{5.1}$$

Then $f(z) \in \Sigma^*_M(\alpha)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \text{ with } \lambda_n \geq 0 \text{ and } \sum_{n=0}^{\infty} \lambda_n = 1 . \tag{5.2}$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \text{ with } \lambda_n \geq 0 \text{ and } \sum_{n=0}^{\infty} \lambda_n = 1 .$$

Since

$$\sum_{n=0}^{\infty} \lambda_n f_n(z) = \lambda_0 f_0 + \sum_{n=0}^{\infty} \lambda_n f_n(z) =$$

$$= (1 - \sum_{n=1}^{\infty} \lambda_n) f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z) =$$

$$= (1 - \sum_{n=1}^{\infty} \lambda_n) \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n (\frac{1}{z} + \frac{1-\alpha}{n+\alpha} =$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \frac{1-\alpha}{n+\alpha} z^n$$

and

$$\sum_{n=1}^{\infty}\lambda_n+\frac{1-\alpha}{n+\alpha}\frac{n+\alpha}{1-\alpha}=\sum_{n=1}^{\infty}\lambda_n=1-\lambda_0\leq 1,$$

the coefficients of $f(z)$ satisfy the coefficient inequality (2.1). Thus from Theorem 2, $f(z)\in\Sigma_M^*(\alpha)$.

Conversely, suppose $f(z)\in\Sigma_M^*(\alpha)$. Since

$$a_n\leq\frac{1-\alpha}{n+\alpha}\text{ for }n=1,2,\ldots,\text{ setting }\lambda_n=\frac{n+\alpha}{1-\alpha}a_n, n=1,2,\ldots\text{ and }\lambda_0=1-\sum_{n=1}^{\infty}\lambda_n,$$

it follows that

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem.

6. Convolution properties of functions in $\Sigma_M^*(\alpha)$. It was shown by Robertson [8] that if

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\text{ and }g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_n z^n$$

are in Σ_S then so is their convolution

$$f(z)*g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n b_n z^n.$$

For functions in $\Sigma_M^*(\alpha)$ much more can be said.

Theorem 7. *If*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n,\qquad g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_n z^n$$

are elements of $\Sigma_M^*(\alpha)$, *then*

$$h(z)=f(z)*g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n b_n z^n$$

is an element of

$$\Sigma_M^*\left(\frac{2\alpha}{1+\alpha^2}\right).$$

The result is best possible.

Proof. Since $f(z)$ and $g(z)$ are in $\Sigma_M^*(\alpha)$, (2.1) gives

$$\sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}a_n\leq 1\tag{6.1}$$

and

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} b_n \leq 1. \tag{6.2}$$

Since $f(z)$ and $g(z)$ are regular in $D = \{z : 0 < |z| < 1\}$, so is $f(z) * g(z)$. Further,

$$\sum_{n=1}^{\infty} \frac{n + \frac{2\alpha}{1+\alpha^2}}{1 - \frac{2\alpha}{1+\alpha^2}} a_n b_n = \sum_{n=1}^{\infty} \frac{n(1+\alpha^2)+2\alpha}{(1-\alpha)^2} a_n b_n \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{n+\alpha^2}{(1-\alpha)^2} a_n b_n \leq \left(\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n\right)\left(\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} b_n\right) \leq 1$$

in view of (6.1) and (6.2). Thus, by Theorem 1,

$$h(z) \in \Sigma_M^*\left(\frac{2\alpha}{1+\alpha^2}\right).$$

The result is sharp with equality for

$$f(z) = g(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z\,.$$

Remark. If $f(z)$ and $g(z)$ are in $\Sigma_M^*(0)$, then, according to above theorem, $f(z) * g(z)$ is also in $\Sigma_M^*(0)$, the result being best possible. This is in sharp contrast with corresponding result for $T^*(0)$ (cf. Remark following Theorem 1 of Schild and Silverman[10]).

Theorem 8. *If $f(z) \in \Sigma_M^*(\alpha)$ and $g(z) \in \Sigma_M^*(\gamma)$ then*

$$f * g \in \Sigma_M^*\left(\frac{\alpha+\gamma}{1+\alpha\gamma}\right);$$

the result being best possible.

Proof. The lines of proof are the same as thoses of Theorem 7. In fact:

$$\sum_{n=1}^{\infty} \frac{n + \frac{\alpha+\gamma}{1+\alpha\gamma}}{1 - \frac{\alpha+\gamma}{1+\alpha\gamma}} a_n b_n = \sum_{n=1}^{\infty} \frac{n(1+\alpha\gamma)+(\alpha+\gamma)}{(1-\alpha)(1-\gamma)} a_n b_n \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{(n+\alpha)(n+\gamma)}{(1-\alpha)(1-\gamma)} a_n b_n \leq \left(\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n\right)\left(\sum_{n=1}^{\infty} \frac{n+\gamma}{1-\gamma} b_n\right) \leq 1$$

since $f(z) \in \Sigma_M^*(\alpha)$ and $g(z) \in \Sigma_M^*(\gamma)$. Thus

$$f * g \in \Sigma_M^*\left(\frac{\alpha+\gamma}{1+\alpha\gamma}\right).$$

The result is best possible for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha}z\,, \qquad g(z) = \frac{1}{z} + \frac{1-\gamma}{1+\gamma}z\,.$$

Theorem 9. *If*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_M^*(\alpha) \text{ and } g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ with } |b_n| \le 1 \text{ for } n = 1, 2, \ldots$$

then $f * g \in \Sigma_M^*(\alpha)$.

Proof.

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|a_n b_n| = \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n |b_n| \le \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n \le 1\,.$$

Hence by Theorem 1, $f * g \in \Sigma^*(\alpha)$.

Here it may be noted that $g(z)$ need not even be univalent. For example, if $g(z) = \frac{1}{z} - \frac{2}{3}z^2$, then $\left|-\frac{2}{3}\right| < 1$, but $g'(z) = -\frac{1}{z^2} - \frac{4}{3}z = -\frac{1}{z^2}\left(1 + \frac{4}{3}z^3\right) = 0$ for $z = \left(-\frac{3}{4}\right)^{1/3}$ which lies inside D.

Corollary. If

$$f(z) \in \Sigma_M^* \text{ and } g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$$

with $0 \le b_n \le 1$, for $n = 1, 2, \ldots$, then $f * g \in \Sigma_M^*(\alpha)$.

Theorem 10. *If* $f(z)$ *and* $g(z)$ *are in* $\Sigma_M^*(\alpha)$ *for* $3 - 2\sqrt{2} \le \alpha \le 1$, *then*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\, z^n \in \Sigma_M^*\left(\frac{4\alpha - (1-\alpha)^2}{4\alpha + 3(1-\alpha)^2}\right).$$

The result is sharp for the functions

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha}z = g(z)\,.$$

Proof. Since $f(z) \in \Sigma_M^*(\alpha)$ we have:

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n \le 1,$$

therefore

$$\sum_{n=1}^{\infty} \left(\frac{n+\alpha}{1-\alpha}\right)^2 a_n^2 \le \left(\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n\right)^2 \le 1.$$

Similarly

$$\sum_{n=1}^{\infty}\left(\frac{n+\alpha}{1-\alpha}\right)^2 b_n^2 \le 1;$$

hence

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{n+\alpha}{1-\alpha}\right)^2 (a_n^2+b_n^2) \le 1. \qquad (6.3)$$

We want to find largest $\beta = \beta(\alpha)$ such that

$$\sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}(a_n^2+b_n^2) \le 1. \qquad (6.4)$$

On comparing this with (6.3) we see that (6.4) is satisfied if

$$\frac{n+\beta}{1-\beta} \le \frac{1}{2}\left(\frac{n+\alpha}{1-\alpha}\right)^2$$

or

$$\beta \le \frac{\left(\frac{n+\alpha}{1-\alpha}\right)^2 - 2n}{\left(\frac{n+\alpha}{1-\alpha}\right)^2 + 2} \quad \text{for } n = 1, 2, \dots . \qquad (6.5)$$

The right side of (6.5) is an increasing function of n, hence the minimum value is obtained by setting $n = 1$. This gives

$$\beta \le \beta(\alpha) = \frac{\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 2}{\left(\frac{1+\alpha}{1-\alpha}\right)^2 + 2} = \frac{4\alpha-(1-\alpha)^2}{4\alpha+3(1-\alpha)^2}.$$

Remark. It may be noted that the quantity $\frac{4\alpha-(1-\alpha)^2}{4\alpha+3(1-\alpha)^2}$ is negative for $0 \le \alpha < 3 - 2\sqrt{2}$ (compare with Theorem 8 of Schild and Silverman [10]).

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STRESZCZENIE

Niech Σ oznacza klasę funkcji $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, regularnych w $D = \{z : 0 < |z| < 1\}$. Przez $\Sigma^*(\alpha)$, $0 \leq \alpha \leq 1$ oznacza się podklasę tej klasy, składająca się z funkcji f, spełniających warunek:

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in D,$$

a przez $\Sigma^*_M(\alpha)$ podklasę klasy $\Sigma^*(\alpha)$ składająca się z funkcji o nieujemnych współczynnikach.

W pracy podano warunek konieczny i wystarczający na to, aby funkcja $f \in \Sigma$ należała do klasy $\Sigma^*_M(\alpha)$. Wykorzystując ten warunek otrzymano twierdzenie dotyczące oszacowania współczynników, twierdzenie o zniekształceniu i wypukłości funkcji $f \in \Sigma^*_M(\alpha)$ oraz pewne rezultaty dotyczące kombinacji liniowych i splotu Hadamarda funkcji klasy $\Sigma^*_M(\alpha)$.

РЕЗЮМЕ

Пусть Σ обозначает класс функций $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, регулярных в $D = \{z : 0 < |z| < 1\}$, а $\Sigma^*(\alpha), 0 \leq \alpha \leq 1$ класс функций f таких, что $\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha$, $z \in D$. Через $\Sigma^*_M(\alpha)$ обозначим подкласс класса $\Sigma^*(\alpha)$ состоящий из функций с позитивными коэффициентами. В работе рассматривается конечное и достаточное условие на то, чтобы функция из Σ принадлежала к классу $\Sigma^*_M(\alpha)$. Авторы получают оценки коэффициентов. теоремы о деформировке и выпуклости функций из $\Sigma^*_M(\alpha)$ и другие результаты.